Comments on Mak's Closure for CISK in Geostrophic Systems

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1. Introduction

Recently a new closure condition for conditional instability of the second kind (CISK) in quasi-geostrophic systems was proposed by Mak (1982, 1983). It is the latest of many attempts to resolve the problem of the short-wave cutoff of CISK. Earlier attempts include the concept of movable CISK (Charney, 1973), Ekman damping (Chang and Williams, 1974), and frequency dependent heating parameter (Hayashi, 1971; Kuo, 1975).

Mak's closure condition was based on the traditional assumption that the condensational heating due to cumulus clouds is sustained by moisture convergence in the subcloud layer. But instead of relating the amount of heating to the total vertical velocity per se, Mak proposed that it is related only to the component of the vertical velocity induced by geostrophic dynamic processes. The vertical velocity induced by cumulus heating does not contribute to heating itself, according to this hypothesis proposed by Mak.

At a first look, Mak's hypothesis appears physically unreasonable. It is difficult to explain why moisture convergence induced by cumulus heating should not contribute to the heating itself. The purpose of this note is to further examine this question. It will be shown that the amount of moisture convergence in the subcloud layer induced by cumulus heating is not sufficient to sustain the heating mechanism under all conditions. Unless certain conditions are satisfied, additional forcing mechanism is needed to sustain cumulus activity. In this situation, Mak's hypothesis can be justified with a slight modification.

Since the analyses presented in this note are closely parallel to those made by Mak, all the notations used here will be the same as those given in Mak (1982) for ease of comparison.

2. Vertical circulation induced by condensational heating

In a geostrophic system, the vertical velocity field is governed by the omega equation:

\[ S \nabla^2 \omega + f_0^2 \frac{\partial^2 \omega}{\partial p^2} = \tilde{F}(\phi) + \tilde{F}(Q) \]  

where

\[ \tilde{F}(\phi) = f_0 \frac{\partial}{\partial p} \left[ \nabla \cdot \nabla (\phi + f) \right] - \nabla^2 \left[ \nabla \cdot \nabla \frac{\partial \phi}{\partial p} \right] \]  

is the geostrophic dynamic forcing function and

\[ \tilde{F}(Q) = -\frac{R}{C_p \rho} \nabla^2 Q \]

represents the thermal forcing due to cumulus clouds. The two components of the vertical velocity field induced by these forcing functions will be denoted as \( \tilde{\omega} \) and \( \tilde{\omega} \), respectively. Mak proposed that the cumulus heating function \( Q \) should be proportional to the mass convergence in the subcloud layer caused by dynamic forcing alone, i.e., \( \tilde{\omega} \) at the base level, denoted as \( \tilde{\omega}_b \), instead of the total vertical velocity at the cloud base height, denoted as \( \tilde{\omega} + \tilde{\omega}_b \).

To fully understand Mak's hypothesis, it would be useful to examine the component of vertical circulation not included in his formulation, namely \( \tilde{\omega} \). In the context of his instability analysis, we write

\[ \tilde{\omega} = \tilde{W}(p) \exp(ikx - \sigma t). \]

The equation governing \( \tilde{W} \) is given by

\[ \frac{d^2 \tilde{W}}{dp^2} - \frac{Sk^2}{f_0^2} \tilde{W} = \frac{k^2 R}{C_p \rho} Q. \]  

It is well known from cumulus parameterization studies (Ooyama, 1971; Arakawa and Schubert, 1974; Cho, 1977) that

\[ \frac{R}{C_p \rho} Q = SM_c \]

where \( M_c \) is the total upward cloud mass flux. Equation (5) can also be written as

\[ \frac{d^2 \tilde{W}}{dp^2} - \frac{Sk^2}{f_0^2} \tilde{W} = \frac{Sk^2}{f_0^2} M_c. \]  

The solution for (5) or (7) can be obtained by the method of Green's function used by Mak. Alternatively, one can also write

\[ \tilde{W} = \tilde{W}_c + \tilde{W}_p \]
where $\tilde{W}_p$ is the particular solution and $\tilde{W}_c$ the complementary solution of the equation. In the layer between the surface ($p = p_s$) and the cloud base level ($p = p_*$), $M_c = 0$ and therefore $\tilde{W}_p = 0$. Similarly, in the layer between the cloud top ($p = p_{**}$) and the top of the domain ($p = p_t$), $\tilde{W}_p = 0$ as well. Within the cloud layer ($p_{**}, p_*$), $\tilde{W}_p$ will depend on the shape of the $M_c$ profile.

