The Existence and Stability of Steady Circulations in a Conditionally Symmetrically Unstable Basic Flow

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ABSTRACT

By treating the latent heating as an energy source which is implicitly related to the motion field, the existence of steady nonlinear circulations in a flow susceptible to Conditionally Symmetric Instability (CSI) is studied. Steady viscous symmetric circulations are shown to be unique and asymptotically stable, when the latent heat sources are weak and insensitive to the motion perturbations.

1. Introduction

Conditional Symmetric Instability (CSI) has been studied as a possible formative mechanism of frontal rainbands by many authors (Bennetts and Hoskins, 1979; Emanuel, 1983; Parson and Hobbs, 1983; Bennetts and Ryder, 1984; Xu, 1986a). However, most of the theoretical studies on CSI were linear and so probably only applied to explaining the initiation of some mesoscale rainbands. Bennetts and Hoskins (1979) integrated nonlinear viscous CSI circulations numerically for 24 hours. Their model circulations showed fairly good resemblance to some observed rainbands, but the growth of the model circulations was monotonic and did not show a tendency to be quasi-steady in the time period of integration. The growth in the later times seemed less realistic in comparison with some observed quasi-steady rainbands. Two possible sources of this discrepancy are: (i) the vertical velocity and its related latent heat release in the model were exaggerated due to the hydrostatic assumption (in which case, for an unstable stratification, the inviscid linear growth rate increases with the wavenumber boundlessly) and thus the energy sources in the model could be too strong, and/or (ii) the viscosity (including the numerical diffusion) was not large enough to represent the eddy viscosity in the matured quasi-steady rainbands, although arbitrarily increasing the eddy viscosity is no guarantee that realism will be attained for the resolved motion.

The quasi-steady nature was often observed in matured rainbands. For example, Bennetts and Ryder (1984) reported in a recent case study that "the bands are identifiable over a fairly long period" and the mesoscale convective "rolls do not propagate relative to the mean flow and so the streamlines are parcel trajectories." By comparing the observed mesoscale structure with various theoretical models, they found that CSI was the most competitive mechanism. We believe that these just mentioned and many other observed frontal rainbands manifest the existence and stability of a class of nonlinear steady symmetric circulations in which the latent heat sources are generated either purely internally or partially internally and partially due to external forcings (e.g., frontogenetic forcings). Here we simply call them "symmetric circulations" and, specifically, by CSI circulations we mean the former (i.e., those with purely internally generated latent heat sources). Based on the linear theory (Xu, 1986a) which indicated the necessity of viscosity for a marginal CSI linear mode, we may speculate that viscosity would also be indispensible for the stable steadiness of a nonlinear symmetric circulation, or specifically, a CSI circulation. As we will see, in order to maintain a steady nonlinear symmetric circulation stable, the latent heating must be weak and/or the (eddy) viscosity must be large enough. The purpose of this paper is to examine the existence, uniqueness and stability of these steady nonlinear viscous symmetric circulations from a theoretical viewpoint.

In the following section, we briefly discuss the viscous nonlinear circulations of (dry) SI (Symmetrical Instability). An existence theorem for steady, nonlinear symmetric circulations with bounded rates of latent heat release is given in section 3, but most mathematical details of the proof appear in the Appendix. The uniqueness and stability of these circulations are discussed in section 4. Some physical interpretations follow in section 5.

2. Steady viscous SI circulations

It was shown (Xu and Clark, 1985) that the viscous nonlinear SI problem with a uniform basic state and unit Prandtl number can be transformed into a viscous nonlinear BI (Buoyancy or Convective Instability) or II (Inertial Instability) problem via a proper coordinate rotation. Moreover, as a generalized conductive solution, the basic state should also satisfy the governing equations. It is easy to see that any SI basic state with uniform spatial gradients satisfies the viscous governing equations and thus the governing equations of the SI problem is equivalent to the equations of the BI problem in a rotating atmosphere if the Prandtl number is 1. However, since the boundaries are no longer horizontal when the SI problem is transformed into a BI problem, the existence of steady viscous SI circulations and certain related properties can not be simply deduced from the already known BI theory (Chandrasekhar, 1961 and Veronis, 1966, where the primary bifurcation point was found to be a steady one for Pr = 1). Nevertheless, the existence of steady viscous SI circulations for a fairly broad range of the basic parameters has been demonstrated numerically by Miller (1984).

For a steady SI circulation, viscosity (and thermal conductivity) has two effects: (i) it dissipates the energy of the SI circulation, and (ii) it conducts both thermal and inertial energy down gradients. Before the primary bifurcation occurs, only the conductive regime exists. Thus, only the second effect of viscosity (and thermal conductivity) is active. After the primary bifurcation, the convective regime is established and the circulation transports energy dominantly in the interior of the fluid. In this case, the first effect constrains the strength of the SI circulation while the second effect transports the external (thermal and inertial) energy to the SI circulation through the surface layers, i.e., the vicinity of the boundaries.

Furthermore, if we assume the coefficient of dynamic viscosity μ to transverse motion is a constant, while we denote by χ the coefficient of thermal conductivity and the coefficient of viscosity to the motion which is parallel to the basic shear, then we can eliminate the second effect of viscosity by setting $\chi = 0$. In this case, with the anelastic approximation we have [cf. (2.4)–(2.7) of Xu, 1986b]

$$\frac{d}{dt}\mathbf{v} - \mathbf{h} = -\nabla(\alpha_0 p) + \alpha_0 \mu \Delta \mathbf{v},$$

$$\frac{d}{dt}\mathbf{h} + \Pi \mathbf{v} = 0,$$

$$\nabla \cdot \rho_0 \mathbf{v} = 0, \tag{1}$$

with boundary condition

$$\mathbf{v} \rightarrow 0 \quad \text{as} \quad \mathbf{x} \rightarrow \partial D. \tag{2}$$

Here $\mathbf{v} \equiv (u, w)$, $\mathbf{h} \equiv (fv, \theta)$, $\mathbf{x} \equiv (x, z)$, $\nabla \equiv (\partial/\partial x, \partial/\partial z)$, $\Delta \equiv \nabla \cdot \nabla$, $d/dt \equiv (\partial/\partial t) + \mathbf{v} \cdot \nabla$, $(u, v, w, \theta, p) \equiv (u', v', w', g\theta'/\Theta_0, p')$, and the primes represent perturbation quantities. $\rho_0(\mathbf{x}) \equiv 1/\alpha_0(\mathbf{x})$ is the density of the background atmosphere. Strictly speaking, if the molecular viscosity is considered, then $\alpha_0 \frac{1}{3}\mu\nabla(\nabla \cdot \mathbf{v})$ should be included in the viscous term unless ρ_0 is constant. However, for the eddy viscosity, there is no exact form and the viscous term used in (1) or (4) (with $\rho_0 \neq$ constant) is an approximation. The stability tensor

$$\Pi = \begin{pmatrix} F^2 & S^2 \\ S^2 & N^2 \end{pmatrix}$$

contains three basic frequencies, i.e., the Brunt–Väisälä frequency N, the baroclinic frequency S, and the inertial frequency F. They are defined as

$$N^2 \equiv \frac{g}{\Theta_0}\frac{\partial \Theta_0}{\partial z}, \quad S^2 \equiv \frac{g}{\Theta_0}\frac{\partial \Theta_0}{\partial x} = f\frac{\partial V}{\partial z}, \quad F^2 \equiv f\left(f + \frac{\partial V}{\partial x}\right),$$

where the basic wind $V = [0, \mathbf{V}(\mathbf{x}), 0]$ and potential temperature Θ_0 fields satisfy the thermal wind relation.

