Another Look at Downslope Winds. Part II: Nonlinear Amplification
beneath Wave-Overturing Layers

DALE R. DURRAN*
Department of Meteorology, University of Utah, Salt Lake City, UT 84112

JOSEPH B. KLEMP
National Center for Atmospheric Research,** Boulder, CO 80307
(Manuscript received 3 February 1987, in final form 1 June 1987)

ABSTRACT

Numerical mountain wave simulations have documented that intense lee-slope winds frequently arise when wave-overturning occurs above the mountain. Explanations for this amplification process have been proposed by Clark and Peltier in terms of a resonance produced by linear-wave reflections from a self-induced critical layer, and by Smith in terms of solutions to Long's equation for flow beneath a stagnant well-mixed layer. In this paper, we evaluate the predictions of these theories through numerical mountain-wave simulations in which the level of wave-overturning is fixed by a critical layer in the mean flow. The response of the simulated flow to changes in the critical-layer height and the mountain height is in good agreement with Smith's theory. A comparison of Smith's solution with shallow-water theory suggests that the strong lee-slope winds associated with wave-overturning are caused by a continuously stratified analog to the transition from subcritical to supercritical flow in conventional hydraulic theory.

1. Introduction

The idea that there is a fundamental similarity between downslope winds and hydraulic jumps was proposed by Long (1953a) over thirty years ago. However, it has been difficult to confirm this hypothesis because there are potentially significant differences between the atmosphere and the simple fluid systems described by hydraulic theory. One difference, which has been a major source of uncertainty, is that gravity waves can transport energy vertically in the atmosphere but not in fluids bounded by a rigid lid or a free surface. In Part I, Durrant (1986) presented evidence from numerical simulations suggesting that vertical energy propagation plays only a minor role in most downslope windstorms, and that there are striking similarities between downslope winds and supercritical flows.

The analogy between hydraulic fluids and the atmosphere is most clear in those cases where there is a pre-existing layering in the atmospheric stability upstream of the obstacle; these are the cases which were examined most extensively in Part I. However, strong downslope winds also develop in an atmosphere with constant mean windspeed and stability (i.e., without a pre-existing layered structure) whenever the wave amplitude is sufficient to produce overturned streamlines (breaking). Since evidence from observations and numerical models suggests that wave-breaking plays an important role in many downslope windstorms, it is natural to ask whether the wave-breaking amplification mechanism is fundamentally different from that described by hydraulic theory. That question is the focus of this paper.

Wave-breaking has been studied extensively by Clark and Peltier (Clark and Peltier 1977, 1984; Peltier and Clark 1979, 1983; Clark and Farley, 1984), who were the first to discover its importance. They have noted that when wave-overturning occurs, the overturned layer contains a region of locally reversed flow in which the Richardson number (Ri) is less than ¼. They proposed that this “wave-induced critical layer” acts as a boundary which reflects upward propagating waves back toward the mountain. Using linear theory to analyze these reflections, they found that if the depth of the cavity between the critical layer and the mountain is suitably tuned, the reflections should produce a resonant wave which amplifies linearly with time and ultimately produces very strong winds.

An alternative description of the wave-breaking amplification process has recently been proposed by Smith (1985). Like Clark and Peltier, Smith has assumed that the breaking region traps the wave energy within the underlying flow. His treatment of the upper boundary

* Present affiliation: Dept. of Atmospheric Sciences, University of Washington, Seattle, WA 98195.
** The National Center for Atmospheric Research is sponsored by the National Science Foundation.

© 1987 American Meteorological Society
depends on two principal assumptions: 1) that the turbulent flow in the breaking region is well mixed, forming a stagnant overturned layer that has constant potential temperature throughout, and 2) that pressure perturbations in the flow above the breaking level are negligible. Note that the presence of a critical layer is implicitly included in the second assumption since without the critical layer disturbances will propagate a significant distance into the well-mixed region.

Thus the theories of Peltier and Clark, and of Smith are both based on the same physical picture, i.e., that the wave-overturning region behaves like a localized critical layer in which the Richardson number is small. However, there are significant differences in the mathematical formulation of each theory and in the resulting predictions. Peltier and Clark (1983) solved the time-dependent linear wave equations subject to a linearized free-surface condition, whereas Smith solved Long’s equation for the nonlinear steady flow beneath a deformable upper boundary along which the windspeed is constant. Peltier and Clark’s results imply that a low-Richardson-number critical layer will produce amplification only when it is positioned approximately 1/4 + n/2, n = 0, 1, · · · vertical wavelengths above the topography. On the other hand, Smith’s results suggest that amplification is possible over the entire range of critical layer heights between (1/4 + n) and (3/4 + n) vertical wavelengths.

