Nonlinearities in Low-Frequency Equatorial Waves

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ABSTRACT

The nonlinear response to low-latitude, temporally periodic forcing centered on the equator is studied. Comparisons are made with a linear solution containing no advections; for both solutions the model consists of a spectral, equatorial β-plane shallow-water system. The imposed forcing period is 40 or 16 days, designed to simulate respectively the global-scale atmospheric oscillation and the 10–20 day monsoonal oscillation. Forcing zonal wavenumbers are $M = 1$ or 2 (40-day period) and $M = 5$ (16-day period).

Results for all cases show that advection does not greatly influence the time evolution of the long-term response, although particular features at a given time may be noticeably affected. For the $M = 1$ (propagating forcing) experiments, the primary changes are in the geopotential field off the equator (consistent with gradient wind effects), and in the low-latitude zonal wind $u$. The low-latitude wind influences help cause significant displacement of divergence from forcing in the nonlinear, as opposed to linear, runs. In the real atmosphere, such displacement could provide a mechanism for nonsteady propagation of convection.

For standing wave $M = 2$ forcing, no eastward propagation is observed in either the linear or nonlinear solutions, in contrast with the actual 40-day oscillation. Therefore, advection does not appear to be a determining factor in explaining the eastward movement. Adveotive effects for propagating $M = 5$ forcing are mostly similar to those of corresponding $M = 1$ cases, the main exception being in the divergence field where no phase shift relative to forcing is seen. Thus, the nonlinearly produced divergence shift appears to be scale-dependent.

1. Introduction

Low-frequency periodic motions are of great importance in the tropics. One type which has received a great deal of recent attention is the 40–50 day oscillation, first discussed in detail by Madden and Julian (1971). It consists of an approximately zonal wave-number 1 perturbation in cloudiness (latent heating), zonal wind and geopotential, which propagates eastward and has maximum amplitude near the equator. Numerous attempts have been made to provide a theoretical explanation for this phenomenon, particularly with regard to the direction of propagation. A frequently mentioned hypothesis is that the eastward progression is due to equatorial Kelvin waves; this is also consistent with the fact that the largest anomalies are found near the equator and involve mainly the pressure and zonal wind. One of the first attempts to link the 40–50 day oscillation to Kelvin waves was the observational study of Parker (1973). However, as pointed out by Chang (1977) and others, a fundamental difficulty in applying Kelvin wave theory to the observed motions is that the phase speed predicted from linear theory (based upon the observed vertical wavelength) is a factor of three larger than the actual speed. Chang's solution was to include Rayleigh damping (intended to simulate cumulus mixing) and Newtonian cooling in a theoretical analysis, from which he obtained results in agreement with observations. Stevens and White (1979) in turn demonstrated shortcomings in Chang's theory, specifically that the agreement with observed phase speed occurs only for a fairly narrow range of dissipation values, and that the presence of Newtonian cooling (as opposed to cumulus friction) is crucial.

Other theoretical and numerical studies include those of Yamagata and Hayashi (1984) and Anderson and Stevens (1987). In Yamagata and Hayashi, a forcing function centered on the equator, with a specified 40-day period, was applied to a shallow-water model. Anderson and Stevens, using a 20-layer system, allowed a 40-day time scale to develop as a result of advection by the mean Hadley cell. Certain results of the above studies are consistent with observations; for example, Yamagata and Hayashi discovered an east–west phase jump in the zonal wind component near the forcing, with lack of a corresponding jump in the pressure field. Anderson and Stevens noted eastward propagation of zonal wind anomalies at 250 mb. However, these findings cannot be regarded as a complete explanation of the observed 40-day phenomenon; Yamagata and Hayashi did not see significant eastward movement (westward propagation was just as evident), and Anderson and Stevens could not ascertain any eastward propagation in the lower troposphere.

One characteristic of much previous modeling work

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on the 40-day oscillation is linearity. This is often done in order to permit analytical solutions (e.g., Yamagata and Hayashi, 1984). However, as pointed out by Van Tuyl (1986), among others, nonlinear effects can be important in the tropics for a number of reasons. Some of these include the large Rossby number (due to the small Coriolis parameter) and the significant ratio of parcel speed to internal wave speed (Froude number). One nonlinear study of the 40-day oscillation is that of Goswami and Shukla (1984), performed with a zonally symmetric general circulation model (GCM). A difficulty in interpreting this and other GCM simulations, however, is the presence of many degrees of freedom and physical processes, which may obscure fundamental causes. (Also, we note that the observed oscillation is not zonally symmetric; e.g., see Lau and Chan, 1985.)

The approach in this paper follows that of Van Tuyl (1986) (hereafter referred to as VT), in which the response to prescribed stationary forcing was examined using a simple, but fully nonlinear, barotropic model. Such a method makes it possible to more effectively isolate the main influences of advective nonlinearity. Because the model is barotropic, interaction of latent heating with the flow cannot be considered. In this work, we allow the forcing to have a component which is periodic in time; otherwise the model and method of solution are exactly as in VT. Although here we concentrate primarily on the 40-day oscillation, other periodic tropical phenomena, which have not been the subject of much modeling study to date, will also be briefly discussed. For example, a feature which propagates westward (relative to the mean flow) has been described by Lim and Chang (1983); it apparently results from transient cold air surges over Indonesia during the Northern Hemisphere winter, which create Rossby waves in a process analogous to geostrophic adjustment. (See the theoretical interpretation in Lim and Chang, 1981.) In addition, there exist smaller scale (wavenumber 4–6) westward propagating features in the summer monsoon region, with a 10–20 day period (Krishnamurti et al., 1985); they are observed to interact with the monsoon and the larger (eastward moving) oscillation on intraseasonal time scales. Due to possible resonant effects, one might expect different nonlinear influences for otherwise identical eastward and westward propagating forcing cases.