Assume that $\mathbf{v}$ is smooth enough and the domain D is bounded at least in one direction, i.e., D is bounded by two parallel lines in the (x, z) plane. Then similarly to (4.1) of Xu (1986b), we have

$$\frac{d}{dt}(\{\{K_2\}\} + \{\{A\}\} - \{\{R\}\}) = -\mu\{\{|\nabla \mathbf{v}|^2\}\} \leqslant -C\{\{K_2\}\}, \tag{3}$$

where $\{\{\quad\}\} \equiv \Omega^{-1}\int_D (\quad)d\mathbf{x}$, $\Omega = \int_D d\mathbf{x}$, $K_2 = (\rho_0/2)|\mathbf{v}|^2$ is the two-dimensional kinetic energy of transverse motion,

$$A = \frac{\rho_0}{2}\mathbf{l} \cdot (\Pi \mathbf{l}) \equiv \frac{\rho_0}{2}(F^2\xi^2 + 2S^2\xi\zeta + N^2\zeta^2)$$

is the generalized potential energy, and $R = \rho_0 \mathbf{l} \cdot \mathbf{h}^0 \equiv \rho_0(\xi f v^0 + \xi\theta^0)$ is the energy associated with the initial value $\mathbf{h}^0 \equiv (fv^0, \theta^0)$. Here $\mathbf{l} \equiv (\xi, \zeta)$ is the parcel displacement, i.e., $\mathbf{v} \equiv d\mathbf{l}/dt$. In (3) the last inequality is the modified Poincaré inequality (see lemma A1 in the Appendix), $C = \mu[K \min\rho_0(\mathbf{x})]^{-1}$ and the constant K depends on the domain D. When the domain D is bounded, any parcel displacement $\mathbf{l}$ is bounded and thus $-\{\{A\}\} + \{\{R\}\}$ must be bounded above. In this case, the integration of (3) would give a contradiction

$$0 \leqslant \{\{K_2\}\} \leqslant \{\{K_2{}^0\}\} - \{\{A\}\} + \{\{R\}\} - C\int_0^t \{\{K_2\}\}dt' \rightarrow -\infty \quad \text{as} \quad t \rightarrow \infty,$$

unless $\{\{K_2\}\} \rightarrow 0$ rapidly enough as $t \rightarrow \infty$. Thus, we can see that without the thermal–inertial conductivity (i.e., $\chi = 0$) and with only transverse viscosity, the SI circulation will vanish. In other words, the maintenance of either steady or unsteady viscous SI circulations crucially depends on thermal–inertial energy transport through the boundaries of the domain.

3. Steady viscous CSI circulations

For a steady CSI circulation, the heating field must also remain steady. Thus, we can treat the latent heating associated with precipitation as an interior heat source which is implicitly related to the motion field. In this way, we can examine circulations of a more general type, i.e., the "symmetric circulations", as defined earlier, in which the latent heat sources are generated either purely internally or partially internally. Being different from that of the SI circulations, as we will see later, the maintenance of these steady symmetric circulations depends on both the interior precipitation heating and the boundary transport of the thermal–inertial energy (the heating depends ultimately on the moisture transport through the boundaries but this process is beyond the description of our present model).

Now considering the precipitation heating implicitly, we have the following governing equations:

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla-\alpha_0\mu\Delta\right)\mathbf{v}-\mathbf{h}=-\nabla(\alpha_0 p), \quad (4a)$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla-\alpha_0\mu\Delta\right)\mathbf{h}+\Pi\mathbf{v}=\mathbf{q}, \quad (4b)$$

$$\nabla\cdot\rho_0\mathbf{v}=0, \quad (4c)$$

where $\mathbf{q} = (0, q)$ and the latent heating rate $q(w, \Theta_0 + \theta, \Theta_e + \theta_e) \approx q(w, \Theta_0, \Theta_e) \approx q(w, z)$ is assumed to be Lipschitz-continuous as a function of w [cf. (13)]. Here, as we mentioned before, the gradient of basic state must be uniform, i.e., Π is a constant matrix, because of the presence of viscosity. Furthermore, since we already consider the precipitation heating, the stability tensor Π in (4) has coefficients appropriate for unsaturated air, and thus positive definite when symmetric instability is conditional on latent heating.

To specify the domain D and boundary condition for (4), we notice that mesoscale rainbands and their adjacent regions are mostly confined to mesoscale areas in the troposphere, i.e.,

$$D=\{\mathbf{x}=(x,z)\,|\,-L<x<L, 0<z<H\}, \quad (5a)$$

where H is the scale height of the circulations which usually does not exceed the height of the tropopause, and $L \gg H$ or even $L \to \infty$. Thus, D may be either bounded or unbounded. We assume the boundary conditions

$$(\mathbf{v},\mathbf{h})=0 \quad \text{on } \partial D. \quad (5b)$$

Here the top, bottom and lateral boundaries act as nonslip and perfect thermally conducting rigid surfaces.

For steady symmetric circulations we may multiply (4b) by Π^{-1} to obtain

$$\alpha_0\mu\Delta\mathbf{v}-\mathbf{v}\cdot\nabla\mathbf{v}+\mathbf{h}-\nabla(\alpha_0 p)=0, \quad (6a)$$

$$\alpha_0\mu\Delta\Pi^{-1}\mathbf{h}-\mathbf{v}\cdot\nabla(\Pi^{-1}\mathbf{h})-\mathbf{v}+\Pi^{-1}\mathbf{q}=0, \quad (6b)$$

$$\nabla\cdot\rho_0\mathbf{v}=0, \quad (6c)$$

where Π^{-1} is the inverse matrix of Π, i.e., $\Pi^{-1}\Pi = \mathbf{I}$. Since the matrix Π is positive definite (dry stability tensor), Π^{-1} is also positive definite. Obviously here $q = q(w, z)$ is also steady and independent of time. To study the existence of steady solutions of (5)–(6) and their stability in the sense of Lyapunov, we need to introduce the following terminology.

We denote by $L_p(D)$ the Banach space of p-summable functions f on D with norm

$$\|f\|_p=\left(\int|f|^p d\mathbf{x}\right)^{1/p}, \quad (7a)$$

where the integral is over D. Let $W_{k,p}(D)$ be the Sobolev space of functions defined on D with p-summable derivatives $\partial^\alpha f$ of order $\alpha \leqslant k$ and with norm

$$\|f\|_{k,p}=\left(\sum_{\alpha=0}^{k}\int|\partial^\alpha f|^p d\mathbf{x}\right)^{1/p}. \quad (7b)$$

When $p = 2$, and $k = 0$, (7a) is equivalent to the scalar norm in the Hilbert space $L_2(D)$; in this case we simply omit the subscript.

By $\mathring{J}(D)$ we denote the space of all smooth four-dimensional vector fields (ψ, Φ) on D, with $\psi = (\psi_1, \psi_2)$ and $\Phi = (\Phi_1, \Phi_2)$ having compact support in D and ψ satisfying the solenoidal-like condition $\nabla\cdot\rho_0\psi = 0$. Since Π^{-1} is positive definite, we can define on $\mathring{J}(D)$ the inner product

$$[(\psi,\Phi),(\psi',\Phi')]\equiv\int\{\nabla\psi\cdot\nabla\psi'+\nabla\Phi\cdot(\Pi^{-1}\nabla\Phi')\}d\mathbf{x}. \quad (8)$$

Here and hereafter we use the summation convention (see page 129 of Dutton, 1976) in two dimensions and thus $\nabla\psi\cdot\nabla\psi' \equiv (\nabla\psi_i)\cdot(\nabla\psi'_i) = \partial_l\psi_i\partial_l\psi'_i$ and $\nabla\Phi\cdot(\Pi^{-1}\nabla\Phi') = \partial_l\Phi_i\Pi_{ij}\partial_l\Phi'_j$. We now define $\mathring{W}(D)$ to be the completion of $\mathring{J}(D)$ in the norm associated with the inner product (8), which we denote by $\|(\psi, \Phi)\|_{\mathring{W}}$. Here $\mathring{W}(D)$ is (componentwise) a closed subspace of the Sobolev space $W_{1,2}(D)$ of four-dimensional vector functions on D.