We begin by describing the numerical model in section 2. Section 3 describes a series of numerical experiments in which the height of the overturning layer is fixed by a critical layer in the mean flow, and the heights of both this layer and the mountain are varied. The results of these experiments are compared with the theories of Clark and Peltier, and Smith in section 4. The close connection between Smith’s solutions and hydraulic theory, for the conventional shallow-water system, is explored in section 5. Section 6 contains the conclusions.

2. The numerical model

The following numerical experiments were conducted with the nonlinear mountain-wave model developed by Durran and Klemp (1983). The model is nonhydrostatic, two-dimensional, and the Coriolis force is neglected. We have eliminated the decrease in mean density with height from the model equations, thereby removing the monotonic increase in wave amplitude with height and allowing the results to be interpreted without unnecessary complications. With this modification, the model equations become:

\[
\begin{align*}
\frac{Du}{Dt} + c_s \theta_0 \frac{\partial \pi}{\partial x} &= D_u, \\
\frac{Dw}{Dt} + c_s \theta_0 \frac{\partial \pi}{\partial z} &= g \frac{\theta - \bar{\theta}}{\theta_0} + D_w,
\end{align*}
\]

(1) (2)

The presence of a critical layer is implicitly included in the second assumption since without the critical layer disturbances will propagate a significant distance into the well-mixed region. Thus the theories of Peltier and Clark, and of Smith are both based on the same physical picture, i.e., that the wave-overturning region behaves like a localized critical layer in which the Richardson number is small. However, there are significant differences in the mathematical formulation of each theory and in the resulting predictions. Peltier and Clark (1983) solved the time-dependent linear wave equations subject to a linearized free-surface condition, whereas Smith solved Long’s equation for the nonlinear steady flow beneath a deformable upper boundary along which the windspeed is constant. Peltier and Clark’s results imply that a low-Richardson-number critical layer will produce amplification only when it is positioned approximately 1/4 + n/2, n = 0, 1, · · · vertical wavelengths above the topography. On the other hand, Smith’s results suggest that amplification is possible over the entire range of critical layer heights between (1/4 + n) and (3/4 + n) vertical wavelengths.

We begin by describing the numerical model in section 2. Section 3 describes a series of numerical experiments in which the height of the overturning layer is fixed by a critical layer in the mean flow, and the heights of both this layer and the mountain are varied. The results of these experiments are compared with the theories of Clark and Peltier, and Smith in section 4. The close connection between Smith’s solutions and hydraulic theory, for the conventional shallow-water system, is explored in section 5. Section 6 contains the conclusions.

\[
\begin{align*}
\frac{\partial \pi}{\partial t} + \frac{R}{c_s} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) - \frac{R \pi_0}{c_s \theta_0} \frac{\partial \theta}{\partial t} &= 0, \\
\frac{D \theta}{D t} &= D_\theta,
\end{align*}
\]

(3) (4)

where

\[
\begin{align*}
D &= \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z} \\
\bar{\pi} + \pi &= \left( \frac{p}{\rho_0} \right)^{R/c_p} \\
= & - \frac{g}{c_s \theta_0}.
\end{align*}
\]

In the above, p is pressure, \rho_0 = 1000 mb, R is the gas constant for dry air, c_p is the specific heat of dry air at constant pressure, c_s is the specific heat at constant volume, \theta is the potential temperature, and u and w are the horizontal and vertical velocity components. Overbars denote the undisturbed, horizontally homogeneous, mean state; and \theta_0 and \pi_0 are representative constant values. The D_\theta, D_u, and D_w terms contain the contributions from subgrid-scale mixing, which is parameterized as a function of the stratification and shear; the mathematical formulae for these terms are given in Durran and Klemp, (1983). A free-slip boundary condition is imposed at the surface and the surface topography is included through a coordinate transformation.

At steady state (3) is identical to the incompressible continuity equation, and the system (1)–(4) is formally equivalent to the conventional Boussinesq system. Equation (3) is derived by taking the local time derivative of the ideal gas law written in terms of the Exner function \pi:

\[
\bar{\pi} + \pi = \left( \frac{R}{\rho_0 \theta} \right)^{R/c_p},
\]

(5)

replacing \partial \rho/\partial t from the compressible continuity equation, and approximating the density and the undifferentiated Exner function and potential temperature by their constant mean values. The use of this “compressible Boussinesq” continuity equation results in a numerical scheme which is similar to the artificial compressibility technique of Chorin (1970). In fact, these simulations are not sensitive to the inclusion of the \partial \theta/\partial t term in (3).

The retention of a prognostic pressure equation allows the system to be integrated in a computationally efficient manner by treating the sound wave modes separately on a shorter time step. The bulk of the computation is done less frequently on a larger time step. In the large time step, the time differencing is leapfrog, horizontal advection is fourth-order and vertical advection is second-order. Buoyancy, diffusion and coordinate transformation terms are computed to at least second-order accuracy. Wave permeable lateral and upper boundaries are specified according to the numerical prescriptions of Orlanski (1976) and Klemp and Durran (1983). (A complete description of the nu-
merical model and a thorough discussion of its testing and verification can be found in Durran and Klemp, 1983.)