One distinction that can be made is that between advective and nonlinear effects; since many possible linearizations exist, there is no unique definition of a nonlinear influence. For the present study, advection of waves by a mean flow created primarily by zonally symmetric mass forcing is defined as a linear advective effect. (This occurs here only for the experiment described in section 4.) All wave–wave interactions, as well as interactions with a mean flow created primarily by wave–wave interactions, are designated nonlinear advective effects. The above definitions are, of course, somewhat arbitrary. It should be pointed out that the expressions “nonlinear solution” and “linear solution” will be used in the future to refer to, respectively, the model response with and without advections, even though the so-called “nonlinear solution” may contain large linear advective influences.

A brief discussion of the model and forcing structure (both spatial and temporal) appears in section 2; more details may be found in VT. Here, we concentrate on forcing symmetric about the equator, because many observed periodic features tend to possess such configuration. In section 3, the 40-day oscillation is modeled by imposing propagating, global-scale forcing (planetary wavenumber 1); subsequent experiments in the section involve changing the direction of propagation (from eastward to westward), and adding a stationary wave forcing component to the eastward case. For all three runs, the complete nonlinear solution is compared, as in VT, to a solution including no advections. Section 4 examines the response to standing-wave forcing (local wavenumber 2) in an experiment analogous to that of Yamagata and Hayashi (1984). Since this case has no imposed direction of propagation, we give particular attention to the possible role of advective nonlinearity in determining propagation characteristics. Section 5 contains the results of two experiments designed to simulate the 10–20 day monsoonal oscillation; here, a propagating wave forcing of wavenumber 5 and 16-day periodicity is specified. We make brief comparisons with the results of corresponding experiments in section 3 (to test the scale-dependence of some of our findings). Section 6 summarizes the main results and conclusions of this work. An important point to stress is that we do not attempt to provide a full theoretical explanation for the 40-day oscillation (or other phenomena), instead concentrating on the role of advections only.

2. Model description

The model used in this study is identical to that of VT. It is comprised of the nonlinear shallow-water system of equations on the equatorial β-plane. In nondimensional form (refer to VT) the equations are

\[
\frac{\partial u}{\partial t} - \theta v + \frac{\partial \phi}{\partial \lambda} = -Ro \left( \frac{\partial u}{\partial \lambda} + \frac{\partial u}{\partial \theta} \right) - \epsilon u \quad (1)
\]

\[
\frac{\partial v}{\partial t} + \theta u + \frac{\partial \phi}{\partial \theta} = -Ro \left( \frac{\partial v}{\partial \lambda} + \frac{\partial v}{\partial \theta} \right) - \epsilon v \quad (2)
\]

\[
\frac{\partial \phi}{\partial t} + \kappa^{-1} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial v}{\partial \theta} \right) = -Ro \left( \frac{\partial (u \phi)}{\partial \lambda} + \frac{\partial (v \phi)}{\partial \theta} \right) - \epsilon \phi + Q, \quad (3)
\]

where \( \epsilon, Q \) are the dissipation constant and mass forcing, with

\[
\kappa = \frac{4\Omega^2 a^2}{gH}, \quad Ro = \frac{U}{2\Omega a}.
\]
(Here \( \Omega \) and \( a \) are the earth's angular frequency and radius.) The value of \( \epsilon \) is \((5.8 \) day\(^{-1}\)). We choose \( U = 18.5 \) m s\(^{-1}\) and \( H \) such that \((\epsilon H)^{1/2} = 50 \) m s\(^{-1}\); thus \( \kappa = 343 \) and \( \text{Ro} = 0.02 \). (Note that \( \text{Ro} \), although it resembles a Rossby number, is not a measure of the relative magnitudes of advective and Coriolis accelerations.)

In this paper \( Q \) is a function of time, with the mathematical form

\[
Q(\lambda, \xi, t) = A\Psi_0(\xi)F_z(t) + B\Psi_0(\xi)F_w(\lambda, t). \tag{4}
\]

The parameters \( A \) and \( B \) are amplitudes, \( \Psi_0 \) is the parabolic cylinder function of order zero (corresponding to a Gaussian forcing distribution about the equator), and \( F_z \) and \( F_w \) are respectively the zonal and wave time-dependent contributions. In VT we use \( \lambda \) for longitude and \( \xi \) for "stretched" latitude \( \kappa^{1/4}\theta \), where \( \theta \) is true latitude. The various forms of \( F_z \) and \( F_w \), as well as values of \( A \) and \( B \), are given in sections 3 through 5.

For the predictive model, the equations are expressed in spectral form by expanding the vector of dependent variables (in the terms of the spatial eigenfunctions of Matsuno (1966)). This spectral representation is identical to that employed in VT, as is the corresponding truncation. The truncation includes three zonal wavenumber values \((m = 0, m = M, m = 2M)\), where \( M \) is the nonzero forcing wavenumber), all values of latitudinal index \( n \leq 2 \), and all mode types (Rossby or gravitational) for a given \((m, n)\) combination. The restriction to small values of \( n \) means that only rather equatorially trapped modes are retained; see for example Matsuno (1966). Later, we shall refer to a particular \( m = M \) mode as the "forced wave" and its \( m = 2M \) counterpart as the "harmonic wave"; in this study, the latter is present solely as the result of nonlinearity.