By a classical solution $(\mathbf{v}, \mathbf{h})$ of the steady system (5)–(6) we mean a solution for which all derivatives appearing in (5)–(6) exist, are continuous, and satisfy the equations pointwise on ∂D and in D. Accordingly, by a generalized solution we mean a vector field $(\mathbf{v}, \mathbf{h}) \in \mathring{W}(D)$ such that

$$-\mu\int\{\nabla\psi\cdot\nabla\mathbf{v}+\nabla\Phi\cdot(\Pi^{-1}\nabla\mathbf{h})\}d\mathbf{x} - \int\rho_0\{\psi\cdot(\mathbf{v}\cdot\nabla\mathbf{v})+\Phi\cdot[\mathbf{v}\cdot\nabla(\Pi^{-1}\mathbf{h})]\}d\mathbf{x} + \int\rho_0\{\psi\cdot\mathbf{h}-\Phi\cdot\mathbf{v}\}d\mathbf{x}+\int\rho_0\Phi\cdot(\Pi^{-1}\mathbf{q})d\mathbf{x}=0 \quad (9)$$

for all $(\psi, \Phi) \in \mathring{J}(D)$. Obviously, (9) is obtained by integrating $\rho_0\psi\cdot(6a) + \rho_0\Phi\cdot(6b)$ and using the solenoidal condition on ψ.

Definition 1: A steady solution $(\mathbf{v}^s, \mathbf{h}^s)$ of system (4)–(5) is symmetrically asymptotically stable in the sense of Lyapunov if and only if it is stable (cf. definition 1 of Xu, 1986b) and for enough small $\|(\mathbf{v}', \mathbf{h}')\|_{t=0}$

$$\|(\mathbf{v}', \mathbf{h}')\| \to 0 \quad \text{as} \quad t \to \infty,$$

where $(\mathbf{v}', \mathbf{h}') = (\mathbf{v} - \mathbf{v}^s, \mathbf{h} - \mathbf{h}^s)$ and $(\mathbf{v}, \mathbf{h})$ is the solution of (4)–(5). Here, as mentioned before, the subscript $p = 2$ is omitted in the norm $\|\cdot\|$.

Theorem 1: When the (latent) heat source $q = 0$, the trivial solution $(\mathbf{v}, \mathbf{h}) = 0$ of system (4)–(5) is symmetrically asymptotically stable in the sense of Lyapunov.

Proof: We define the energy of any solution $(\mathbf{v}, \mathbf{h})$ of system (4)–(5) as

$$E = \frac{1}{2}\int \rho_0\{|\mathbf{v}|^2 + \mathbf{h}\cdot(\Pi^{-1}\mathbf{h})\}d\mathbf{x}. \tag{10}$$

By the transport theorem, the time rate of change of E is

$$\begin{aligned}\frac{dE}{dt} &= \int \rho_0\left\{\mathbf{v}\cdot\frac{d}{dt}\mathbf{v} + \mathbf{h}\cdot\left(\Pi^{-1}\frac{d}{dt}\mathbf{h}\right)\right\}d\mathbf{x} \\ &= -\mu\int\{|\nabla\mathbf{v}|^2 + \nabla\mathbf{h}\cdot(\Pi^{-1}\nabla\mathbf{h})\}d\mathbf{x} - B \\ &= -\mu\|(\mathbf{v}, \mathbf{h})\|_{\mathring{W}}{}^2 < 0,\end{aligned} \tag{11}$$

where (4)–(5) and integration by parts are used. Here B represents non-negative dissipative effects on the boundaries (§11.1.2 of Dutton, 1976) and $B = 0$ when the first-order derivatives of $(\mathbf{v}, \mathbf{h})$ have finite values at the boundaries.

Since the constant matrix Π and thus Π^{-1} are positive definite, the energy E given by (10) is also positive definite, while its time rate of change (11) is negative definite. Thus, considering E as a Lyapunov function, we can conclude that the trivial solution is symmetrically asymptotically stable in the sense of Lyapunov. QED.

Obviously, the asymptotically stable property in theorem 1 crucially depends on the presence of viscosity. Theorem 1 also suggests that when the heat source (due to precipitation, etc.) q is steady and sufficiently weak, there would exist steady symmetric circulations which are themselves stable. This problem will be examined through the proofs of the following theorems. Our method of proof is based on that developed by Ladyzhenskaya (1969) for a similar problem involving the Navier-Stokes equations for incompressible flow. For the most part our proof will follow that of Dutton and Kloeden (1983) where Ladyzhenskaya's method is modified and applied to a rotational and Boussinesq-compressible flow containing the Hadley convective regimes due to the external heating. In the Appendix, we prove the following theorem.

Theorem 2: If the latent heating is bounded, i.e., $\|q\| \leqslant Q$, then there exists at least one steady symmetric circulation which is a generalized solution $(\mathbf{v}, \mathbf{h}) \in \mathring{W}(D)$ of the steady system (5)–(6). Such a solution satisfies the bound

$$\|(\mathbf{v}, \mathbf{h})\|_{\mathring{W}} \leqslant \sqrt{K}F(\nu\Lambda)^{-1}\|q\| \leqslant (\pi\nu\Lambda)^{-1}FHQ, \tag{12}$$

where $\nu^{-1} = \mu^{-1}\max\rho_0(\mathbf{x})$, $\Lambda^2 \equiv \det(\Pi) = N^2f^2 - S^4$, the constant $K = (H/\pi)^2$, and H is the depth of the domain D in (5a).

Here Q is independent of the solution $(\mathbf{v}, \mathbf{h})$ and assumed to be bounded. The latter assumption is crucial for the proof in the Appendix. It, however, is not a serious limitation for theorem 2, because the boundedness of Q can be shown as follows. The latent heating related to an external forcing is obviously bounded, so we only need to consider the case of purely internally generated heating, i.e., the case of CSI circulations. Let Γ_0 be the Lipschitz constant for function $q(w)$. Thus, for an idealized latent heat source, we have

$$q\begin{cases}\leqslant\Gamma_0 w, & \text{when} \quad w>0 \\ =0, & \text{otherwise.}\end{cases} \tag{13}$$

Practically, Γ_0 can be estimated by $\max(N^2 - N_w{}^2)$, where $N_w{}^2 = (g/\Theta_0)(\partial\Theta_e/\partial z)$ and Θ_e is the equivalent potential temperature. By (13) we have $Q \leqslant \Gamma_0\|w\| \leqslant \Gamma_0\|\mathbf{v}\|$. Thus, to see the boundedness of Q, we only need to show that the integrated transverse kinetic energy $\frac{1}{2}\|\mathbf{v}\|^2$ is bounded. According to (13), the buoyancy θ generated by latent heating is bounded by $\Gamma_0 H$, so that $\{\{\theta w\}\}$, i.e., the conversion from buoyant energy due to latent heating to the transverse kinetic energy is at most proportional to $\|w\|$ ($< \|\mathbf{v}\|$), especially when $\|\mathbf{v}\|$ is large. On the other hand, as shown by (3), the viscous dissipation rate of the transverse kinetic energy is at least proportional to $\|\mathbf{v}\|^2$ (rather than $\|\mathbf{v}\|$). The thermal–inertial conductivity transports the (moist) thermal–inertial energy through the boundary to the interior domain, maintains the conditional instability of the basic state, but gives no direct contribution to the transverse kinetic energy. The dry basic state is stable and gives only a negative contribution to the generation of the transverse kinetic energy. Therefore, for a viscous CSI circulation, the net generation of the transverse kinetic energy is positive when the circulation is weak, but it will become negative as the circulation is strong enough. This simply means that $\|\mathbf{v}\|^2/2$ is bounded and so is Q. Thus, we have the following corollary.

Corollary 2: In response to an internally generated steady latent heat source, there exists at least one steady CSI circulation.

Here the prerequested condition is that a nonzero steady latent heat source can be generated internally. This condition is equivalent to that the primary bifurcation point of (4)–(5) with $q = q(w)$ is a steady one. The latter seems to be true because both the numerical integration of viscous CSI circulation (Bennetts and Hoskins, 1979) and the linear CSI modes with bulk viscosity approximation (Xu, 1986a) have shown that the initial growths of CSI circulations are monotonic. However, to prove this rigorously seems difficult for an analytical method, although the similar problem for conditionally (upright) convective instability has been solved successfully (Bretherton, 1987).