3. Downslope windstorms in the presence of a critical layer

Although wave-breaking produces only a local critical layer of rather limited horizontal extent, most previous investigators have assumed that the influence of local critical layers is similar to that produced when the horizontal extent of the critical layer is much larger than a horizontal wavelength. Therefore, we will examine the nominally simpler situation in which there is a critical layer in the mean flow crossing a mountain. This approach allows the sensitivity of the disturbance to the height of the wave-overturning region to be systematically investigated by changing the position of the critical layer in the mean flow.

In the following numerical simulations the Brunt-Väisälä frequency $N$ is 0.01047 s$^{-1}$; the mean cross mountain wind $U$ is 20 m s$^{-1}$ between the ground and a 2 km deep shear layer. The mean wind decreases linearly from 20 m s$^{-1}$ to 0 across the shear layer, and remains zero above the critical level. It follows that the mean Richardson number in the shear layer is 1.1; the mean vertical wavelength $L_z$ is 12 km below the shear layer, and the thickness of the shear layer is $L_z/6$. Here $L_z = 2\pi/l$, where $l = N/U$ is the vertical wavenumber below the shear layer. The mountain is specified by a “Witch of Agnesi” curve:

$$h(x) = \frac{h_m a^2}{x^2 + a^2},$$

with a half-width $a = 10$ km.

The mountain is centered in a computational domain which is 120 km wide. The horizontal grid interval is 1.5 km; the vertical interval is 333 m. The top of the domain was located at least 4 km above the critical level. The solutions were not sensitive to the location of the upper boundary since disturbances did not propagate significantly above the critical layer. The solutions showed only a slight sensitivity to the size of the horizontal and vertical grid intervals. The large and small time steps were 12.5 and 4.17 s. Each simulation was continued until the solution in the vicinity of the mountain reached a nearly steady state. The time required for the flow to become nearly steady varied from about 2.5 hours when the critical layer was at a height of 5 km, to about 14 hours when the critical layer was at 19 km.

Flows that produce strong lee-slope winds also exert a strong pressure drag on the mountain. The steady-state pressure drag exerted by the disturbance on the mountain was calculated in each simulation and entered in Table 1 as a function of the mountain height and the height of the critical layer. The pressure drag has been normalized by the drag which would be associated with linear hydrostatic waves forced by an identical mountain when the mean windspeed and stability are constant and equal to the values found below the shear layer. This normalization indicates the extent to which nonlinear effects and complexities in the atmospheric structure modify the strength of the disturbance.

The entries in Table 1 divide sharply into high-drag and low-drag regimes. Note that it is possible to produce large drafts (and strong downslope winds) over a broad range of critical layer heights and, that when the elevation of the critical layer is less than $\frac{1}{4}$ of a vertical wavelength, the mountain height required to produce a strong response increases as the height of the critical layer increases. There is a one-to-one correspondence between solutions which produce a strong drag and solutions which exhibit strong lee-slope winds.

<table>
<thead>
<tr>
<th>Critical level height (km)</th>
<th>Mountain height ($N h_m / U$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4</td>
</tr>
<tr>
<td>0.5</td>
<td>0.7</td>
</tr>
<tr>
<td>0.9</td>
<td>2.0</td>
</tr>
</tbody>
</table>

The transition between high and low drag regimes is illustrated in Figs. 1 and 2. Figure 1 shows the isentropes and Fig. 2 shows the horizontal velocity field in simulations for three different mountain heights where the initial upstream profiles of $N$ and $U$ are held fixed with the critical layer at 7 km (i.e., shear layer located between 5 and 7 km). When $h_m = 0.3$ (Figs. 1a, 2a), the windspeed maximum is weak and centered near the top of the mountain; the isentropes show a weak lee wave train and a pronounced dip over the mountain which resembles the free surface displacement in a subsurface flow. However, when $h_m = 0.4$ (Figs. 1b, 2b), the flow above the lee slope is distinctly different. The horizontal windspeed increases, and the isentropes continue to descend, until the flow reaches the base of the mountain. The flow then recovers farther downstream in an abrupt transition that bears strong resemblance to a hydraulic jump. A similar jump develops when $h_m = 0.5$ (Figs. 1c, 2c) except that the jump (where the high speed flow recovers to the ambient conditions) propagates downstream more rapidly so that it has passed through the downstream boundary at the time shown in Figs. 1 and 2. Although
the increase in nonlinearity from $l_h = 0.4$ to $l_h = 0.5$ has increased the propagation speed of the jump, the normalized drag has decreased slightly, as shown in Table 1.