Notation for the spectral model follows that of VT:

\[
\dot{D}_\gamma + i\sigma D_\gamma = -\frac{\text{Ro}}{N_\gamma} \sum_\alpha \sum_\beta W_{\gamma,\alpha,\beta} D_\alpha D_\beta - \epsilon D_\gamma + \tilde{Q}_\gamma t(t). \tag{5}
\]

Here, the forcing (last) term now contains a time-dependent factor \( f(t) \); the quantity \( \tilde{Q}_\gamma \) corresponds to the spatial projection of the forcing onto spectral component \( \gamma \). In the present work \( f(t) \) is the same for all forced modes in a given simulation (i.e., it does not depend on \( \gamma \)), although there is of course no reason why this must be true in general. The different functions \( f(t) \), corresponding to transformed versions of (4), appear in the following sections.

As in VT, the forcing is applied at \( t = 0 \) to a state of rest and the model integrated with a leapfrog time differencing scheme (\( 2\Delta t = 15 \) minutes). To prevent the development of computational instability the dissipation term in (5) is not directly evaluated at time level \( n \); here we use the discretization

\[
\frac{D^{(n+1)} - D^{(n-1)}}{2\Delta t} = -\epsilon \left( \frac{D^{(n+1)} + D^{(n-1)}}{2} \right).
\]

[This was not necessary in VT because, as demonstrated by comparative experiments (not discussed), the instability had no effect for the relatively short integration time of 20 days.] Here, the integration times are 80 days for 40-day forcing and 48 days for 16-day forcing.

3. Forty-day oscillation, wavenumber 1 forcing

Madden and Julian (1972) postulated that the 40–50 day oscillation results from an eastward propagating convection (latent heating) anomaly in the tropical Pacific region; observational evidence for such a feature may be found in, for example, Weickmann (1983). Therefore, as a first approximation we choose to model the oscillation by imposing a latitudinally symmetric, wave-only forcing function with zonal wavenumber \( M = 1 \), which propagates around the globe in 40 days. (This corresponds to a phase speed of \( 11.6 \) m s\(^{-1}\).) Note that although both eastward and westward phase velocities are considered in this section, only the former has direct relevance to the observed 40-day oscillation; thus we concentrate mainly on the eastward case.

a. Eastward propagating wave

The mathematical form of the heating function for eastward propagating forcing is given by

\[
Q(\lambda, \xi, t) = B e^{-\epsilon^2/2} \cos(\lambda - \omega t), \tag{6}
\]

where \( \omega = 2\pi/(40 \) days) and \( M \) has been set equal to 1. Thus \( F_w(\lambda, t) \) in (4) is \( \cos(M\lambda - \omega t) \). Here \( B \) corresponds to 1.25 K day\(^{-1}\), or 36 m day\(^{-1}\) when expressed as a mass source \( \dot{Q}/g \); the proportionality factor relating the two quantities is the gas constant \( R \) (refer to VT for further explanation). The spectral counterpart of (6) [in (5)] is

\[
\dot{Q}_\gamma(t) = \tilde{Q}_\gamma e^{-\epsilon t}, \quad \text{that is,} \quad f(t) = e^{-\epsilon t}.
\]

Before considering numerical results, it may be instructive to study some linear analytical solutions briefly in order to determine what resonant effects might be expected. The relevance of linear resonant influences here is that they help to determine the flow strength (and structure), and thereby the importance of nonlinearity. Turning to (5), the linearized version takes the form

\[
\dot{D} + i\sigma D = -\epsilon D + \tilde{Q}(t),
\]

where the subscript \( \gamma \) has been dropped. If we define \( S \) as the ratio of solution amplitude for traveling heating \( f(t) = e^{-\epsilon t} \) to that for steady heating \( f(t) = 1 \), it is easy to show that in the limit of large \( t \):

\[
S = \left[ \frac{\epsilon^2 + \sigma^2}{\epsilon^2 + (\sigma - \omega)^2} \right]^{1/2}. \tag{7}
\]
For the eastward case \( \omega > 0 \); substituting values for \( \omega \), \( \epsilon \) (dissipation constant) and \( \sigma \) (mode eigenfrequency) gives \( S_K = 1.27 \) (Kelvin mode) and \( S_R = 0.68 \) (Rossby mode). As a result, we expect that the total solution will be slightly weaker for the propagating than for the steady case, since the Rossby mode is taken somewhat further from resonance than the Kelvin mode is brought towards it. In addition, the structure of the propagating solution (wind and geopotential) should more closely resemble that of a Kelvin wave.

After being integrated from a state of rest, the model attains its limit cycle, both for the linear and nonlinear runs, in approximately 20 days. Subsequently, the wind and geopotential fields maintain their shape and intensity while propagating east at the speed of the forcing. Figure 1 shows the nonlinear solution at day 80. The linear solution (not shown) basically resembles that of Matsuno’s (1966) steady experiment, the main difference being that here the strongest geopotential centers are on the equator. This is consistent with our expectation of Kelvin wave resonance. Changes due to nonlinearity are rather small and mostly resemble those of the \( M = 2 \) steady case in VT (i.e., \( S_2 \)). Some of the changes here include a slight (<0.5 m s\(^{-1}\) strengthening (weakening) of the equatorial easterly (westerly) jet; sharpening (flattening) of the equatorial trough (ridge); intensification (diminishing) of the midlatitude cyclonic (anticyclonic) circulation. The latter effect is, of course, consistent with gradient wind influences.