4. A criterion of uniqueness and stability

Since q is a Lipschitz-continuous function of w, for $w' = w - w^s$, the perturbation to q can be expressed as

$$q'\begin{cases}\leq \Gamma w', & \text{when} \quad w^s + w' > 0 \\ =0, & \text{otherwise.}\end{cases} \quad (14)$$

If q is proportional to w, then (14) is just a perturbation of (13) and $\Gamma = \Gamma_0$. In this case, as shown later, the problems of uniqueness and stability can not be answered by the theorems developed in this section. However, if the rate of moisture supply is limited and/or the latent heating is partially related to an external forcing (e.g., frontogenetic forcing), then $\Gamma <$ or $\ll \Gamma_0$. In this case, we have the following theorems.

Theorem 3: When the latent heating is weak and insensitive to the motion perturbation, specifically when

$$\|q\| + \nu\Gamma r_0 < \pi^2\nu^2\Lambda(2FH^2)^{-1}, \quad (15)$$

the problem (5)–(6) has no more than one generalized solution. Here Γ and r_0 are given in (14) and (17), respectively.

Proof: Suppose that there were two solutions of (9), e.g., $(\mathbf{v}', \mathbf{h}')$ and $(\mathbf{v}'', \mathbf{h}'')$. Then the difference $(\mathbf{v}^*, \mathbf{h}^*) = (\mathbf{v}', \mathbf{h}') - (\mathbf{v}'', \mathbf{h}'')$ would belong to $\mathring{W}(D)$. If we set $(\boldsymbol{\psi}, \boldsymbol{\Phi}) = (\mathbf{v}^*, \mathbf{h}^*)$ in (9) for each of $(\mathbf{v}', \mathbf{h}')$ and $(\mathbf{v}'', \mathbf{h}'')$, subtract and integrate by parts, then we obtain

$$\mu\int[|\nabla\mathbf{v}^*|^2 + \nabla\mathbf{h}^*\cdot(\Pi^{-1}\nabla\mathbf{h}^*)]d\mathbf{x} = \int\rho_0\{\mathbf{v}'\cdot(\mathbf{v}^*\cdot\nabla\mathbf{v}^*) + \mathbf{h}'\cdot[\mathbf{v}^*\cdot\nabla(\Pi^{-1}\mathbf{h}^*)]\}d\mathbf{x} + \int\rho_0\mathbf{h}^*\cdot(\Pi^{-1}\mathbf{q}^*)d\mathbf{x}.$$

Hence similar to the proof of lemmas A2 and A4, we have

$$\begin{aligned}\nu\|(\mathbf{v}^*, \mathbf{h}^*)\|_{\mathring{W}}^2 &\leq 2\sqrt{K}\|(\mathbf{v}', \mathbf{h}')\|_{\mathring{W}}\|(\mathbf{v}^*, \mathbf{h}^*)\|_{\mathring{W}}^2 + \sqrt{K}(F/\Lambda)\|q^*\|\ \|\nabla(\Pi^{-1/2}\mathbf{h}^*)\| \\ &\leq 2KF(\nu\Lambda)^{-1}(\|q'\| + \nu\Gamma r_0)\|(\mathbf{v}^*, \mathbf{h}^*)\|_{\mathring{W}}^2, \end{aligned} \quad (16)$$

where (12) and (14) are used for $(\mathbf{v}', \mathbf{h}')$ and q^* respectively, and

$$r_0 = \frac{1}{2}\max_t \frac{\left(\int_{D_0}|\nabla\mathbf{v}^*|^2 d\mathbf{x}\right)^{1/2}\left(\int_{D_0}|\Pi^{-1/2}\nabla\mathbf{h}^*|^2 d\mathbf{x}\right)^{1/2}}{\|(\mathbf{v}^*, \mathbf{h}^*)\|_{\mathring{W}}^2} < \frac{1}{4}, \quad (17)$$

$D_0 \subset D$ is the subdomain where $q' \neq 0$. Note that $K = (H/\pi)^2$ and q' in (16) means q in (15). Thus, when (15) is satisfied, (16) implies that $\|(\mathbf{v}^*, \mathbf{h}^*)\|_{\mathring{W}}^2 = 0$, i.e., the two generalized solutions $(\mathbf{v}', \mathbf{h}')$ and $(\mathbf{v}'', \mathbf{h}'')$ coincide in $\mathring{W}(D)$. QED.

Theorem 4: A steady circulation that is a solution $(\mathbf{v}, \mathbf{h}) = (\mathbf{v}^s, \mathbf{h}^s)$ of the system (5)–(6) is symmetrically asymptotically stable in the sense of Lyapunov if

$$\|(\mathbf{v}^s, \mathbf{h}^s)\|_{\mathring{W}} + (\pi\Lambda)^{-1}FH\Gamma r_0 < \pi\nu(2H)^{-1} \quad (18)$$

or (15) is satisfied.

Proof: In the same manner as in (9) with $(\mathbf{v}, \mathbf{h}) \in \mathring{W}(D)$ for all $t \geq 0$, we define the generalized solution of the nonsteady system (4)–(5). The perturbation $(\mathbf{v}', \mathbf{h}') = (\mathbf{v}, \mathbf{h}) - (\mathbf{v}^s, \mathbf{h}^s)$ with pressure $p' = p - p^s$ satisfies in the generalized sense the system of

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \mathbf{v}'\cdot\nabla\right)\mathbf{v}' - \mathbf{h}' = -\mathbf{v}^s\cdot\nabla\mathbf{v}' - \mathbf{v}'\cdot\nabla\mathbf{v}^s - \nabla(\alpha_0 p') + \alpha_0\mu\Delta\mathbf{v}', \\ &\left(\frac{\partial}{\partial t} + \mathbf{v}'\cdot\nabla\right)\Pi^{-1}\mathbf{h}' + \mathbf{v}' = -\mathbf{v}^s\cdot\nabla(\Pi^{-1}\mathbf{h}') - \mathbf{v}'\cdot\nabla(\Pi^{-1}\mathbf{h}^s) + \alpha_0\mu\Delta\Pi^{-1}\mathbf{h}' + \Pi^{-1}\mathbf{q}', \\ &\nabla\cdot\rho_0\mathbf{v}' = 0, \\ &(\mathbf{v}', \mathbf{h}') = 0 \quad \text{on } \partial D. \end{aligned} \quad (19)$$

Similarly to (10) the energy of $(\mathbf{v}', \mathbf{h}')$ can be defined as

$$E' = \frac{1}{2}\int\rho_0\{|\mathbf{v}'|^2 + \mathbf{h}'\cdot(\Pi^{-1}\mathbf{h}')\}d\mathbf{x} > 0 \quad (20)$$

because Π^{-1} is positive definite. On differentiating E' with respect to time t, substituting for the total derivatives of $\mathbf{v}'$ and $\Pi^{-1}\mathbf{h}'$ from (19), integrating by parts and using the continuity equation and boundary conditions, we obtain

$$\frac{dE'}{dt} = -\nu\int\rho_0\{|\nabla\mathbf{v}'|^2 + \nabla\mathbf{h}'\cdot(\Pi^{-1}\nabla\mathbf{h}')\}d\mathbf{x} + \int\rho_0\{\mathbf{v}^s\cdot(\mathbf{v}'\cdot\nabla\mathbf{v}') + \mathbf{h}^s\cdot[\mathbf{v}'\cdot\nabla(\Pi^{-1}\mathbf{h}')]\}d\mathbf{x} + \int\rho_0\mathbf{h}'\cdot(\Pi^{-1}\mathbf{q}')d\mathbf{x}. \quad (21)$$

By the same reasoning as in (16) we can estimate the upper bound of the rhs of (21) and thus have

$$\frac{dE'}{dt} < \{-\nu + 2\sqrt{K}\|(\mathbf{v}^s, \mathbf{h}^s)\|_{\mathring{W}} + 2KF\Lambda^{-1}\Gamma r_0\}\|(\mathbf{v}', \mathbf{h}')\|_{\mathring{W}}^2. \quad (22)$$

When the steady solution $(\mathbf{v}^s, \mathbf{h}^s)$ satisfies the bound (18), the time derivative of the energy E' in (22) will be negative. Considering E' as a Lyapunov function, we prove the theorem for condition (18) immediately.