Figures 1 and 2 display the numerically simulated disturbances that produce the first three drag entries listed on the second row in Table 1. The wind-speed and isentrope fields associated with all the other low-drag entries in Table 1 are qualitatively similar to those shown in Figs. 1a and 2a; the disturbance below the critical layer resembles a linear gravity wave. The other high-drag entries in Table 1 are associated with disturbances which resemble those shown in Figs. 1b, 1c and 2b, 2c; the flow exhibits a transition to a shallow jet structure as it crosses the mountain peak.

When the height of the critical level is less than a vertical wavelength $L_z$, the stratified flow beneath the critical level exhibits a pronounced visual similarity to the hydraulic systems described by shallow-water theory. However as the height of the critical level is increased above one vertical wavelength, internal perturbations appear in the flow which do not occur in the shallow-water system. These internal perturbations are illustrated by the isentropes (Fig. 3) and horizontal windspeed (Fig. 4) which develop when the shear layer beneath the critical level is centered at a height of $4/3 L_z$ (16 km). The drag for these cases appears in row eight of Table 1. When $l_h = 0.1$ (Figs. 3a, 4a) the normalized drag is small, and the disturbance consists of a vertically propagating internal gravity wave which is absorbed at the critical level. The perturbations in Figs. 3a and 4a have been multiplied by a factor of 5 so that they can be more readily compared with the case in Figs. 3b and 4b where $l_h = 0.5$. In this second case, the flow in the lee of the mountain consists of alternating horizontal layers in which the fluid is either accelerated or decelerated. The layer nearest the mountain is accelerated. The downstream edge of each layer continually propagates away from the mountain.

4. Comparison with the theories of Clark and Peltier, and Smith

According to the linear-resonance amplification mechanism proposed by Peltier and Clark (1983), one would expect that high drag states should develop only when the height of the critical layer was $\frac{1}{4} L_z, \frac{3}{4} L_z, \frac{5}{4} L_z, \ldots$ vertical wavelengths above the ground. There is some ambiguity in defining these levels because the vertical wavelength $2\pi U(z)/N(z)$ approaches zero as $z$ approaches the critical level. In the case of a linear wind shear, there are an infinite number of vertical wavelengths between the ground and the critical layer. In their analysis, Peltier and Clark base the vertical wavelength calculation on the constant values of $N$ and $U$

---

Fig. 1. Isentropes for a flow with a critical layer at 7 km (shear layer centered at 6 km) when $l_h$ is (a) 0.3, (b) 0.4, and (c) 0.5. In this and all subsequent figures, the airflow is from left to right.
Fig. 3. Isentropes for a flow with a critical layer at 17 km (shear layer centered at 16 km) when $h/m$ is (a) 0.1 and (b) 0.5. The perturbations in (a) have been multiplied by five so that the wave amplitude is scaled for easy comparison with (b).

$U$ which are found below the critical layer. Following their approach, one would expect that the large amplitude responses should be confined to just two rows in Table 1, row 3 (critical level at $0.75L_d$) and row 7 (critical level at $1.25L_d$). In fact, the maximum normalized drags which occur in these rows are not at all exceptional.

Although, the resonant amplification mechanism of Peltier and Clark does not explicitly depend on wave amplitude, it does depend on the development of some region in the flow that reflects upward propagating waves back toward the surface. When $N$ and $U$ are constant with height, the reflecting region does not appear until the waves reach overturning amplitude. As a consequence, the normalized drag is very sensitive

Fig. 2. As in Fig. 1 except for horizontal windspeed.
to the mountain height when that height is near the threshold required to force breaking waves. On the other hand, once the mountain becomes large enough to force breaking waves, the normalized drag is relatively insensitive to further changes in the mountain height (see Fig. 14 in Durrant, 1986). In the present simulations, reflections are produced when the waves encounter the pre-existing critical layer. As demonstrated by the high drag entries in the first line of Table 1, energy can be trapped in the cavity between the critical layer and the ground when \( h_m \geq 0.2 \), suggesting that the critical layer begins to act as a reflector when the mountain height is rather small. Therefore, one might expect the nondimensional drag in our numerical simulations to be relatively insensitive to the mountain height once \( h_m \) exceeds a threshold value around 0.2. Our results, however, remain highly sensitive to \( h_m \). In two simulations in which \( h_m = 0.5 \) and 0.7, and the critical level is at a height of \( 0.75L_z \) (9 km), the Richardson number rapidly drops below \( \frac{1}{4} \) in the portion of the critical layer directly over the mountain, and the horizontal winds are locally reversed; yet a strong response develops only in the \( h_m = 0.7 \) case. This dependence on mountain height appears to be inconsistent with the resonant reflection mechanism of Peltier and Clark.