The difference (nonlinear minus linear) solution, not surprisingly, is similar to that of \( S_2 \) also. For example, there are harmonic (here, \( m = 2 \)) troughs in the location of the primary troughs and ridges of the nonlinear solution, with \( m = 2 \) ridges between. However, the phases of the maximum wind and geopotential anomalies are somewhat different, and there is a tendency for the greatest geopotential anomalies to occur along and near the equator. This is because the harmonic \( (m = 2) \) Kelvin mode here is important (equal in amplitude to the \( m = 2 \) Rossby mode), whereas in \( S_2 \) the Rossby component dominated by nearly a factor of three. More significantly, the latter relation between harmonic Kelvin and Rossby amplitudes also holds for the \( M = 1 \) steady solution (\( S_1 \); not shown). Thus, we see that resonance can influence the nonlinearly produced components as well as the primary (forced) ones, meaning that propagation direction has an impact on the overall effect of nonlinearity. Specifically, the greater harmonic Kelvin presence in the eastward (relative to steady) case means that nonlinear influences should tend to be more confined near the equator. Additionally, the flow is \( \sim 25\% \) weaker here than in the steady experiment \( S_1 \), implying that the magnitude of the nonlinear effects should be reduced.

The divergence in both the linear and nonlinear cases, as expected, remains unchanged as it propagates. However, there is some phase shift between the maximum divergence (or convergence) and maximum forcing; for the linear solution both the convergent and divergent regions are centered slightly west of the forcing maximum, while in the nonlinear solution the convergent region is shifted eastward and the divergent region westward. As a result the nonlinear convergent (divergent) region is more (less) in phase with the forcing. The reason why the two areas behave differently is unclear, but for the nonlinear divergent region at least, the phase discrepancy is large enough to be significant.

b. Steady with eastward propagating wave forcing

In the previous subsection, the forcing consisted of a steadily propagating wave only; it is not surprising that the response also showed steady propagation. (We note that for the nonlinear case, however, this relation between forcing and response need not always be true.) In the real atmosphere, differential heating over continents and oceans, among other processes, gives rise
to thermal forcing which is nearly steady but still dependent on longitude. We now add to our previous forcing a time-mean background component, which will be assumed to possess a nonzero zonal wavenumber. The maximum total forcing will then be a function of time as the propagating wave moves into and out of preferred regions in the stationary wave (i.e., there will alternately be reinforcement and cancellation).

In order to obtain results as realistic as possible, we desire to select a mean state representative of that actually observed at low latitudes. One important contribution to the tropical steady flow consists of forced Kelvin waves (e.g., see Webster, 1972). Following Webster and Holton (1982), we adopt a background "Kelvin wave-type" forcing of the form

\[ Ae^{-y^2/\nu^2} \cos MA, \]

which supports an equilibrium wind and geopotential response. Here \( y_0 \) is set equal to \( \sqrt{2} \) times the deformation radius \( (c/\beta)^{1/2} \) (approximately twice Webster and Holton's \( y_0 \) value of 1000 km), and \( M = 1 \). Thus our background state is simply the nonlinear solution of \( S_1 \), which of course includes modes other than the Kelvin. (For planetary scales we expect the Kelvin contribution to be important, however.) In terms of our previous notation, the total forcing is

\[ Q(\lambda, \xi, t) = Be^{-kt^2/2}[\cos \lambda + \cos(\lambda - \omega t)], \quad (8) \]

with \( Q_0(t) = \bar{Q}_0(1 + e^{it}) \). An important consequence of using the heating function (8) is that the mean flow is then zonally asymmetric; this can be crucial in determining meridional wave energy propagation (refer to Webster and Holton, 1982).

The magnitudes of the steady and propagating forcing components are each assumed to be 1.25 K day\(^{-1}\), so the maximum over the domain of the total forcing varies at different times from zero to 2.5 K day\(^{-1}\). (Recall that in the previous subsection the maximum amplitude was always 1.25 K day\(^{-1}\).) As before, all forcing is applied at \( t = 0 \) to a state of rest; in an experiment where the flow was initialized to be in equilibrium with the steady heating, no difference in the final limiting state was observed.

The nonlinear solution for the steady plus eastward propagating case \((EM_1, \text{where } M \text{ refers to mean})\), every ten days from day 50 through day 80, is presented in Fig. 2. Note that the wind and geopotential response grows, decays, and dies while maintaining approximately the same position relative to the maximum forcing amplitude. (The strength of the maximum forcing, of course, is not constant.) As in the previous run, nonlinear effects influence mainly the shape and intensity of particular features at a given time (as opposed to, for example, propagation or overall time evolution). Here, in addition to the usual midlatitude gradient wind correction, nonlinearity most significantly tends to strengthen the equatorial jets, as well as the maximum zonal equatorial pressure gradient, at and somewhat after the time of maximum heating.

Around the time of minimum heating, the linear and nonlinear solutions differ very little, because the flow fields then are rather weak, which in turn is due to the equivalence of the forcing and dissipation time scales (both approximately six days since the inverse forcing frequency is \( 40/2\pi \) days). The largest wind speed in the nonlinear case decreases from 31 m s\(^{-1}\) at maximum forcing to 5 m s\(^{-1}\) at minimum forcing, while the geopotential amplitude decreases from 65 to 16 m. In the difference fields the corresponding reductions are 7 to 1 m s\(^{-1}\) and 22 to 2 m.