In fact, the stability criterion (18) is also a uniqueness criterion which is implied by the first inequality of (16). Moreover, by theorem 2, all existing steady solutions satisfy (12), so that criterion (15), as a combination of (12) and (18), guarantees both the uniqueness and stability of the generalized steady solutions. QED.

By replacing $(\mathbf{v}^s, \mathbf{h}^s)$ and Γ with (0, 0) and Γ_0 respectively, we obtain immediately that the basic state is stable to CSI perturbations if

$$\Gamma_0 r_0 < \nu\Lambda(\pi/H)^2(2F)^{-1}. \tag{23}$$

In other words, the inverse proposition of (23) is the necessary condition for the initiation of CSI circulations. Thus, for the extreme case of $\Gamma = \Gamma_0$, (15) and (18) become trivial and useless, because the stability required by (15) and (18) are stronger than (23). In this case, the stability problem for the nonlinear CSI circulation needs more detailed numerical studies. According to the theory of conditionally convective instability (Bretherton, 1987), the similarity between CSI and conditionally convective instability, and the linear theory of CSI (Xu, 1986a), we may speculate that, if $\Gamma = \Gamma_0$ and CSI is weak enough, steady CSI circulations may be stable but may not be unique. In this case, the second term on the rhs of (21), i.e., the nonlinear interaction term between ($\mathbf{v}^s$, $\mathbf{h}^s$) and ($\mathbf{v}'$, $\mathbf{h}'$) should be negative, rather than positive as estimated in (22).

However, for the extreme case of $\Gamma = 0$, (15) and (18) are nontrivial and do indicate the uniqueness and stability of the symmetric circulations. This latter situation will occur if the latent heat release reaches the limit allowed by the large-scale moisture supply and/or if the latent heat source is generated externally (e.g., by frontogenetical forcing). As the situation changes from $\Gamma = 0$ to $\Gamma = \Gamma_0$, (15) and (18) become less useful and the informations contained in theorems 3 and 4 depreciate, perhaps rapidly, to zero. The real frontal rainband situations are more likely between the above two extreme cases, because CSIs are often associated with either strong or weak frontogenetic forcings and a pure CSI was rarely observed in frontal rainbands. Thus, we expect that theorems 3 and 4 may find a use to the situations where CSI and frontogenetic forcing coexist. This problem is under our current study.

5. Some physical implications

From the physical viewpoint, (12) reflects the fact that the energy dissipation rate of a steady symmetric circulation is bounded by the energy production rate of the latent heat source. It is simply a consequence of energy conservation. We may see this directly from the rearranged form of (12) (cf. lemma A2)

$$\sqrt{2}\frac{\nu}{H}\|(\nabla\mathbf{v}, \nabla\Pi^{-1/2}\mathbf{h})\| \leqslant \|\Pi^{-1/2}\mathbf{q}\|.$$

On the other hand, the derivations of (21)–(22) imply that (18) results from a comparison between dissipation, nonlinearity and/or sensitivity of the heating to the motion perturbation. Here we only consider the extreme case $\Gamma = 0$. In this case (18) can be rewritten as

$$\mathrm{Re} < 1, \tag{25}$$

where $\mathrm{Re} = 2H(\pi\nu)^{-1}\|(\mathbf{v}, \mathbf{h})\|_{\mathscr{W}}$ is the generalized Reynolds number. Therefore a symmetric circulation will be steady with a "laminarlike" regime if the heat source is not adequate enough to support a "turbulentlike" regime for which at least $\mathrm{Re} \geqslant 1$.

By (12) we may say that a steady symmetric circulation will be weaker if the heat source becomes weaker and/or viscosity becomes larger, and by (18) we would say that a weaker steady symmetric circulation with larger viscosity and smaller sensitivity of the heating to the motion perturbation should be more stable. Thus, as a combination of the two, (15) implies that to keep a symmetric circulation steady, the heat source and its sensitivity to the motion perturbation should not be too strong and the (eddy) viscosity should not be too small. In other words, inviscid symmetric circulations, including CSI circulations, may not be steady—the conjecture which we mentioned earlier.

The factor Λ/F in (15) represents the effect of the (dry symmetrically stable) basic state on the stability of steady symmetric circulations. This factor was originally introduced in the derivation of (12), or equivalently (A17), when (A1) was used. Physically it estimates the constraint of the dry symmetrically stable basic state on the motion fields induced by purely thermal sources, i.e., heterotropic vector sources $\mathbf{q} = (0, q)$. Note that $\Lambda^2 = N^2F^2 - S^4$ and the factor Λ/F is just the square root of the dry potential vorticity (Bennetts and Hoskins, 1979) which increases as the baroclinicity $|S^2| = |f\partial V/\partial z|$ decreases and/or the stratification N^2 and/or inertial stability increase, or say, as the symmetric stability of the (dry) basic state increases. Thus, the implication of (15) in this aspect is that with same heat source a steady symmetric circulation will be less stable or even become unstable (growing and so on) when the basic states becomes less stable symmetrically. When the baroclinicity vanishes, i.e., $|S^2| = f|\partial V/\partial z| \to 0$, then $\Lambda/F \to N$. In this case, the physical meaning of this factor is the intuitive one that the constraint of the basic state on the motion fields induced by a purely thermal source depends solely on the stratification.

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APPENDIX A

Proof of Theorem 2

In order to prove theorem 2 we modify for the present study the proof used by Dutton and Kloeden (1983) or originally by Ladyzhenskaya (1969). Thus, we need some properties of Π^{-1}. First, it is easy to be verified that

$$\Pi^{-1}=\begin{pmatrix} N^2 & -S^2 \\ -S^2 & F^2 \end{pmatrix}\Lambda^{-2}$$

$$\text{and}\quad \mathbf{q}\cdot(\Pi^{-1}\mathbf{q})=|q|^2(F/\Lambda)^2 \tag{A1}$$

where $\Lambda^2 \equiv \det(\Pi) = N^2F^2 - S^4$ and $\mathbf{q} = (0, q)$. Also, we can have

$$R^T\Pi^{-1}R=\begin{pmatrix} \hat{N}^2 & 0 \\ 0 & \hat{F}^2 \end{pmatrix}\Lambda^{-2}\equiv\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where

$$R\equiv R(\hat{\alpha})=\begin{pmatrix} \cos\hat{\alpha} & -\sin\hat{\alpha} \\ \sin\hat{\alpha} & \cos\hat{\alpha} \end{pmatrix},\quad \hat{\alpha}=\frac{1}{2}\arctan\left(\frac{-2S^2}{N^2-F^2}\right),$$

$$\hat{N}^2\equiv\frac{1}{2}(F^2+N^2+B),\quad \hat{F}^2\equiv\frac{1}{2}(F^2+N^2-B)$$

$$\text{and}\quad B=\mathrm{sgn}(N^2-F^2)[(N^2-F^2)^2+4S^4]^{1/2}.$$

Obviously, $R^T(\alpha) = R(-\alpha) = R^{-1}(\alpha)$, so that with the following notation

$$\Pi^{-r}\equiv R\begin{pmatrix} \lambda_1{}^r & 0 \\ 0 & \lambda_2{}^r \end{pmatrix}R^T,$$

we have

$$\lambda_{\min}|\mathbf{\Phi}|^2\leqslant\mathbf{\Phi}\cdot(\Pi^{-1}\mathbf{\Phi})=|\Pi^{-1/2}\mathbf{\Phi}|^2\leqslant\lambda_{\max}|\mathbf{\Phi}|^2, \tag{A2}$$

$$\lambda_{\min}|\Pi^{-1/2}\mathbf{\Phi}|^2\leqslant|\Pi^{-1}\mathbf{\Phi}|^2\leqslant\lambda_{\max}|\Pi^{-1/2}\mathbf{\Phi}|^2, \tag{A3}$$

where $\lambda_{max} = \max(\lambda_1, \lambda_2)$, $\lambda_{min} = \min(\lambda_1, \lambda_2)$, and $\mathbf{\Phi}$ is an arbitrary two-dimensional vector.

