Streamfunction and Velocity Potential Representation of Equatorially Trapped Waves

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1. Introduction

In this note the velocity potential and streamfunction representations of linear equatorially trapped waves are derived. The motivation for this study is two-fold. First, insight into the dynamics of the trapped waves can be gained by the decomposition of the wind field into its divergent and nondivergent components. This decomposition will lead to a novel explanation for the lack of any meridional wind for the Kelvin wave. The relative magnitude of the divergent and nondivergent wind components will also give insight as to whether it is better to study tropical waves in the observations using velocity potential or streamfunction. Secondly, the recent availability of global datasets with streamfunction and velocity potential representations of the horizontal winds has led to many studies of planetary-scale fluctuations using these quantities. In particular Lorenc (1984), using empirical orthogonal functions (EOF) of velocity potential, portrayed the 40–50 day oscillation as an eastward propagating planetary-scale phenomenon centered on the equator. Because the velocity potential represents the largest scale of the divergence, it is not obvious in Lorenc's study whether the 40–50 day oscillation was truly a planetary-scale fluctuation or possibly largely confined to the tropics. By examining the linear equatorially trapped Kelvin wave we will demonstrate that it is indeed possible for an equatorially trapped wave to appear global in extent when represented by its velocity potential and streamfunction. This is not to imply that the 40–50 day oscillation seen by Lorenc was a Kelvin wave; rather, that due caution needs to be exercised when interpreting the results. Indeed, any equatorially trapped divergence field will possess a much larger-scale velocity potential because of their inverse Laplacian relationship.

The format of this note thus is to derive the velocity potential and streamfunction for known analytic linear trapped waves. Particular emphasis will be given to the Kelvin wave. A discussion of the dependence of the trapping, horizontal structure and relative magnitude of the divergent and nondivergent wind components on the frequency and wavenumber follows. Finally, the implications for future studies of equatorially trapped waves using the now readily available velocity potential and streamfunction are discussed.

2. Derivation of velocity potential and streamfunction

Following Matsuno (1966), the shallow water equations, linearized about a basic state at rest on an equatorial beta plane, are written as follows:

$$\frac{\partial u}{\partial t} - \beta y v + \frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + \beta y u + \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial \phi}{\partial t} + c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

where u, v are the perturbation velocities, c^2 ($=gH$) is the square of the pure gravity wave speed for a mean depth H, and ϕ ($=gh$) is the perturbation geopotential where h is the small surface deviation. Wave motion propagating in the east–west direction is assumed such that u, v and ϕ all have the factor $e^{i(kx-\omega t)}$.

For the Kelvin wave solution, $v \equiv 0$ and the solutions for u and ϕ are well known:

$$u = U_0 \exp(-ay^2)$$

$$\phi = cU_0 \exp(-ay^2)$$

where $a = \beta/2c$ and U_0 is an arbitrary amplitude. For our purposes $U_0 = 1$. Kelvin waves propagate nondispersively eastward according to the dispersion relation

$$\omega = kc.$$

The e-folding decay width is given by

$$Y_L = a^{-1/2}.$$

The velocity potential, χ, (assumed to also have the factor $e^{i(kx-\omega t)}$) is derived by expressing the horizontal divergence in terms of the Laplacian of χ:

$$\nabla \cdot V = \frac{\partial^2 \chi}{\partial y^2} - k^2 \chi = ik \exp(-ay^2).$$

The boundary conditions on χ are

$$\chi = 0 \quad \text{at} \quad y = \pm\infty.$$

Thus the total solution is

$$\chi = \frac{1}{4}\left(\frac{\pi}{a}\right)^{1/2} \exp(-i\pi/2 + k^2/4a) \times \left\{\exp(ky)\left[1 - \operatorname{erf}\left(\sqrt{a}y + \frac{k}{2\sqrt{a}}\right)\right] + \exp(-ky)\left[1 + \operatorname{erf}\left(\sqrt{a}y - \frac{k}{2\sqrt{a}}\right)\right]\right\}.$$