The complementary solution $\tilde{W}_c$ can be written as

$$
\tilde{W}_c = \begin{cases} 
Ae^{mp} + Be^{-mp}, & p_ < p < p_2 \\
Ce^{mp} + De^{-mp}, & p_{**} < p < p_* \\
Ee^{mp} + Fe^{-mp}, & p_1 < p < p_{**}
\end{cases}
$$

(9)

where $m = kS^{1/2}/f_0$. The coefficients $A, B, C, D, E$ and $F$ can be determined from the boundary conditions $\tilde{W}(p_s) = \tilde{W}(p_t) = 0$ and the condition that $\tilde{W}$ and $\partial\tilde{W}/\partial p$ be continuous at $p = p_s$ and $p_{**}$.

From the point of view of moisture supply in the subcloud layer, quantities of particular interest are the magnitude of $\tilde{W}$ and $M_c$ at the cloud base level, $\tilde{W}_*$ and $M_{c,*}$ respectively. For a well-mixed subcloud layer with water vapor mixing ratio $\tilde{q}_d$, the rate of moisture supply due to convergence induced by cumulus thermal forcing is $-\tilde{W}_d\tilde{q}_d$, while the rate of upward moisture transport in cumulus clouds is $M_{c,*}\tilde{q}_d$. If $-\tilde{W}_d < M_{c,*}$, then clouds transport moisture away from the boundary layer at a rate faster than the rate of moisture supply by convergence induced by cumulus thermal forcing itself. In the absence of additional forcing, this will lead to a decreased moisture content of the boundary layer, and hence less intense convective activities. On the other hand, if $-\tilde{W}_d > M_{c,*}$, the moisture content in the boundary layer will increase even in the absence of other forcing mechanisms. This will lead to ever increasing levels of convective activities. In this case, it is more likely for convective heating alone to cause the growth of convective activities at the large scale.

To incorporate the effect of this process into a dynamic model requires explicit considerations of the moisture budget. In simple analytic studies of CISK, the moisture budget of the boundary layer usually is not considered explicitly. Instead, the amount of cumulus heating is related directly to the mass convergence rate below the cloud base. Due to the processes just discussed, additional conditions need to be introduced in such closure schemes in order to avoid spurious instability processes.

To examine the relationship between $\tilde{w}_*$ and $M_{c,*}$, let us note that the complementary solution $\tilde{W}_c$ can be expressed in terms of the particular solution $\tilde{W}_p$. Using the following notations for simplicity:

$$
G_1 = e^{mp}, \quad G_2 = e^{mp_2}, \\
G_* = e^{mp_*} \quad \text{and} \quad G_{**} = e^{mp_{**}},
$$

(10)

we find

$$
\tilde{W}_* = \frac{1}{2} \left[ \frac{1}{1 - G_1^2/G_2^2} \left( 1 + G_1^2/G_2^2 \right) \tilde{W}_p(p_*) - \frac{1}{m} \frac{\partial \tilde{W}_p(p_{**})}{\partial p} \right] \\
- \frac{1}{m} \frac{G_{**}}{G_*} \left[ (1 + G_1^2/G_2^2) \tilde{W}_p(p_{**}) - \frac{1}{m} \frac{\partial \tilde{W}_p(p_{**})}{\partial p} \right]. 
$$

(11)

The particular solution $\tilde{W}_p$ depends on the magnitude and the vertical distribution of cumulus heating, and therefore, on those of the total cloud mass flux $M_c$. Two specific vertical profiles of $M_c$ will be considered in the following as examples. To give numerical illustration of the results we adopt from Mak (1982) the following set of values for the various relevant parameters:

$$
p_1 = 150 \text{ mb}, \quad p_2 = 1000 \text{ mb} \\
p_* = 900 \text{ mb}, \quad p_{**} = 400 \text{ mb} \\
S = 0.04 \text{ m}^2 \text{s}^{-2} \text{ mb}^{-1}, \quad f = 10^{-4} \text{ s}^{-1}.
$$

(12)

Case I:

We adopt the constant heating profile used by Mak (1982) as the first example. Assuming a constant stability parameter $S$, this implies that $M_c = M_{c,*}$ is also a constant in the cloud layer ($p_{**}, p_*$). The particular solution for (7) is

$$
\tilde{W}_p = \begin{cases} 
-M_{c,*} & \text{for } p_{**} < p < p_* \\
0 & \text{for } p < p_{**}, \ p > p_*
\end{cases}
$$

(13)

which gives

$$
\tilde{W}_* = \frac{1}{2} \left[ \frac{1}{1 - G_1^2/G_2^2} \left( 1 + G_1^2/G_2^2 \right) \\
- \frac{G_{**}}{G_*} \left( 1 + G_1^2/G_2^2 \right) \right] (-M_{c,*}). 
$$