Lemma A1: For any component f of a vector field $(\psi, \mathbf{\Phi}) \in \mathring{W}(D)$,

$$\int f^2 d\mathbf{x}\leqslant K\int|\nabla f|^2 d\mathbf{x} \tag{A4}$$

$$\int f^4 d\mathbf{x}\leqslant 2\left(\int f^2 d\mathbf{x}\right)\left(\int|\nabla f|^2 d\mathbf{x}\right), \tag{A5}$$

where the constant K depends only on the domain D and in our case of a strip domain (5a), $K = (H/\pi)^2$.

Proof: Since $(\psi, \mathbf{\Phi}) = 0$ at $z = 0$, f can be expanded in a Fourier sine series:

$$f=\sum_{n=1}^{\infty}a_n(x)\sin\frac{n\pi z}{H}.$$

Thus,

$$\iint f^2 dxdz=\frac{H}{2}\sum\int a_n{}^2(x)dx\leqslant\frac{H^3}{2\pi^2}\sum\int\frac{n^2\pi^2}{H^2}a_n{}^2(x)dx$$

$$=\frac{H^2}{\pi^2}\iint\left(\frac{\partial f}{\partial z}\right)^2 dxdz\leqslant\frac{H^2}{\pi^2}\int|\nabla f|^2 d\mathbf{x}$$

proves (A4). This is similar to the proof of lemma 3.3 of Kloeden (1985). For the proof of (A5), see section 1.1 of Ladyzhenskaya (1969). QED.

Now we define the linear functionals

$$J_1(\psi,\mathbf{\Phi})=\int(\rho_0/\rho_m)\mathbf{\Phi}\cdot(\Pi^{-1}\mathbf{q})d\mathbf{x}, \tag{A6}$$

$$J_2(\psi,\mathbf{\Phi})=\int(\rho_0/\rho_m)(\psi\cdot\mathbf{h}-\mathbf{\Phi}\cdot\mathbf{v})d\mathbf{x}, \tag{A7}$$

$$J_3(\psi,\mathbf{\Phi})=\int(\rho_0/\rho_m)\{\psi\cdot(\mathbf{v}\cdot\nabla\mathbf{v})+\mathbf{\Phi}\cdot[\mathbf{v}\cdot\nabla(\Pi^{-1}\mathbf{h})]\}d\mathbf{x} \tag{A8}$$

on $\mathring{J}(D)$ for fixed $\mathbf{q} = (0, q)$, $q \in L_2(D)$ and $(\mathbf{v}, \mathbf{h}) \in \mathring{W}(D)$. Here $\rho_m = \max\rho_0(\mathbf{x})$. In the following three lemmas we show that these functionals are bounded in $\mathring{J}(D)$.

Lemma A2: The functional J_1 is bounded on $\mathring{J}(D)$.

Proof: On using (A1)–(A2), (A4) and the Cauchy-Schwarz inequality, we have

$$|J_1(\psi,\mathbf{\Phi})|\leqslant\int|(\Pi^{-1/2}\mathbf{\Phi})\cdot(\Pi^{-1/2}\mathbf{q})|d\mathbf{x}$$

$$\leqslant\left(\int|\Pi^{-1/2}\mathbf{\Phi}|^2 d\mathbf{x}\right)^{1/2}\left(\int|\Pi^{-1/2}\mathbf{q}|^2 d\mathbf{x}\right)^{1/2}$$

$$\leqslant K^{1/2}\left(\int|\nabla(\Pi^{-1/2}\mathbf{\Phi})|^2 d\mathbf{x}\right)^{1/2}\left[\int\mathbf{q}\cdot(\Pi^{-1}\mathbf{q})d\mathbf{x}\right]^{1/2}$$

$$\leqslant\sqrt{K}F\Lambda^{-1}\|(\psi,\mathbf{\Phi})\|_{\mathring{W}}\|q\|$$

for any $(\psi, \mathbf{\Phi}) \in \mathring{J}(D)$. QED.

Lemma A3: The functional J_2 is bounded on $\mathring{J}(D)$.

Proof: First we introduce the following elementary inequality

$$\sqrt{ab}+\sqrt{cd}\leqslant\sqrt{a+d}\sqrt{b+c}, \tag{A9}$$

where a, b, c and d are arbitrary non-negative numbers. The relationship can be easily verified by squaring both sides of (A9). Again, by using the Cauchy-Schwarz inequality, (A2), (A4) and (A9), we have

$$|J_2(\psi,\mathbf{\Phi})|\leqslant\int|\psi\cdot\mathbf{h}|d\mathbf{x}+\int|\mathbf{\Phi}\cdot\mathbf{v}|d\mathbf{x}$$

$$\leqslant K\left(\int|\nabla\psi|^2 d\mathbf{x}\right)^{1/2}\left(\int|\nabla\mathbf{h}|^2 d\mathbf{x}\right)^{1/2}+K\left(\int|\nabla\mathbf{\Phi}|^2 d\mathbf{x}\right)^{1/2}\left(\int|\nabla\mathbf{v}|^2 d\mathbf{x}\right)^{1/2}$$

$$\leqslant K\lambda_{\min}^{-1/2}\left\{\left(\int|\nabla\psi|^2 d\mathbf{x}\right)^{1/2}\left[\int\nabla\mathbf{h}\cdot(\Pi^{-1}\nabla\mathbf{h})d\mathbf{x}\right]^{1/2}+\left(\int|\nabla\mathbf{v}|^2 d\mathbf{x}\right)^{1/2}\left[\int\nabla\Phi\cdot(\Pi^{-1}\nabla\Phi)d\mathbf{x}\right]^{1/2}\right\}$$

$$\leqslant K\lambda_{\min}^{1/2}\|(\psi,\Phi)\|_{\mathring{W}}\|(\mathbf{v},\mathbf{h})\|_{\mathring{W}}$$

for any $(\psi, \Phi) \in \mathring{J}(D)$. QED.

Lemma A4: The functional J_3 is bounded on $\mathring{J}(D)$.

Proof: First, integrating by parts, we have

$$J_3(\psi,\Phi)=\int(\rho_0/\rho_m)[\psi_i(\mathbf{v}\cdot\nabla v_i)+\Phi_i(\mathbf{v}\cdot\nabla\Pi_{ij}^{-1}h_j)]d\mathbf{x}$$

$$=-\int(\rho_0/\rho_m)[v_i\mathbf{v}\cdot\nabla\psi_i+\Pi_{ij}^{-1}h_j\mathbf{v}\cdot\nabla\Phi_i]d\mathbf{x}$$

for any $(\psi, \Phi) \in \mathring{J}(D)$.