The error function is given by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)dt.$$

The streamfunction, ψ, is derived by expressing the meridional velocity in terms of χ and ψ:

$$v = ik\psi + \frac{\partial \chi}{\partial y} = 0.$$

Thus,

$$\psi = \frac{1}{4}\left(\frac{\pi}{a}\right)^{1/2} \exp(k^2/4a)\{\exp(ky)[1 - \operatorname{erf}(\sqrt{a}y + k/2\sqrt{a})] - \exp(-ky)[1 + \operatorname{erf}(\sqrt{a}y - k/2\sqrt{a})]\}.$$

The ratios of the nondivergent ($\mathbf{V}_\psi$) to divergent ($\mathbf{V}_\chi$) components of the wind are

$$\frac{v_\psi}{v_\chi} = -1, \quad \text{everywhere}$$

$$\frac{u_\psi}{u_\chi} = -1 + \exp(-ay^2)\left\langle k/4\left(\frac{\pi}{a}\right)^{1/2} \exp(k^2/4a) \times \left\{\exp(ky)\left[1 - \operatorname{erf}\left(\sqrt{a}y + \frac{k}{2\sqrt{a}}\right)\right] + \exp(-ky) \times \left[1 + \operatorname{erf}\left(\sqrt{a}y - \frac{k}{2\sqrt{a}}\right)\right]\right\}\right\rangle^{-1}.$$

Away from the equator $u_\psi/u_\chi = -1$, as does the ratio of the meridional velocities. The lack of meridional velocity for a Kelvin wave can be understood as the perfect cancellation of the divergent meridional wind with the nondivergent wind.

For purposes of example, the horizontal structure of an eastward propagating wavenumber 1 Kelvin wave with phase speed 15 m s^{-1} will be displayed. If the basic state were easterly at 5 m s^{-1}, then this wave would have period ~45 days. The *e*-folding width $Y_L = a^{-1/2} \approx 10°$ latitude.

The total horizontal wind field [$\exp(-ay^2)$, 0] is displayed in Fig. 1a. One wavelength in the zonal direction and the equivalent distance from South to North pole in the meridional direction are shown. Note that because $v = 0$, the u component of the wind is in exact geostrophic balance with the meridional pressure gradient. The corresponding divergence and vorticity are displayed in Figs. 1b and 1c. For this choice of frequency and wavenumber, the relative magnitude of the vorticity is about four times that of the divergence. The equatorial trapping is clearly seen in the wind, divergence and vorticity.