Clark and Peltier (1984, hereafter denoted CP84) have conducted numerical experiments in which the height of a critical layer in the mean flow impinging on a mountain was varied, while the mountain height was held fixed. They concluded that the results of those experiments support their resonance hypothesis. This conclusion appears to be at variance with our current findings; however, it does appear possible to reconcile the numerical results presented in CP84 with those presented in this paper. No experiments are presented in CP84 in which the height of the critical layer is below \( 0.75L_z \); however, it is precisely at the levels below \( 0.75L_z \) where we find the greatest conflict between the resonance hypothesis and the results in Table 1. When the height of the critical layer is between \( 0.75L_z \) and \( 1.25L_z \), their results are entirely consistent with those in Table 1. When the critical level is above \( 1.25L_z \), there is only one simulation (critical level at \( 1.6L_z \)) where CP84 exhibit results different from those in the current study. The differences in this case can perhaps be attributed to differences in the mean states and the lengths of each simulation. (In CP84 the shear layer below the critical level is much wider, density decreases with height, and the nondimensional termination time, \( Ut/\alpha \), is less than half that used in the current simulations).

Another analysis of wave breaking has been proposed by Smith (1986), who solved the hydrostatic Long's equation (Long, 1953b) for flow beneath the breaking layer. He derived an upper-boundary condition for these solutions by assuming that the disturbance is in hydrostatic balance, that turbulence produces a well-mixed stagnant region within the breaking layer, that no disturbances are transmitted through this layer, and that it is possible to identify a dividing streamline which separates the well-mixed region from the laminar flow below. Under these assumptions, the horizontal velocity is constant along the dividing streamline (Smith, 1986).

Smith's solution for the displacement of the dividing streamline from its undisturbed position \( \delta \) is plotted as a function of local topography height \( h \) in the family of curves shown in Fig. 5. The different curves show how the system responds to changes in the undisturbed height of the dividing streamline. Here, we have plotted only those heights between \( (1/4 + n)L_z \) and \( (3/4 + n)L_z \), where the most interesting behavior is possible. As the flow encounters a mountain, the dividing streamline is displaced downward. After passing the crest of a small mountain, it begins to rise and returns to its initial elevation. If the mountain peak is sufficiently high, the
Fig. 5. Smith’s solution for deflection of the dividing streamline \( \delta \) as a function of terrain height \( lh \). Each curve is for a different upstream height of the dividing streamline \( H_0 \). Curves in the fourth quadrant describe flows that can transition to “supercritical” over positive terrain. (After Smith, 1985.)

Streamline displacement reaches the turning point where \( \partial h / \partial \delta = 0 \), and the flow can transition to a new regime in which the dividing streamline continues to descend along the lee slope (a situation reminiscent of the transition from subcritical to supercritical flow). As the initial height of the dividing streamline increases, the height of the mountain required to produce this transition also increases. The theory yields no predictions for the case in which the mountain height is greater than that required to produce a transition. Pierrehumbert (personal communication) has suggested that in those cases, the solution technique fails because the upstream conditions adjust to a state in which the stability and windspeed are not constant with height between the surface and the dividing streamline; under these circumstances, Long’s equation no longer governs the flow.

The undisturbed height of the dividing streamline, produced by a breaking wave, cannot be determined a priori. However in the present application, the undisturbed height of the dividing streamline can be related to the elevation of the critical layer in the upstream flow. This relationship is illustrated in Fig. 6, which shows the streamline originating at the 5.5 km level upstream and the 18 m s\(^{-1}\) isotach in two cases where the height of the critical level is 7 km (shear layer between 5 and 7 km) and \( lh_m \) is either 0.3 or 0.5. In both the weak (Fig. 6a) and strong (Fig. 6b) drag cases, the 5.5 km streamline approximates the idealized “dividing streamline” along which the windspeed is constant.

If the undisturbed height of the dividing streamline \( H_0 \) is estimated to be at the middle of the shear layer, the relationship documented in Table 1 between the height of the critical layer and the value of \( lh_m \) required to produce a transition from the low drag to the high drag regime is in good agreement with Smith’s predictions, as summarized in Fig. 5. In further accordance with Smith’s theory, only low-drag disturbances developed in those numerical simulations where the shear layer was centered between 0.75Lz and 1.25Lz; however, these low-drag results do not necessarily generalize to more nonlinear situations. Solutions to Long’s equation for an atmosphere with constant mean wind and stability (no critical layer) indicate that wave-breaking will occur at the 0.75Lz level when \( lh_m \) exceeds 0.85. Thus if the mountain is sufficiently high and the elevation of the mean-flow critical layer exceeds 0.75Lz,
wave-breaking may still develop beneath the mean- 
flow critical layer, producing a strong response. 