Now we introduce another elementary inequality

$$\sqrt{a}+\sqrt{b}\leqslant\sqrt{2(a+b)} \qquad \text{(A10)}$$

where a and b are arbitrary non-negative numbers. On using the Cauchy-Schwarz inequality, (A2)–(A5) and (A9)–(A10), we obtain for any $(\psi, \Phi) \in \mathring{J}(D)$

$$|J_3(\psi,\Phi)|\leqslant\int|v_i\mathbf{v}\cdot\nabla\psi_i|d\mathbf{x}+\int|\Pi_{ij}^{-1/2}h_j\mathbf{v}\cdot\nabla\Pi_{ki}^{-1/2}\Phi_k|d\mathbf{x}$$

$$\leqslant\left(\int v_i^4 d\mathbf{x}\right)^{1/4}\left(\int v_n^2v_n^2 d\mathbf{x}\right)^{1/4}\left(\int|\nabla\psi_i|^2 d\mathbf{x}\right)^{1/2}$$

$$+\left(\int|\Pi_j^{-1/2}h_j|^4 d\mathbf{x}\right)^{1/4}\left(\int v_n^2v_n^2 d\mathbf{x}\right)^{1/4}\left(\int|\Pi_{ik}^{-1/2}\nabla\Phi_k|^2 d\mathbf{x}\right)^{1/2}$$

$$\leqslant\left(\int v_n^2v_n^2 d\mathbf{x}\right)^{1/4}\left\{\left(\int v_i^4 d\mathbf{x}\right)^{1/2}+\left(\int|\Pi_{ij}^{-1/2}h_j|^4 d\mathbf{x}\right)^{1/2}\right\}^{1/2}\left\{\int(|\nabla\psi_i|^2+|\Pi_{ik}^{-1/2}\nabla\Phi_k|^2)d\mathbf{x}\right\}^{1/2}$$

$$\leqslant\sqrt{2}\left(\int v_n^2v_n^2 d\mathbf{x}\right)^{1/4}\left\{\left(\int v_i^2v_i^2 d\mathbf{x}\right)^{1/2}+\left(\int|\Pi_{ij}^{-1/2}h_j|^2|\Pi_{ik}^{-1/2}h_k|^2 d\mathbf{x}\right)^{1/2}\right\}^{1/2}\|(\psi,\Phi)\|_{\mathring{W}}$$

$$\leqslant\sqrt{2}\left\{\left(\int v_i^2v_i^2 d\mathbf{x}\right)^{1/2}+\left(\int|\Pi_{ij}^{-1/2}h_j|^2|\Pi_{ik}^{-1/2}h_k|^2 d\mathbf{x}\right)^{1/2}\right\}\|(\psi,\Phi)\|_{\mathring{W}}$$

$$\leqslant 2\left\{\left(\int v_i^2 d\mathbf{x}\right)^{1/2}\left(\int|\nabla v_i|^2 d\mathbf{x}\right)^{1/2}+\left(\int|\Pi_{ij}^{-1/2}h_j|^2 d\mathbf{x}\right)^{1/2}\left(\int|\Pi_{ik}^{-1/2}\nabla h_k|^2 d\mathbf{x}\right)^{1/2}\right\}\|(\psi,\Phi)\|_{\mathring{W}}$$

$$=2\sqrt{K}\|(\mathbf{v},\mathbf{h})\|_{\mathring{W}}^2\|(\psi,\Phi)\|_{\mathring{W}}. \quad \text{QED.}$$

Now, we can apply the Riesz representation theorem on the Hilbert space $\mathring{W}(D)$ for each of the bounded linear functionals (A6)–(A8) to find unique elements $A_1(\mathbf{v}, \mathbf{h})$, $A_2(\mathbf{v}, \mathbf{h})$ and $A_3(\mathbf{v}, \mathbf{h})$ of $\mathring{W}(D)$, respectively such that

$$J_1(\psi,\Phi)=[(\psi,\Phi),A_1(\mathbf{v},\mathbf{h})], \qquad \text{(A11)}$$

$$J_2(\psi,\Phi)=[(\psi,\Phi),A_2(\mathbf{v},\mathbf{h})], \qquad \text{(A12)}$$

$$J_3(\psi,\Phi)=[(\psi,\Phi),A_3(\mathbf{v},\mathbf{h})] \qquad \text{(A13)}$$

for all $(\psi, \Phi) \in \mathring{J}(D)$. Here the inner product [,] has been defined in (8).

Now by (A6)–(A8) and (A11)–(A13) we can rewrite (9) as

$$(\mathbf{v},\mathbf{h})=\nu^{-1}T(\mathbf{v},\mathbf{h}), \qquad \text{(A14)}$$

for all $(\psi, \Phi) \in \mathring{J}(D)$, where $\nu = \mu/\rho_m$ and T is defined for fixed $q \in L_2(D)$ and all $(\mathbf{v}, \mathbf{h}) \in \mathring{W}(D)$ as

$$T(\mathbf{v},\mathbf{h})=(A_1+A_2-A_3)(\mathbf{v},\mathbf{h}). \qquad \text{(A15)}$$

Moreover, for any $0 \leqslant \gamma \leqslant \nu^{-1}$, any solution $(\mathbf{v}, \mathbf{h})$ in $\mathring{W}(D)$ of

$$(\mathbf{v},\mathbf{h})=\gamma T(\mathbf{v},\mathbf{h}) \qquad \text{(A16)}$$

satisfies

$$0=[(\mathbf{v},\mathbf{h}),(\mathbf{v},\mathbf{h})-\gamma T(\mathbf{v},\mathbf{h})]$$

$$=[(\mathbf{v},\mathbf{h}),(\mathbf{v},\mathbf{h})-\gamma(A_1+A_2-A_3)(\mathbf{v},\mathbf{h})]$$

$$=\|(\mathbf{v},\mathbf{h})\|_{\mathring{W}}^2-\gamma\int(\rho_0/\rho_m)\mathbf{h}\cdot(\Pi^{-1}\mathbf{q})d\mathbf{x}$$

because

$$[(\mathbf{v},\mathbf{h}),A_2(\mathbf{v},\mathbf{h})]=\int(\rho_0/\rho_m)(\mathbf{v}\cdot\mathbf{h}-\mathbf{h}\cdot\mathbf{v})d\mathbf{x}=0$$

$[(\mathbf{v},\mathbf{h}),A_3(\mathbf{v},\mathbf{h})]$

$$=\int(\rho_0/\rho_m)\{\mathbf{v}\cdot(\mathbf{v}\cdot\nabla\mathbf{v})+\mathbf{h}\cdot[(\mathbf{v}\cdot\nabla(\Pi^{-1}\mathbf{h})]\}d\mathbf{x}$$

$$=\frac{1}{2}\int(\rho_0/\rho_m)\mathbf{v}\cdot\nabla[|\mathbf{v}|^2+\mathbf{h}\cdot(\Pi^{-1}\mathbf{h})]d\mathbf{x}=0.$$

Hence, the solutions of (A16) satisfy

$$\|(\mathbf{v},\mathbf{h})\|_{\mathring{W}}{}^2=\gamma\int\mathbf{h}\cdot(\Pi^{-1}\mathbf{q})d\mathbf{x}$$

$$\leqslant\gamma\sqrt{K}\left(\int|\nabla(\Pi^{-1/2}\mathbf{h})|^2dx\right)^{1/2}(F/\Lambda)\|q\|$$

$$\leqslant\sqrt{K}F(\nu\Lambda)^{-1}\|(\mathbf{v},\mathbf{h})\|_{\mathring{W}}\|q\|,$$

that is

$$\|(\mathbf{v},\mathbf{h})\|_{\mathring{W}}\leqslant\sqrt{K}F(\nu\Lambda)^{-1}\|q\|\leqslant(\nu\Lambda)^{-1}\sqrt{K}FQ \quad (A17)$$

uniformly in $0\leqslant\gamma\leqslant\nu^{-1}$, where Q is the upper bound of $\|q\|$ as assumed in (12). Here (A17) is valid either when the domain D is bounded or not (but D should be a strip domain).

In order to proceed with our proof, we need now to consider solutions of (9) separately by whether D is bounded or not (but should be a strip). When D is bounded, we can further show that A_1, A_2 and A_3 are compact operators on $\mathring{W}(D)$; that is, they map bounded sequences in $\mathring{W}(D)$ into sequences in $\mathring{W}(D)$ that have convergent subsequences. As we mentioned before, $\mathring{W}(D)$ is a closed subspace of the Sobolev space $W_{1,2}$, while the latter for bounded D is compactly embedded in the spaces $L_2(D)$ and $L_4(D)$ (Theorem 5.4B of Adams, 1975). Thus, when D is bounded, a closed ball $B(R)$ of radius R in $\mathring{W}(D)$ is a compact subset of the space $L_2(D)$ and $L_4(D)$. This property will be used in the following proof.

Lemma A5: The operators A_1, A_2 and A_3 are compact on $\mathring{W}(D)$ if the domain D is bounded.