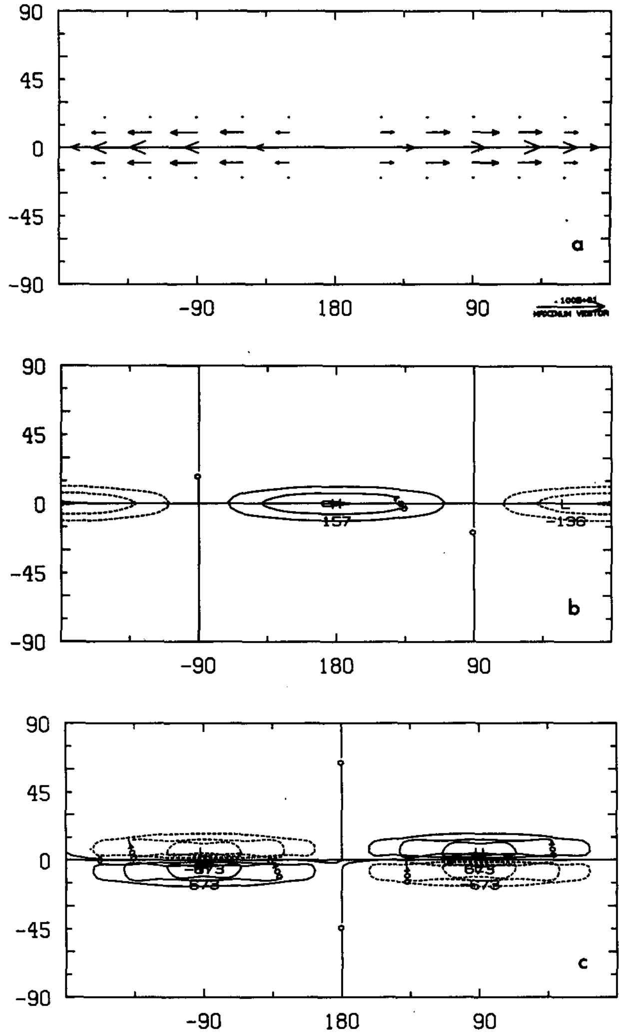

FIG. 1. (a) The velocity field, (b) the divergence and (c) the vorticity for the wavenumber 1 Kelvin wave wiith phase speed 15 m s^{-1}. The Y axis, labeled in degrees, represents an equivalent distance from South to North Pole. One wavelength is shown in the x direction. The maximum wind in (a) has unit magnitude (m s^{-1}) and the scale is shown in the lower right. The contour interval in (b) is 0.5 × 10^{-7} s^{-1} and in (c) 0.2 × 10^{-6} s^{-1}.

The corresponding velocity potential and streamfunction are displayed in Fig. 2. Both fields give the impression of significant velocity perturbations away from the equator despite the known trapping within 10°. The divergent and nondivergent wind components are displayed in Fig. 3. Away from the equator $u_\psi \approx -u_\chi$ and $v_\psi = -v_\chi$. This gives the appearance of significant amplitude in the individual components

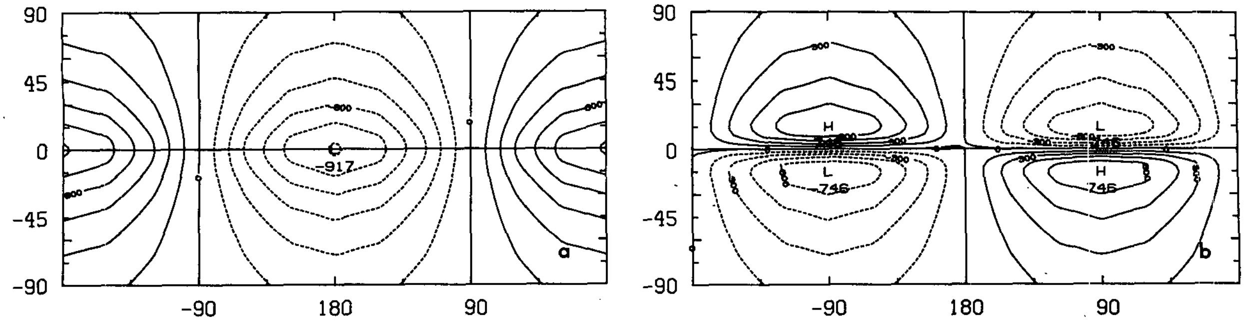

FIG. 2. (a) The velocity potential and (b) streamfunction for the Kelvin wave. The contour interval is 0.15×10^5 m^2 s^{-2}.

away from the equator. The ratio of the nondivergent to divergent zonal wind reaches a positive maximum along the equator (in this case $\approx +10$). Thus despite the fact the u is completely geostrophic and thus probably very divergent along the equator, the Kelvin wave is mostly nondivergent where the total velocity amplitude is large (near the equator) and most nondivergent on the equator.