The similarities between our numerical results and 
Smith's solutions are also indicated in Fig. 7. The 
results in Table 1 were characterized as high drag (nor-
malized drag greater than 2) or low drag (normalized 
drag less than 2) and plotted with an $H$ or an $L$ in Fig. 
7 as a function of critical-level height $H_0$ and mountain 
height $h_m$. The solid curve in Fig. 7 is the critical curve 
along which Smith's solutions undergo a transition to 
a shallow high velocity flow at the crest of the 
mountain. According to Smith, all points to the right of 
the critical curve in the range $(1/2 + 2n)\pi < H_0 < (3/2 + 2n)\pi, n = 0, 1, \ldots$ should be associated with high 
drag; all others should be low drag. The agreement be-
tween the numerical results and Smith's theory is rather 
good, especially when one considers that the dividing 
streamline used in Smith's solutions is an idealization of 
the conditions which are actually present beneath the 
critical layer in the numerical simulations.

5. Connection between Smith's solutions and conven-
tional hydraulic theory

As mentioned previously, the transition in Smith's 
solution that allows the dividing streamline to continue 
to descend on the lee side of the mountain has quali-
tative similarities with the transition to supercritical 
flow that occurs in hydraulic theory. In this section, 
we will more rigorously examine the relationship be-
tween Smith's solutions and analogous solutions to the 
steady state shallow-water equations.

We consider an inviscid shallow-water system in 
which a fluid layer, having an initial depth $H_0$ and 
velocity $U$, flows over terrain defined by the profile $h(x)$. 
At steady state the shallow-water equations are given by:

$$
\frac{\partial u}{\partial x} + g \frac{\partial \delta_c}{\partial x} = 0, 
\frac{\partial \phi}{\partial x} + \frac{\partial u}{\partial x} = 0, 
$$

where $\delta_c$ is the displacement of the free-surface from 
its position upstream and $\phi = H_0 + \delta_c - h$ is the total 
thickness of the fluid layer. For a layer capped by a free 
surface, $g'$ is the gravitational acceleration $g$, while in 
atmospheric applications, $g' = (\Delta\theta/\theta_0)g$ is a reduced 
gravity. Here $\theta_0$ is the potential temperature of the fluid 
layer, $\Delta\theta$ is the jump in potential temperature at the 
top of this layer, and $\theta_1 = \theta_0 + \Delta\theta$. Equations (7) 
and (8) may be combined to yield the following relationship 
between the slope of the upper boundary and the slope of 
the terrain (Long, 1954):

$$
(1 - F^{-2}) \frac{\partial \delta_c}{\partial x} = \frac{\partial h}{\partial x}, 
$$

where the Froude number $F$ is defined as

$$
F^{-2} = \frac{g' \phi}{u^2}. 
$$

As discussed in section 4, Smith solved the hydro-
static Long's equation for the flow of a continuously 
stratified fluid beneath a hypothetical "dividing 
streamline" which passes just beneath the wave-over-
turning region. These solutions apply to the case where 
the Brunt-Väisälä frequency $N$ and the windspeed $U$ 
are constant with height far upstream. Smith's expres-
sion for the displacement of the dividing streamline $\delta_c$ 
from its upstream position may be written in the form:

$$
\delta_c = h(\cos\phi)^{-1}, 
$$

where $l = N/U$. A relationship comparable to (9) is 
obtained by differentiating (11) with respect to $x$ which 
yields:

$$
\left\{ \frac{\cos\phi - l_0 \sin\phi}{1 - l_0 \sin\phi} \right\} \frac{\partial \delta_c}{\partial x} = \frac{\partial h}{\partial x}. 
$$

Equation (12) has a form that is identical to (9) if a 
Froude number $F$ for the continuously stratified case 
is defined such that

$$
F^{-2} = \frac{1 - \cos\phi}{1 - l_0 \sin\phi}. 
$$

Smith's solution implies that the horizontal windspeed 
at the level of the terrain, $u_t = u(x, h)$ is equal to
Thus (13) may be written in the alternate form:
\[
\mathcal{F}^{-2} = \frac{1}{2} \frac{N^2 \phi^2}{U \mu} \left( \frac{\sin(1/2 \phi)}{1/2 \phi} \right)^2.
\]
(14)

A comparison of the shallow-water relationship (9) with the continuously stratified result (12)–(13) shows that at any point where \( \mathcal{F} = \mathcal{F} \), the slope of each fluid's upper boundary will have an identical dependence on the slope of the underlying terrain. In hydraulic theory, if \( \mathcal{F} < 1 \), the flow is subcritical and the interface descends where the terrain rises; if \( \mathcal{F} > 1 \), the flow is supercritical and the interface ascends. Following hydraulic theory, we refer to those regimes in continuously stratified flows in which \( \mathcal{F} < 1 \) as subcritical and those in which \( \mathcal{F} > 1 \) as supercritical. The upstream Froude number may be evaluated from (14):
\[
\mathcal{F}_0 = \left[ \sqrt{2} \sin(1/2 H_0) \right]^{-1},
\]
(15)

implying that the continuously stratified flow far upstream will be subcritical when \((1/2 + 2n)\pi < H_0 < (3/2 + 2n)\pi, n = 0, 1, \cdots \). This is exactly the same interval over which rising topography can produce transitions to faster flow in Smith’s solutions (see Fig. 5).