Proof: Let $\{\mathbf{v}_n,\mathbf{h}_n\}$ be a sequence in $B(R)$ for some fixed radius R. By many steps similar to those in the proof of lemma A3, we have

$$|[(\psi,\Phi),A_2(\mathbf{v}_n,\mathbf{h}_n)-A_2(\mathbf{v}_m,\mathbf{h}_m)]|$$

$$\leqslant\int|\psi\cdot(\mathbf{h}_n-\mathbf{h}_m)|dx+\int|\Phi\cdot(\mathbf{v}_n-\mathbf{v}_m)|dx$$

$$\leqslant(K/\lambda_{\min})^{1/2}\|(\psi,\Phi)\|_{\mathring{W}}\|(\mathbf{v}_n,\mathbf{h}_n)-(\mathbf{v}_m,\mathbf{h}_m)\|.$$

Thus, by setting above $(\psi,\Phi)=A_2(\mathbf{v}_n,\mathbf{h}_n)-A_2(\mathbf{v}_m,\mathbf{h}_m)$, we have

$$\|A_2(\mathbf{v}_n,\mathbf{h}_n)-A_2(\mathbf{v}_m,\mathbf{h}_m)\|_{\mathring{W}}$$

$$\leqslant(K/\lambda_{\min})^{1/2}\|(\mathbf{v}_n,\mathbf{h}_n)-(\mathbf{v}_m,\mathbf{h}_m)\|. \quad (A18)$$

From this and the remark preceeding the statement of the lemma, we conclude that A_2 is a compact operator on $\mathring{W}(D)$. Similarly, for operator A_1,

$$|[(\psi,\Phi),A_1(q_n)-A_1(q_m)]|$$

$$\leqslant\sqrt{K}F\Lambda^{-1}\|(\psi,\Phi)\|_{\mathring{W}}\|q_n-q_m\|$$

$$\leqslant\sqrt{K}F\Lambda^{-1}\Gamma_0\|(\psi,\Phi)\|_{\mathring{W}}\|w_n-w_m\|,$$

where Γ_0 is the Lipschitz constant for q as a function of w [cf. (13)]. Thus, we can obtain an inequality similar to (A18) and prove the compactness on $\mathring{W}(D)$ of the operator A_1. Also similarly for operator A_3,

$$|[(\psi,\Phi),A_3(\mathbf{v}_n,\mathbf{h}_n)-A_3(\mathbf{v}_m,\mathbf{h}_m)]|\leqslant\int|(v_{i,n}\mathbf{v}_n-v_{i,m}\mathbf{v}_m)\cdot\nabla\psi_i|dx+\int|(h_{i,n}\mathbf{v}_n-h_{i,m}\mathbf{v}_m)\cdot\nabla\Pi_{ij}{}^{-1}\Phi_j|d\mathbf{x}$$

$$\leqslant\int|(v_{i,n}-v_{i,m})\mathbf{v}_n\cdot\nabla\psi_i|d\mathbf{x}+\int|v_{i,m}(\mathbf{v}_n-\mathbf{v}_m)\cdot\nabla\psi_i|d\mathbf{x}$$

$$+\int|(h_{i,n}-h_{i,m})\mathbf{v}_n\cdot\nabla\Pi_{ij}{}^{-1}\Phi_j|dx+\int|h_{i,m}(\mathbf{v}_n-\mathbf{v}_m)\cdot\nabla\Pi_{ij}{}^{-1}\Phi_j|dx$$

$$\leqslant\|v_{i,n}-v_{i,m}\|_4\|\mathbf{v}_n\|_4\|\nabla\psi_i\|+\|v_{i,m}\|_4\|\mathbf{v}_n-\mathbf{v}_m\|_4\|\nabla\psi_i\|$$

$$+\|h_{i,n}-h_{i,m}\|_4\|\mathbf{v}_n\|_4\|\nabla\Pi_{ij}{}^{-1}\Phi_j\|+\|h_{i,m}\|_4\|\mathbf{v}_n-\mathbf{v}_m\|_4\|\nabla\Pi_{ij}{}^{-1}\Phi_j\|$$

$$\leqslant M_1\{\|\mathbf{v}_n-\mathbf{v}_m\|_4(\|\nabla\psi\|+\|\nabla\Pi^{-1}\Phi\|)+\|\mathbf{h}_n-\mathbf{h}_m\|_4\|\nabla\Pi^{-1}\Phi\|\}$$

$$\leqslant M_1\lambda_{\max}^{1/2}\|(\psi,\Phi)\|_{\mathring{W}}\|(\mathbf{v}_n,\mathbf{v}_n)-(\mathbf{v}_m,\mathbf{h}_m)\|_4,$$

where the constant $M_1\geqslant2\|\mathbf{v}_n\|_4+\|\mathbf{h}_n\|_4$ for any n. This leads to an inequality similar to (A18) and in turn to the compactness on $\mathring{W}(D)$ of the operator A_3. QED.

Therefore, by lemma A5, when domain D is bounded, the operator T in (A15) is compact and we can thus apply the Leray-Schauder principle (page 32 of Ladyzhenskaya, 1969), which says that if all possible solutions of Eq. (A16) (with compact operator T) for $0\leqslant\gamma<\nu^{-1}$ lie within some closed ball in $\mathring{W}(D)$ such as (A17), then Eq. (A16) with $\gamma=\nu^{-1}$, or equivalently

(A14) or (9), has at least one solution in this ball. So far, for a bounded domain we have proved theorem 2.

To prove the remaining part of theorem 2, i.e., the part for an unbounded strip domain, we need to consider a monotonically increasing sequence of bounded domains D_k, $k = 1, 2, 3, \cdots$, which has the unbounded strip domain D_∞ as its limit. It is easy to see that if we extend each of the vector $(\mathbf{v}, \mathbf{h})$ belonging to $\mathring{W}(D_k)$ over whole D_∞ by setting $(\mathbf{v}, \mathbf{h})$ equal to zero outside D_k, then $(\mathbf{v}, \mathbf{h})$ will belong to $\mathring{W}(D_\infty)$ and

$$\|(\mathbf{v}, \mathbf{h})\|_{\mathring{W}(D_k)} = \|(\mathbf{v}, \mathbf{h})\|_{\mathring{W}(D_\infty)}$$

for any $(\mathbf{v}, \mathbf{h}) \in \mathring{W}(D_k)$. As we already proved, for each of the bounded domain D_k, Eq. (9) with $D = D_k$ has at least one solution $(\mathbf{v}, \mathbf{h})^{(k)}$ and the estimate (A17) gives

$$\|(\mathbf{v}, \mathbf{h})^{(k)}\|_{\mathring{W}(D_\infty)} = \|(\mathbf{v}, \mathbf{h})^{(k)}\|_{\mathring{W}(D_k)} \leqslant (\nu\Lambda)^{-1}\sqrt{K}FQ. \quad \text{(A19)}$$

The upper bound in (A19) holds for all the $(\mathbf{v}, \mathbf{h})^{(k)}$, so that the sequence of $(\mathbf{v}, \mathbf{h})^{(k)}$ is weakly compact in $\mathring{W}(D_\infty)$. We now show that any weak limit of $(\mathbf{v}, \mathbf{h})^{(k)}$ is a generalized solution of (6) and (5), or say a solution of (9) in $\mathring{W}(D_\infty)$. Since any $(\mathbf{v}, \mathbf{h}) \in \mathring{J}(D_\infty)$ has compact support in D_∞, the identity (9) will hold with any fixed (ψ, Φ) and all $(\mathbf{v}, \mathbf{h})^{(k)}$ for all sufficiently large k. Passing the limit in (9) along a subsequence k' for which $\{(\mathbf{v}, \mathbf{h})^{(k')}\}$ is weakly convergent in $\mathring{W}(D_\infty)$ to $(\mathbf{v}, \mathbf{h})$, we see that $(\mathbf{v}, \mathbf{h})$ satisfies (9) with the fixed (ψ, Φ). This completes the proof of theorem 2.

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