All of the wave types found by Matsuno (1966) can be expressed in terms of velocity potential and streamfunction. As an example, the westward propagating mixed Rossby–gravity wave will be examined. Following Matsuno, the solutions for u and v for the Rossby–gravity wave are

$$v = \exp(-ay^2)$$

$$u = i\frac{\omega}{c}\, y \exp(-ay^2)$$

and the dispersion relation is

$$\omega/c - k - \beta/\omega = 0$$

Solutions for westward propagating waves are valid provided

$$1 + k\omega/\beta > 0.$$

Again the divergence is expressed as the Laplacian of the veliocity potential

$$\nabla \cdot \nabla = \frac{\partial^2 \chi}{\partial y^2} - k^2 \chi = Ay \exp(-ay^2),$$

where

$$A = -(2a + k\omega/c).$$

The total solution for χ is

$$\chi = \frac{A}{8a} \exp(k^2/4a)\left(\frac{\pi}{a}\right)^{1/2} \left\{ \exp(ky)\left[1 - \operatorname{erf}\left(\sqrt{a}y + \frac{k}{2\sqrt{a}}\right)\right] - \exp(-ky)\left[1 + \operatorname{erf}\left(\sqrt{a}y - \frac{k}{2\sqrt{a}}\right)\right]\right\}.$$

Again expressing v in terms of ψ and χ

$$v = ik\psi + \frac{\partial \chi}{\partial y} = \exp(-ay^2)$$

yields

$$\psi = \frac{i\omega}{2ac} \exp(-ay^2) + \frac{iA}{8a} \exp(k^2/4a)\left(\frac{\pi}{a}\right)^{1/2} \times \left\{ \exp(ky) \times \left[1 - \operatorname{erf}\left(\sqrt{a}y + \frac{k}{2\sqrt{a}}\right)\right] + \exp(-ky)\left[1 + \operatorname{erf}\left(\sqrt{a}y - \frac{k}{2\sqrt{a}}\right)\right]\right\}.$$

The e-folding width for the symmetric component of the wind (v) is

$$Y_L = a^{-1/2}.$$

Observed equatorially trapped Rossby-gravity waves (Yanai and Maruyama 1966) have periods 4–5 days

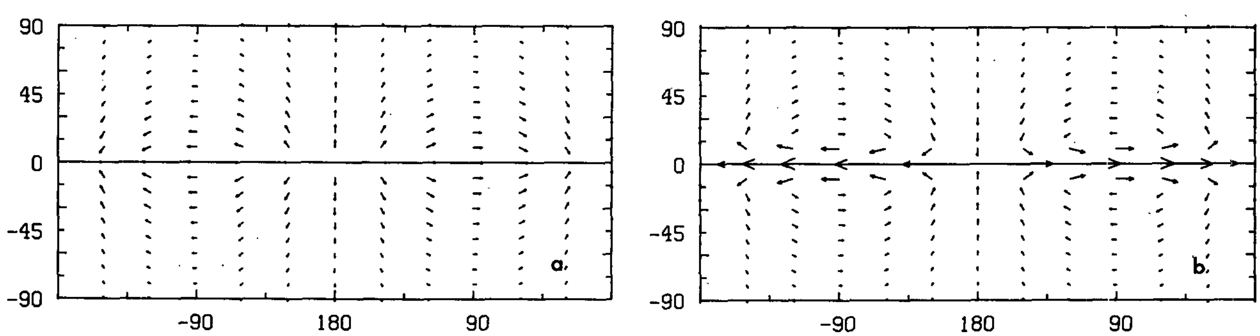

FIG. 3. (a) The divergent and (b) nondivergent components of the wind for the Kelvin wave. The vector scale is as in Fig. 1a.