The anomalous (nonlinear minus linear) wind and geopotential are strongest when the total fields are strongest, that is, at and just after the heating maximum. At these times, an eastward propagating structure is observed; since there is a significant Rossby mode component present (evident from the geopotential centers off the equator), this motion obviously results from the forcing. The \( m = 2 \) trough and ridge pattern here occurs in the same phase relative to the total solution as for the corresponding "switch-on" forcing cases (e.g., \( S_1 \)). At and after the minimum forcing, the difference field is quite weak and propagation characteristics are difficult to determine.

The divergent \((\delta > 0)\) region in the nonlinear solution is found somewhat west of the forcing, particularly at 50 and again at 80 days (i.e., near and after the time when the forcing is maximum). No significant shift of the divergent region occurs in the linear case, and the convergent region tends to coincide with the forcing in both solutions. This is consistent with the results of the previous eastward propagating case (i.e., with no steady forcing component); the rather slight westward displacement of both the convergent and divergent regions in the linear solution is even observed here as well. Thus, the presence of a forced mean flow has little impact on the structure (although not the intensity) of the divergence field in this case.

In addition to studying physical fields (e.g., geopotential, divergence), one may also examine mode trajectories in phase space, that is, real and imaginary parts of spectral coefficients \( D \) as a function of time. In Fig. 3 the complex amplitudes of the \( m = 1 \) Kelvin, \( m = 1 \) Rossby and \( m = 2 \) Rossby modes are plotted from day 40 through day 80 for the present case. Also included are the 40-day averages for the time-dependent curves, as well as the equilibrium solution (nonlinear) of \( S_1 \). Note that the forced Kelvin, forced Rossby and harmonic Rossby trajectories all proceed in a clockwise direction, corresponding to eastward phase propagation. This of course results from the forcing (curve \( Q \)). Other features to note are that the harmonic Rossby amplitude is significant compared to the forced Rossby amplitude, and that the time-mean and corresponding steady forced values (respectively \( M \) and \( S \)) are nearly identical for each curve. The latter phe-
Fig. 2. Wind and geopotential for nonlinear solution of $M = 1$ steady plus eastward propagating forcing case (case $EM_t$),
every 10 days, day 50 through day 80. Geopotential every 20 m. All other symbols and units as in Fig. 1.
wave forcing only, nonlinear changes are small and essentially resemble those for corresponding steady forcing. The addition of a time-mean background forcing causes nonlinear effects to be greater in magnitude (at least partly because the maximum total forcing is larger), but the qualitative nature of these effects is not much different than in the propagating-only case. Our experiments thus far consequently suggest that advective nonlinearity is of secondary importance in influencing the 40-day oscillation. We shall examine this conclusion for standing-wave forcing in section 4.

c. Westward propagating wave

For $M = 1$ westward propagating forcing we have

$$Q(\lambda, \xi, t) = Be^{-t^2/2} \cos(\lambda + \omega t).$$

Also,

$$Q_x(t) = \bar{Q}_x e^{i\omega t}.$$

The linear solution at 80 days (again, other times are not discussed because the fields merely exhibit steady propagation) shows the same features in nearly the same locations as the corresponding eastward case. However, the intensity, particularly of the geopotential field away from the equator, is markedly increased. This may be explained as a result of the previously discussed Rossby mode resonance; the analytical formula (7) predicts strengthening of the flow. (The actual forced Rossby amplitude is increased 37% from the steady solution, while the Kelvin amplitude is decreased by 12%. Corresponding analytical values are respectively 52% and 18%.) Also consistent with this reasoning, the amplitude of the midlatitude geopotential is increased more strongly over the eastward case (from 25 m to 60 m) than is that of the maximum wind speed (from 14 m s$^{-1}$ to 20 m s$^{-1}$). The divergence magnitude is actually somewhat greater in the eastward run.

For the nonlinear solution (Fig. 4) the intensity of the maximum winds and the geopotential amplitude are nearly identical to those of the linear solution. The main differences consist of a slight strengthening of both the easterly and westerly jets (as in $EM_1$), and the previously observed “gradient wind” influence on geopotential. Here, however, the anticyclonic circulation is not merely broadened or weakened; rather, there is a tendency for the high pressure cell to be split in two, although only one distinct center is present. This suggests a strong influence by the $m = 2$ Rossby mode, which in fact is nearly one order of magnitude stronger than in the corresponding eastward case. Thus we again note a significant resonant influence which is nonlinear, and so cannot be fully explained by the analytical relation (7). In the difference map (not shown), the pattern resembles that of the eastward propagating-only case, but the amplitude is considerably larger and the phases of the ridges and troughs are shifted nearly 180°. The linear solution has both the maximum convergence and divergence centered slightly east of the force.
ing, although the phase difference is rather small. As in the preceding case, nonlinearity shifts the convergent region to the east and the divergent region to the west. Here, the divergence is slightly more out of phase with the forcing in the nonlinear solution (i.e., it now lies westward), while the convergence is considerably more out of phase. All three $M = 1$ experiments therefore suggest that, in cases of propagating forcing, the frequently made assumption that maximum heating and divergence coincide (e.g., Julian, 1984) could be incorrect. In fact, preceding comments imply that the phase difference may result partially from linear dynamics, but more fundamentally from nonlinear processes.

4. Forty-day oscillation, standing-wave $M = 2$ forcing

Lau and Chan (1985) report the existence of a 40–50 day oscillation during the northern winter which is comprised of an east–west dipole structure; eastward propagation is observed over the Indian Ocean but the oscillation then becomes stationary (and intensifies) when the dipole centers reach Indonesia and the central equatorial Pacific. Thus a significant standing-wave component is present in the 40-day oscillation, in addition to the propagating contribution.