(14)

The result shows that the following inequality holds independent of the values of the parameters given in (12):

$$
\tilde{W}_*/(-M_{c,*}) \leq \frac{1}{2}.
$$

(15)

The ratio $\tilde{W}_*/(-M_{c,*})$ is a function of the wavelength $\lambda$ of the disturbance. In the limit $\lambda \to 0$, $\tilde{W}_*/(-M_{c,*}) \to 1/2$. As $\lambda \to \infty$, $\tilde{W}_*/(-M_{c,*}) \to 0$. Figure 1 shows the value of $\tilde{W}_*/(-M_{c,*})$ as a function of the wavelength $\lambda$. From this result we may conclude that if the cumulus heating profile is constant in height, condensational heating alone is not likely to lead to spontaneous growth of the disturbance (i.e., no pure...
CISK may take place, if explicit moisture budget of the subcloud layer is considered.

Case 2:

As the second example, we consider a case in which the cumulus heating has a maximum in the midtroposphere. Specifically we assume

$$
M_c = \begin{cases} 
M_{c*} + \Delta M_c \sin[\pi(p_*/p)/(p_* - p_{**})], & p_* > p > p_{**} \\
0, & p_* < p, \quad p < p_{**}.
\end{cases} \tag{16}
$$

The maximum value of $M_c$ is

$$M_{cm} = M_{c*} + \Delta M_c. \tag{17}$$

We use $\alpha$ to denote the ratio between $M_{cm}$ and $M_{c*}$:

$$\alpha = M_{cm}/M_{c*}. \tag{18}$$

The particular solution $\bar{W}_p$ for (7) can be determined easily:

$$
\bar{W}_p = \begin{cases} 
\frac{-M_{c*}}{m^2} \sqrt{[(\pi/p_{**})/(p_* - p_{**})]^2 + m^2} \Delta M_c, & p_* > p > p_{**} \\
\times \sin\left[\frac{\pi(p_* - p)}{p_* - p_{**}}\right], & p_* > p > p_{**} \\
0, & p < p_{**}, \quad p_* < p.
\end{cases} \tag{19}
$$

Then $\bar{W}_*$ is found to be

$$
\bar{W}_* = \frac{1}{1 - G_{**}/G_*^2} \left[\frac{1 + G_1^2/G_{**}^2}{2} \right] \left(1 - G_1^2/G_*^2\right) \left(1 - G_1^2/G_{**}^2\right) \left(1 + G_1^2/G_*^2\right) \left(1 - G_1^2/G_{**}^2\right) \left(-M_{c*}\right) \tag{20}
$$

where

$$\gamma = m \frac{\pi/(p_* - p_{**})}{\sqrt{[(\pi/p_{**})/(p_* - p_{**})]^2 + m^2}}. \tag{21}
$$

The presence of a heating maximum in the midtroposphere increases the value of $\bar{W}_* / (-M_{c*})$. Figure 2 shows the value of this ratio as a function of wavelength $\lambda$ for various values of $\alpha$. Its limiting values for very short and very long wavelength are the same as in case 1, i.e.,

$$\bar{W}_*( -M_{c*}) \rightarrow \begin{cases} 
0 & \text{as } \lambda \rightarrow \infty \\
\frac{1}{2} & \text{as } \lambda \rightarrow 0.
\end{cases} \tag{21}
$$

For the set of parameter values given in (12), $\bar{W}_* / (-M_{c*}) < 1$ for $\alpha < 4$. For $\alpha \geq 4$, the value of this ratio may exceed unity over a certain wavelength range centered around $\lambda \approx 1000$ km. Therefore, unless the maximum value of $M_c$ in the upper levels is far greater than the value of $M_c$ at the cloud base level, the moisture convergence in the subcloud layer induced by cumulus heating alone is not sufficient to sustain continuing development of cumulus activities. Spontaneous growth caused by cumulus heating alone (i.e., pure CISK) is possible only if $\alpha = M_{cm}/M_{c*}$ exceeds a certain critical value. Even then, instability may occur only in a certain wavelength range centered around $\lambda \approx 1000$ km, or what is often referred to as the mesoscale. In any case, from the point of view of moisture supply in the subcloud layer, spontaneous instability caused by cumulus heating alone is not possible at very short and very long wavelength limits (i.e., $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$) in a quasi-geostrophic system.