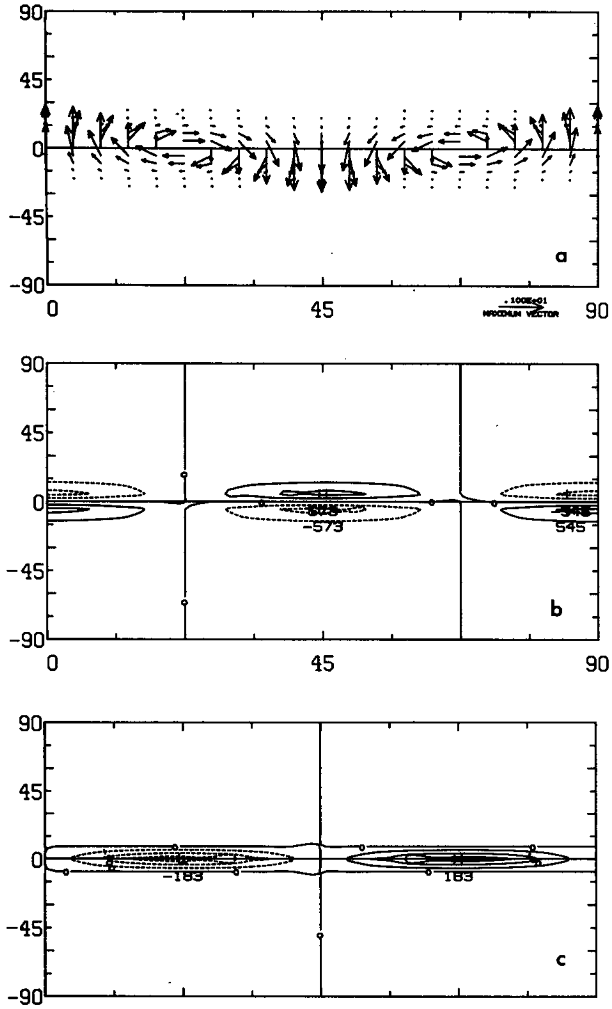

FIG. 4. (a) The velocity field, (b) divergence and (c) vorticity for the wavenumber 4 Rossby gravity wave with phase speed 20 m s^{-1}. One wavelength is shown in the x direction. The maximum wind vector has unit magnitude (m s^{-1}) and the scale is shown in the lower right. The contour interval in (b) is 0.25×10^{-6} s^{-1} and in (c) 0.5×10^{-6} s^{-1}.

and zonal wavenumber 4. The velocity, divergence and vorticity for this case are plotted in Fig. 4. The e-folding width for this example is $Y_L \approx 10°$. The corresponding velocity potential and streamfunction are shown in Fig. 5 and the divergent and nondivergent wind components in Fig. 6. In contrast to the Kelvin wave example, the velocity potential and streamfunction of the Rossby-gravity wave appear much more trapped. This may be partially due to the difference of zonal wavenumbers for each wave. The meridional scale of the streamfunction and velocity potential for the wavenumber 4 Rossby-gravity wave should be smaller than the wavenumber 1 Kelvin wave because of the isotropic nature of the Laplace operator. It must be kept in mind, though, that both these waves possess similar meridional e-folding decay widths for the velocity fields. The streamfunction and velocity potential, indeed, do have a larger meridional scale than the velocity perturbation, as is expected from the fact that the gradient of these fields yield the velocity. As for the Kelvin wave, away from the equator the divergent and nondivergent winds oppose each other. Near the equator, however, $u_\psi = -u_\chi$ and $v_\chi \sim v_\psi$. Thus the depiction of the Rossby–gravity wave depends on the proper determination of both the divergent and nondivergent winds.

3. Discussion

The representation of equatorially trapped waves in terms of velocity potential and streamfunction can give the appearance of the waves having planetary scale. This problem is especially acute for the wavenumber 1 Kelvin wave. The opposition of the divergent and nondivergent meridional winds everywhere and the opposition of the zonal components away from the equator contribute to this problem. The mixed Rossby–gravity wave (using the observed 4–5 day period wavenumber 4) appears much more trapped when expressed in terms of velocity potential and streamfunction. The major point of this note is to elucidate the possibility that planetary-scale velocity potential and streamfunction indeed may represent only an equatorially trapped phenomenon.