As in conventional hydraulic theory, any transition between subcritical and supercritical flow in the steady-state continuously stratified system must occur at the top of the mountain (\( \mathcal{F} = 1 \) implies \( \partial h/\partial x = 0 \)). Following the development in Long (1954), we can also demonstrate that if \( \mathcal{F} = 1 \) at the top of the mountain, there must be a transition from subcritical to supercritical flow. Differentiating (12) with respect to \( x \) and evaluating the result at the point where \( \mathcal{F} = 1 \), yields:
\[
\frac{\partial^2 h}{\partial x^2} = -\frac{2 + I^2 \delta_e^2}{1 - \cos \delta_e \delta_e^2} \frac{h}{\partial \delta_e^2} \left( \frac{\partial \delta_e}{\partial x} \right)^2.
\]
(16)

For positive mountain heights, it follows that, since \( \partial^2 h/\partial x^2 < 0 \) at the mountain crest, \( \delta_e/\delta x \) cannot vanish there. Therefore, if the interface is descending as it approaches a critical state at the mountain crest, it must continue to descend over the lee slope.

In the steady, small-amplitude limit there is an exact correspondence between the shallow-water and the continuously stratified systems. The linearized systems are governed by (9) and by (12)–(13) except that \( F \) and \( \mathcal{F} \) are replaced by the appropriate mean state Froude numbers. In the shallow-water system the mean Froude number is \( F_0 = U \sqrt{g/H_0} \); in the stratified system it is \( \mathcal{F}_0 \) given by (15). Thus, the two systems are equivalent when \( F_0 = \mathcal{F}_0 \). If the depth and the mean windspeed in the two systems are the same, \( F_0 = \mathcal{F}_0 \) implies:
\[
g' = \frac{g (\theta_1 - \theta_0)}{2 \theta_1} \left( \frac{\sin(1/2 H_0)}{1/2 H_0} \right)^2.
\]
(17)

In deriving (17), we have replaced \( N^2 \) by \( g/\theta_1 (\theta_1 - \theta_0)/H_0 \), where \( \theta_1 \) is the potential temperature at the surface in the stratified flow. If the phase shift within the continuously stratified layer is small (\( H_0 \ll 1 \)), the behavior of the stratified flow is identical to that of an unstratified fluid capped by an inversion having a potential temperature increase that is just one-half the total increase across the depth of the stratified fluid: \( \Delta \theta = (\theta_1 - \theta_0)/2 \). If the phase shift within the stratified layer is significant, the effective \( \Delta \theta \) across the inversion in the equivalent shallow water system is reduced by the last factor in (17).

The analogy between \( \mathcal{F} \) and \( F \) extends to the time-dependent equations, in that \( \mathcal{F} = 1 \) in the stratified system, and \( F = 1 \) in the shallow-water system, both imply that the phase speed of the most rapidly propagating linear gravity wave is equal to the speed of the mean flow (i.e., that the flow is exactly critical). In the linear shallow-water system, disturbances propagate at speeds \( U \pm \sqrt{g H_0} \) and the flow becomes critical at \( F_0 = 1 \), when the upstream propagating mode is held just stationary. In the continuously stratified system, linear hydrostatic disturbances propagate at speeds \( U \pm \sqrt{g} k_z \), where \( k_z \) is the vertical wavenumber. Modes propagating upstream of the mountain, over flat ground, must satisfy \( w = 0 \) at the surface and \( \partial w/\partial z = -\partial u/\partial x = 0 \) along the upper boundary, which requires that \( k_z H_0 = (1/2 + n)\pi, n = 0, 1, \cdots \). Thus, the fastest mode propagates upstream at a speed \( U - 2\pi \cdot \sqrt{g} k_z \) and becomes just stationary when \( H_0 = \pi/2 \), which from (15) is equivalent to the condition \( \mathcal{F}_0 = 1 \).

In the nonlinear case, the equivalence between the shallow-water and stratified systems is no longer exact; however, strong quantitative similarities are still evident. The dashed curve in Fig. 8 is the locus of upstream
Froude numbers \( F_0 \) and normalized terrain heights \( h_0 / H_0 \) at which the flow becomes just critical at the crest of the obstacle in the shallow water system (cf. Long, 1954; Houghton and Kasahara, 1968). If a point lies to the left of the curve, the flow remains everywhere subcritical \( (F_0 < 1) \) or everywhere supercritical \( (F_0 > 1) \) as it traverses the obstacle. If a point lies to the right of the curve, the upstream conditions must adjust (by means of a traveling wave when \( F_0 < 1 \) or an upstream-propagating hydraulic jump when \( F_0 > 1 \)) so that, at steady state, \( F = 1 \) at the crest. When a point lies to the right of the critical curve and \( F_0 < 1 \), the horizontal velocities along the lee slope of the terrain will be significantly stronger than those in the undisturbed flow upstream; this is the regime that is associated with downslope windstorms.