A theoretical investigation of the response of a linear shallow water model to 40-day, stationary forcing was made by Yamagata and Hayashi (1984). Their approach follows quite closely that of Gill (1980); significantly, both studies employ the “long-wave” approximation (in which the zonal wind is assumed to be in geostrophic balance). In Yamagata and Hayashi, however, the mass source term in the pressure tendency equation has a factor $e^{iut}$. Since the problem is linear, the resulting solution is simply Gill’s multiplied by $e^{iut}$, with the dissipation constant $\epsilon$ replaced by $\epsilon + i\omega$. Yamagata and Hayashi’s selected zonal scale (i.e., the equivalent of a half-wavelength) of 80° is in good agreement with the observations of Lau and Chan (1985), who show a distance between the dipole centers of 60°–90° of longitude. It should, however, be pointed out that Yamagata and Hayashi consider only the case of one localized center. Some of their results are consistent with the observed motions, in particular the phase jump in $\zeta$ near the heating (that is, $\zeta$ changes from strongly positive to strongly negative over a short distance), and the lack of such a jump in the pressure field.

In this section, we extend Yamagata and Hayashi’s methods to the nonlinear case, although given our low-order spectral approach none of the forcings here are truly isolated. We employ a forcing function which has both a wave and a zonally symmetric component; thus a mass sink is not balanced by an identical adjacent mass source. The nonzero zonal wavenumber of the heating is $M = 2$ (half wavelength of 90°), chosen for consistency with that of Yamagata and Hayashi (1984). Since their meridional scale is also close to ours (one deformation radius, with their value of c about 25% larger), direct comparisons to their results will be made when appropriate. One major difference that should be kept in mind is that the present model contains only two nonzero values of $m$ (zonal wavenumber), rather than the infinite number implicitly present in Yamagata and Hayashi’s analytical solution.

The mathematical formula for the forcing in this section is

$$Q(\lambda, \xi, t) = Ae^{-\xi^2/2} \cos 2\lambda \cos \omega t,$$

$$Q_s(t) = Q, \cos \omega t.$$  \hfill (9)

Here $A = B = 1.25$ K day$^{-1}$; thus the maximum forcing corresponds to 2.5 K day$^{-1}$. In contrast to the previous cases, note that $Q_s$ for the zonal modes is nonzero.

Figure 5 shows the nonlinear solution from day 50 through day 80 at ten-day intervals, in the same manner as for Fig. 2. The nonlinear wind and geopotential at 60 days here are the counterpart to Fig. 4 of Yamagata and Hayashi (1984) (shown as our Fig. 6), although on their map pressure is not plotted. Numerous similarities are apparent, in particular the overall shape of the pat-
Fig. 5. Wind and geopotential for nonlinear and difference equations of $M = 2$ standing wave with zonal case. Every 10 days, day 50 through day 80, geopotential contour interval 20 m (nonlinear solutions $a-d$), 2 m (difference solutions $e$-$h$). All other symbols and units as in Fig. 1.
tern, the two cyclonic centers off the equator just west of the forcing, and the wind maximum on the equator west of these centers (see also Fig. 1 of Gill, 1980). The major difference is that in our solution, a pronounced easterly wind maximum appears east of the forcing; in Yamagata and Hayashi, east–west phase variations in \( u \) are rather small over this region. The maximum occurs in our linear solution also and so is not primarily due to advective effects. The advective effects which are present here are significant, but essentially identical to those in the corresponding steady case [\( S_1, S_2 \) in VT, i.e., the response to (9) with \( at \) fixed at \( \pi \)].

Comparison of the fields at 60 and 80 days (respectively maximum negative and maximum positive forcing) shows the presence of a standing oscillation in the forcing region, consistent with Yamagata and Hayashi’s findings; note the change from equatorially westerly to easterly winds 30° longitude west of the forcing center. Advective changes at 80 days, not surprisingly, greatly resemble those for the steady run \( S_1, S_2 \) [set in (9) equal to zero], the most prominent change being a slight weakening of the flow. One other feature that Yamagata and Hayashi observed was wave propagation (easterly and westerly) away from the location of the heating. Here (refer to Fig. 5), some westward propagation of the equatorial wind maximum \textit{west} of the forcing may be seen; the maximum east of the forcing appears merely to damp out. No eastward propagation of any flow features is ascertainable. Thus in our nonlinear (as well as linear) model, a stationary oscillating heat source does not appear capable of generating significant eastward motions, and so does not well simulate the 40-day oscillation. Examination of mode amplitudes shows that in our solution, either linear or nonlinear, Rossby mode amplitude dominates Kelvin mode amplitude by a factor of between two and ten, thus explaining the lack of eastward propagation. We also note that advection has no effect on other propagation characteristics.

The difference solution (Fig. 5) shows weak apparent westward Rossby wave and eastward Kelvin wave movement (zonal scale \( m = 4 \)) between day 60 and day 70; at other times propagation is somewhat difficult to interpret. Interestingly, the strongest maximum wind and geopotential anomalies occur at day 70, when the forcing is zero. In the divergence field, virtually no phase shift relative to forcing at any time is apparent; this contrasts with the results of the three \( M = 1 \) cases.

Figure 7 shows trajectories for the present experiment. As expected from the lack of imposed propagation direction, the Kelvin mode always progresses to the east and the Rossby modes to the west (respectively clockwise and counterclockwise). This occurs in both the linear and nonlinear solutions. The Kelvin mode is seen to strengthen over part of its cycle as the result of advection; however, the forced Rossby mode is stronger at this time also (around 60 days, i.e., the time of maximum negative forcing). Thus, advective effects do not preferentially strengthen the Kelvin mode, accounting for the failure of eastward propagation to be generated in the nonlinear case. The harmonic Rossby mode is comparatively weak at all times, so its propagation has little impact.