Whether the 40–50 day oscillation detected by Lorenc (1984) is essentially an equatorially trapped eastward propagating wave remains an unanswered question. A similar study by Lau and Chan (1985) using OLR data gave the appearance of the 40–50 day os-

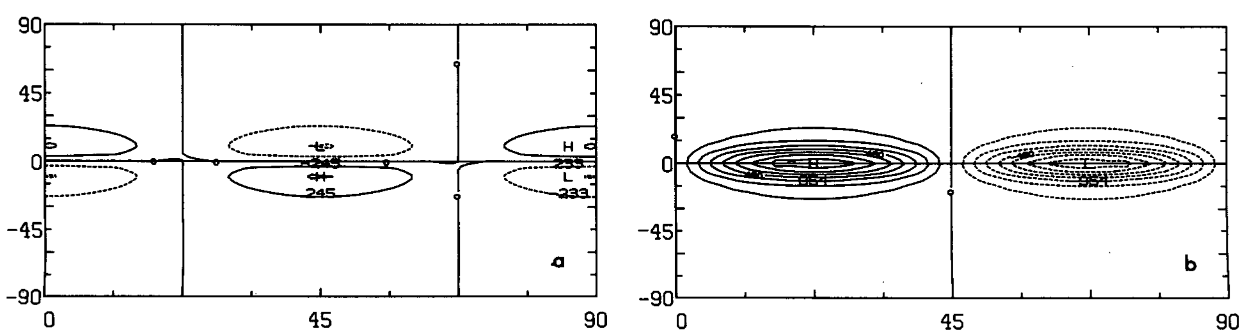

FIG. 5. (a) The velocity potential and (b) streamfunction for the Rossby–gravity wave. The contour interval is 0.12×10^6 m^2 s^{-2}.

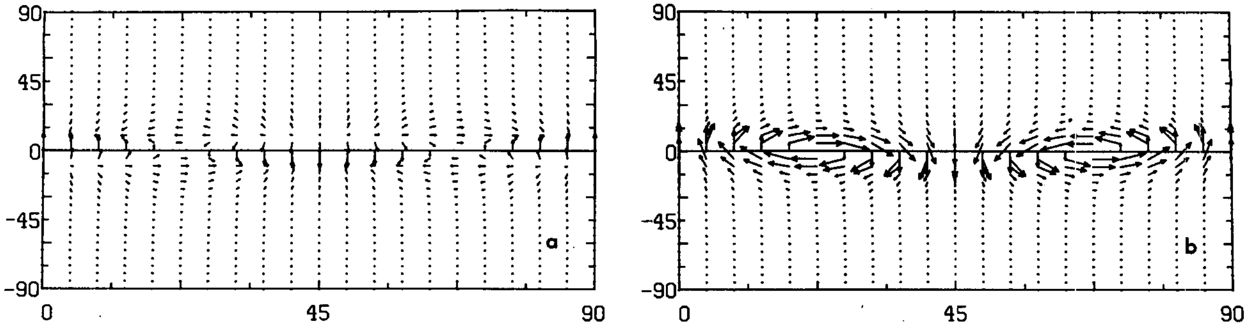

FIG. 6. (a) The divergent and (b) nondivergent components of the wind for the Rossby gravity wave. The vector scale is as in Fig. 4a.

cillation as an eastward propagating divergence–convergence pattern confined to the tropics. Thus, it seems possible that Lorenc's results may indeed be representative of an equatorially trapped wave. Clearly, more work is required. Examination of the streamfunction in a similar manner as the velocity potential in Lorenc's study would be of most interest. For the analytic linear Kelvin wave discussed in this note, the velocity perturbation along the equator was almost completely due to the nondivergent wind. Whether an EOF analysis of the streamfunction would detect this complimentary signal to the velocity potential is not known.

The relevance of the results discussed in this note of course depend on whether the observed atmosphere behaves as if it were linear with a basic state at rest on an equatorial beta plane. The observed atmosphere is spherical, has a nonzero, sheared basic state, and is nonlinear. The present results are hoped to suggest only the possibility that planetary-scale streamfunction and velocity potential may indeed represent just an equatorially trapped phenomenon and that the divergent and nondivergent components of the wind are both necessary to understand equatorial wave dynamics.

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