3. Justification of Mak’s hypothesis

Although it is well known that analysis of CISK is often sensitive to the vertical heating profile, the pos-
sibility that cumulus heating-induced mass convergence may not provide sufficient moisture to sustain the heating mechanism itself is not taken into consideration in most previous studies. The assumption of simple proportionality between cumulus heating and low-level mass convergence excludes the consideration of such a possibility.

The consideration of the problem of moisture supply by Kuo (1975) led him to propose a frequency-dependent cumulus heating parameter. But the relationship between heating and heating-induced moisture convergence should depend on the dynamic systems being considered. We showed here in this study that the relationship in a quasi-geostrophic system is far more complex than that proposed by Kuo, which was made proportional to the wave period.

The value of $\alpha = M_m/M_S$ depends on many factors including the vertical stability and moisture profiles above the cloud base. If the atmospheric conditions are such that $\alpha$ is below a certain critical value so that the moisture convergence induced by heating is not sufficient to sustain the cumulus activities, then sustained growth of cumulus convection is possible only when it is forced by mass convergence induced by other dynamic processes, i.e., $\tilde{\omega}$ in Mak's notation for quasi-geostrophic systems. Therefore, the total heating should be proportional to $\tilde{\omega}$. The moisture convergence induced by heating itself will maintain cumulus activity at a level higher than what it would be if it were forced by $\tilde{\omega}$ alone. But this can be taken into account by choosing an appropriate proportional constant. In other words, using Mak's notation, the heating function should be proportional to $\omega^*$:

$$Q = -ch(p)\tilde{\omega}^* = -ch(p)(\tilde{\omega}^* + \tilde{\omega}^*_0);$$

but

$$\tilde{\omega}^*_0 \propto Q = -ch(p)\omega^*.$$  \hspace{1cm} (22)

Therefore, we write

$$\tilde{\omega}^* = \beta(h)\omega^*.$$  \hspace{1cm} (23)

Here $\beta$ is a linear functional of $h(p)$ which can be determined from the omega equation (1), following the same procedure given in section 2 which leads to (14) and (20). It is also a function of wavelength. Equations (22) and (24) then give

$$\omega^* = \tilde{\omega}^*/(1 - \beta)$$  \hspace{1cm} (25)

$$Q = -\frac{\epsilon}{1 - \beta} h(p)\tilde{\omega}^*.$$  \hspace{1cm} (26)

Thus, we may assume in this situation that $Q$ is proportional to $\tilde{\omega}^*$. The proportional constant $\epsilon/(1 - \beta)$ should depend on the wavelength. This wavelength dependence was ignored in Mak's formulation.

4. Extension to other dynamic systems

Although the discussion presented so far has been limited to quasi-geostrophic systems only, the main conclusion reached can be generalized to some other dynamic systems.

Hoskins and Draghi (1977) showed that in a semi-geostrophic system, the transformed vertical velocity field

$$\omega^* = \frac{f}{(\zeta + f)} \omega$$  \hspace{1cm} (27)

satisfies a Poisson's equation in the geostrophic coordinate identical in form to the quasi-geostrophic $\omega$-equation given by (1), except that the stability parameter $S$ is replaced by the potential vorticity. Therefore the qualitative conclusions we have derived for $\omega$ in a quasi-geostrophic system apply to $\omega^*$ in the semi-geostrophic system as well. Under the geostrophic momentum approximation, the physical vertical velocity field

$$\omega = \frac{(\zeta + f)}{f} \omega^*$$  \hspace{1cm} (28)

is amplified by the factor $(\zeta + f)/f$. A pure CISK is therefore more likely to take place in regions where the air flow is strongly cyclonic. But this nonlinear effect is appreciable only if the relative vorticity $\zeta$ is comparable to $f$, such as the region near the front of a cyclone. In an instability analysis, since we assume the disturbance starts from an infinitesimally small initial amplitude, such a nonlinear effect is not likely to be of major importance.

Schubert and Hack (1983) recently showed that the governing equations for the dynamics of a balanced vortex can be simplified through a coordinate transformation. In the "absolute momentum coordinate", the transformed vertical velocity field $\omega^*$ defined in the same way as given by (28) satisfies an $\omega^*$-equation identical to that derived by Hoskins and Draghi for semi-geostrophic systems. Therefore our conclusion about $\omega$ in a quasi-geostrophic system applies to a balanced vortex as well, except that in such a vortex, the magnitude of the physical vertical velocity is amplified by the factor $(\zeta + f)/f$. The heating induced mass convergence in the subcloud layer can be expected to provide sufficient moisture supply to sustain cumulus activities in a cyclonic vortex if its strength has developed beyond a certain stage.

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