Summarizing the results of this section, our model has been able to reproduce certain of the observed aspects of the 40-day oscillation which were also simulated by Yamagata and Hayashi (1984). These include a standing oscillation in the vicinity of the forcing with a phase jump in \( u \) and westward wave propagation away from the forcing. Our results, however, do not show any noticeable eastward propagation in either the linear or nonlinear solutions, apparently due to insufficient Kelvin wave generation. This may be caused by the previously mentioned strong truncation (with respect to \( m \)). Adveotive effects are again seen to influence details of the flow at a particular time, but not propagation or the overall evolution; thus linear models (in the highly simplified case) appear adequate for simulating the main features of the oscillation.

5. Sixteen-day oscillation, propagating \( M = 5 \) forcing

Oscillations in the Asian monsoon region (30°E–160°E) on time scales of approximately 10–20 days
position of troughs. This phenomenon, consequently, is of practical as well as theoretical interest.

In addition to the westward propagating mode, eastward propagating features of roughly similar space and time scales are also associated with the monsoon, particularly in the Northern Hemisphere winter season. Their phase speed is given as \( \sim 4 \text{ m s}^{-1} \) by Murakami (1981). In the present section, we model both the eastward and westward waves by imposing a propagating forcing function centered on the equator, as for \( M = 1 \) in section 3. Here, we select \( M = 5 \) and \( \omega = 2\pi/(16 \text{ days}) \), resulting in a phase speed (for both eastward and westward experiments) of \( 5.8 \text{ m s}^{-1} \). Our results are not expected to be very sensitive to this parameter. An assumption we are making is that the energy source for these waves lies on or near the equator, which may not necessarily be true; note that the monsoon region is \( \sim 20^\circ \) latitude removed from the equatorial zone.

The experiments in this section are the \( M = 5 \) counterparts to the propagating-only cases of section 3. One purpose in performing them is to examine the scale-dependence of our previous findings, especially with regard to divergence. Recall that for the \( M = 1 \) cases a significant shift of divergence from forcing was found, whereas for \( M = 2 \) no such shift occurred. Since the \( M = 1 \) forcing always had a propagating component while the \( M = 2 \) forcing was stationary (in space), it is difficult to distinguish scale-dependent effects from those of propagation based on these results. Also, there is obviously a clearer scale separation for the \( M = 5 \) vs \( M = 1 \) comparison.

Mathematically the forcing function follows that of section 3:

\[
Q(\lambda, \xi, t) = B e^{-t^{1/2}} \cos(5\lambda - \omega t)
\]

for eastward propagating forcing, and

\[
Q(\lambda, \xi, t) = B e^{-t^{1/2}} \cos(5\lambda + \omega t)
\]

for westward propagating forcing. Again \( B \) corresponds to 1.25 K day\(^{-1}\). Note that \( Q_2(t) = Q_e e^{-\nu t} \) (eastward) or \( Q_2(t) = Q_w e^{\nu t} \) (westward).

Figure 8 presents the nonlinear solution for \textit{westward} propagating, traveling wave forcing at day 48. (As in the corresponding \( M = 1 \) runs, the limit cycle consists of phase propagation only.) Nonlinearity affects the geopotential trough and ridge pattern as before, with a slight hint of double structure here on the eastern equatorward side of the anticyclonic regions. The velocity field shows a slight increase in the amplitude of \( u \) with a much larger (\( \sim 60-70\% \)) corresponding increase for \( v \). This more significant change in the meridional wind is reflected in the difference map, and is consistent with the results for steady \( M = 8 \) forcing in VT. Divergence and mass forcing in the nonlinear solution are nearly exactly in phase, thus implying that the previously observed phase shifts are a function of scale rather than propagation.

The corresponding eastward propagating case pos-
Fig. 8. Wind and geopotential for nonlinear solution of $M = 5$ westward propagating forcing case at 48 days. Forcing period is 16 days. Geopotential every 5 m. All other symbols and units as in Fig. 1.

possesses much weaker amplitudes than does the previous experiment, particularly in the geopotential field; this is identical to what was noted for $M = 1$. Changes due to nonlinearity are extremely small, with maximum height and wind anomalies of 0.3 m and 0.05 m s$^{-1}$. As expected, the divergence field is unaltered as well. Therefore, we may summarize the above results by saying that nonlinearity has some noticeable effect on meridional wind and geopotential for the westward propagating case, and no impact in the eastward run. Scale-dependent differences with $M = 1$ include larger nonlinear changes in $v$ relative to $u$, and lack of a divergence shift from the forcing centers.

6. Summary and conclusions

Perhaps the most important result of this study is that advection (either linear or nonlinear) did not fundamentally affect the evolution in any of the cases. This implies that the linear shallow-water studies of Yamagata and Hayashi (1984) and Anderson and Stevens (1987) are fundamentally correct (within the limitations of the shallow-water model) in terms of their conclusions. Significant advective influences, however, were found here in certain instances. For the $M = 1$ runs, the most important changes occurred in the geopotential field, particularly for the anticyclonic regions in the westward propagating experiment (where nonlinearity created a double structure), and in the divergence field. The nonlinear divergent region generally showed significant westward displacement from the mass source at and shortly after maximum forcing, with the convergent region exhibiting a corresponding eastward displacement mainly in the westward propagating case. These phase shifts were often large enough so that the common assumption of divergence coincident with forcing was invalid. Comparative experiments indicated little impact on the main influences of nonlinearity when a time-mean state was added.