The solid curve in Fig. 8 is the locus of upstream internal Froude numbers \( \mathcal{F}_0 \) and normalized terrain heights \( h_0 / H_0 \) for which the flow in the stratified system becomes just critical at the crest of the mountain. The agreement between the shallow-water critical curve and the critical curve for the stratified system is rather good over the range \( 0 < \lambda H_0 < \pi \). The agreement is particularly good in the vicinity of \( F_0 = \mathcal{F}_0 = 1 \), where the critical condition can be achieved by small disturbances. These results suggest that, when \( 0 < \lambda H_0 < \pi \), the development of “hydraulic jumps” in the stratified system can be predicted with reasonable accuracy by examining an analogous shallow water system in which \( F_0 = \mathcal{F}_0 \).

Significant differences between the two critical curves develop when \( \pi < \lambda H_0 < 3\pi / 2 \). This difference is due to the ways that \( F_0 \) and \( \mathcal{F}_0 \) respond to changes in the depth of the fluid. Equation (15) implies that when \( 0 < \lambda H_0 < \pi \), increases in \( H_0 \) decrease \( \mathcal{F}_0 \); this behavior is similar to that in the shallow-water system where increases in \( H_0 \) decrease \( F_0 \). However, when \( \pi < \lambda H_0 < 2\pi \), increases in \( H_0 \) increase \( \mathcal{F}_0 \) but continue to decrease \( F_0 \). As a result, \( \mathcal{F}_0 \) has a lower limit of \( 1/\sqrt{2} \). Nevertheless, for \( \pi / 2 < \lambda H_0 < 3\pi / 2 \) the upstream flow is subcritical and if a point lies to the right of this branch of the critical curve, the analytic and numerical solutions show that the stratified system will undergo a transition to supercritical flow which is qualitatively analogous to that in the shallow water system.

When \( \lambda H_0 > 3\pi / 2 \) further differences develop between the stratified and shallow water systems. In particular, the stratified flow never reaches critical conditions when \( 3\pi / 2 < \lambda H_0 < 2\pi \). The behavior of the stratified system is also periodic when the upstream elevation of the dividing streamline exceeds one vertical wavelength.

6. Discussion

The importance of wave-breaking as a nonlinear amplification mechanism was first discovered by Clark and Peltier (1977), who suggested that the amplification could be explained by the continuous growth of linear waves in a resonant cavity between the ground and a local critical layer in which \( \mathcal{R}i < 1/6 \) (Peltier and Clark, 1983). More recently, Smith (1985) explained the amplification associated with wave breaking through solutions to Long’s equation for a system in which the horizontal windspeed is constant along the upper boundary of the fluid, beneath the overturning layer. Smith’s theory may be considered as an extension of Peltier and Clark’s analysis to nonlinear steady-state problems. It should be emphasized that both of these theories presuppose the existence of a wave-overturning layer, and that predicting the development and height of wave-overturning layers remains a complex problem. In the general case, where the windspeed and stability functions of height, the development of breaking waves can be regulated by subtle features such as inversions in the upstream flow (Durrant, 1986), and consequently, the onset and position of breaking waves cannot be predicted without recourse to numerical simulation.

In this paper, we have presented numerical simulations in which the position of the wave-overturning region was fixed by a critical layer in the mean flow. The response of the flow beneath the overturned layer to changes in the height of the mean critical layer and the height of the mountain was in good agreement with the nonlinear analytic solutions of Smith. Our results did not appear to be consistent with the linear-amplification mechanism proposed by Peltier and Clark.

We have examined the similarity between Smith’s theory and conventional hydraulic theory in some detail. We demonstrated that an internal Froude number \( \mathcal{F} \) may be defined for Smith’s stratified system, which is analogous to the conventional Froude number \( F \) in the shallow-water system. In the small-amplitude limit there is an exact equivalence between the stratified system and a shallow-water system in which \( F = \mathcal{F} \). When the depth of the fluid is less than one-half vertical wavelength the nonlinear stratified system continues to exhibit a remarkable quantitative similarity to a nonlinear shallow-water system in which \( F = \mathcal{F} \). When the fluid is more than one-half-wavelength deep, internal perturbations in the stratified fluid produce significant quantitative differences in the behavior of the two nonlinear systems, but a qualitative similarity remains. These results suggest that the mechanism which amplifies the pressure drag and the downslope wind in a breaking mountain wave is fundamentally similar to that produced through the transition from subcritical to supercritical flow in shallow-water hydraulic theory.

Acknowledgments. Doug Lilly provided comments which improved the manuscript. DRD’s participation in this research was supported by NSF Grant ATM-8320695 and ATM-8609345. Computing facilities were provided by the Scientific Computing Division of the National Center for Atmospheric Research.
REFERENCES