For standing-wave $M = 2$ forcing, a standing oscillation was discovered in the equatorial region, with accompanying westward propagation of low-latitude features. There was a conspicuous lack of corresponding eastward propagation, however, even though the experiment was analogous to that of Yamagata and Hayashi, in which some eastward propagation was observed. Therefore, this case was not representative of the 40-day oscillation. One possible reason for the difference with Yamagata and Hayashi's results may be our strong spectral truncation, discussed in section 4. It is important to note that advection had no effect on the propagation here, instead acting to alter flow features as in the steady solutions. No significant advective phase shift of divergence relative to forcing was seen.

Propagating forcing for $M = 5$ yielded nonlinear effects similar to those of corresponding $M = 1$ cases, the primary exception being for divergence where no phase shift occurred. Other minor differences with $M = 1$ were due to the geometry of the flow (e.g., $v$ more important compared to $u$).

One major implication of our results, mentioned previously, is that linear models can reasonably well simulate the main features of the response to periodic forcing when no interactive heating is included. The inclusion of interactive latent heat release might be expected to produce radically different nonlinear effects than those observed here; this has also been pointed out by, for example, Lau and Lim (1984). For the 40-
day oscillation in particular, interactive effects would appear to be important since, as previously stated, this phenomenon does not much resemble a simple traveling wave (Anderson et al., 1984). Consequently, one should expect somewhat limited agreement with observations from a study such as this one.

The phase shift of divergence relative to forcing in the \( M = 1 \) experiments suggests that an estimate of divergence based on, for example, cloud top temperatures alone (Julian, 1984) might give misleading results. (The assumption of this method is that coldest cloud tops indicate maximum latent heating, and thus maximum divergence.) It is also interesting that the shift in this study appears related to advective nonlinearity, implying that divergence obtained from linear models may be incorrect in certain cases as well. The differing behavior of the divergent and convergent areas in our nonlinear solutions is difficult to interpret, but could be an artifact of the nonlinear shallow water model. Nevertheless, the presence of a divergence shift at all is significant; in particular, it provides a possible mechanism for nonsteady propagation of convection if interactive heating is important.

Another advective effect is the split of the anticyclonic pressure couplet into four (rather than two) centers, seen mainly in the westward propagating \( M = 1 \) experiment. This demonstrates another instance where the use of linear models for diagnostic purposes may be inadequate. The preference of this particular nonlinear effect for westward moving forcing is most likely due to the larger amplitude of the pressure centers (off the equator) in these cases, resulting from the previously discussed Rossby mode resonance. Therefore, one might be able to observe a similar phenomenon for eastward propagating forcing if the magnitude of the heating is sufficiently large.

The lack of eastward propagation seen in our standing-wave experiment, inconsistent with the 40-day oscillation, may have a number of causes. One could be the previously mentioned strong truncation, which might eliminate modes that are an essential part of the eastward moving feature. Additionally a 40-day period is much closer to the time scale of Rossby waves than Kelvin waves for \( m = 2 \), and especially for \( m = 4 \). A final, very important observation was made by Anderson and Stevens (1987), who discovered a corresponding absence of eastward propagation in a linear shallow-water experiment similar to that of Yamagata and Hayashi, but with more realistic parameters and a nonzero base state. They noted that eastward motion did occur in a 20-level model, at least for the upper troposphere, in a manner consistent with zonal wind observations; therefore, the failure of eastward propagation to be generated in the present case stems at least in part from the limitations of the shallow-water system. For example, the Kelvin wave in Anderson and Stevens, due to the more realistic dissipation scheme of the multilayer model, could have been slowed in the way described by Chang (1977) and brought closer to resonance. The main implication of our standing-wave results with regard to nonlinearity is that horizontal advections, at least within the context of the shallow-water model, do not substantially affect either the standing wave or propagating components of the 40-day oscillation.

The observation that forcing and divergence coincide for all nonlinear \( M = 5 \) runs, while they do not for corresponding \( M = 1 \) cases, is potentially important because it implies that in the real atmosphere, divergence may be locally associated with latent heating mainly on smaller scales. Such behavior is consistent with quasi-geostrophic theory (which of course is not necessarily applicable at low latitudes); also, a similar effect has been demonstrated by Zebiak (1982) for the linear case only. It is not, however, clear why in the present model, a significant divergence shift occurs exclusively for the nonlinear solution. The scale-dependent influence of nonlinearity on divergence here is probably not due merely to the weaker flow for \( M = 5 \), since the divergence has comparable magnitude between \( M = 1 \) and \( M = 5 \) experiments. (This in turn implies that divergence advection, at least by the zonal wind \( u \), should be equally important for both wave-numbers.) Experiments with more realistic models, particularly ones in which thermodynamic forcing can be represented explicitly, would be useful for exploring the relationship between divergence and heating at all scales.

In conclusion, our results for time-dependent forcing are consistent with the steady results of Van Tuyl (1986), in that advective effects can noticeably modify the response to tropical forcing but do not change the fundamental character of this response. The secondary role of nonlinearity in simple tropical forced models was also discussed by Gill and Phillips (1986); their paper was an extension of the analytical work of Gill (1980). It is likely that the allowance of more degrees of freedom, and especially the inclusion of interactive latent heat release, would give rise to nonlinear (or other advective) effects not observed in this study.

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