Radiative Transfer through Arbitrarily Shaped Optical Media. Part II: Group Theory and Simple Closures

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ABSTRACT

This paper presents a formulation of the radiative transfer equation which allows for the distinction between various groups of spatial scales of variation that comprise the radiance field. Such a formulation provides a convenient means for studying the effects of spatial inhomogeneity and scale interaction on the radiative transfer. Notions of scale hierarchy and closure are introduced into the radiative transfer equation, and it is demonstrated how the customary treatment of partial cloudiness based on cloud amount as a weighting parameter is a special form of closure. Discussion of this particular closure and other assumptions relevant to this partial cloud treatment are presented. Another simple example of closure is described which allows for the treatment of spatial inhomogeneities as a new form of optical property. This concept is introduced into a two-stream model to demonstrate, in a gross way, the effects of inhomogeneities on radiative transfer. Comparisons with the more formal calculations of Part I are presented.

1. Introduction

All problems that require a realistic account of the effects of radiation in a cloudy atmosphere must be concerned with the spatial (and temporal) variability of clouds and the impact of this variability on the radiative processes of interest. Many examples readily come to mind. The remote sensing studies of clouds and precipitation using instruments with fields of view that are only partially filled by cloud (such as the so-called "beam filling" problem of the SMMR instrument, Thiele 1987). Other examples can be found in those studies that require the radiation budget of some volume of atmosphere from the subcloud scale (e.g., Curry et al. 1986) to the synoptic scale (Ramanathan et al. 1983).

This has been a concern for some time, and the usual approach to accommodate the spatial inhomogeneity of cloud is to weight the radiation field appropriate to the clear and cloudy portions of the sky in some manner. In this way cloud amount enters into the mathematical description of radiative transfer. The cloud amount parameter has now assumed major significance in world climate programs, a large body of recent research has focused on the definition of its climatology (e.g., Barton 1983; Schiffer and Rossow 1983; Sadler 1969; Warren et al. 1985; among other studies) and the significance of cloud amount to modeled climates (Stephens and Webster 1981; Wetherald and Manabe 1980; and Sommerville and Remer 1984; among others). Despite this large amount of ongoing research, the real relevance of cloud amount to the description of radiative transfer has never been elevated beyond the empirical level. Whether or not cloud amount is a parameter fundamental to the description of the radiative transfer processes that are of interest to such a wide range of problems has never clearly been established, and as demonstrated below, all of the assumptions have not been defined.

The object of this paper is to formulate the transfer problem presented by Stephens (1988, hereafter referred to as Part I) in such a way as to provide some insight into the way in which the scales of variability influence the radiative transfer through an atmosphere that contains a heterogeneous distribution of cloud. This formulation provides a convenient means for studying the effects of spatial structure and scale interaction on the radiative transfer through an inhomogeneous atmosphere and, in particular, for assessing the assumptions pertinent to the use of cloud amount as currently adopted in the atmospheric science research literature.

The outline of this paper is as follows. The transfer equation is presented in operator form in the following section and a scaling analysis is carried out in section 3 which is based on the Fourier analysis described in Part I. New concepts in radiative transfer are introduced in section 4 in the form of scale hierarchy and closure. Section 5 demonstrates, for the first time, how the existing use of cloud amount in radiative transfer is a special and very simple form of closure. Another simple example of closure, described in section 6, al-
lows for the treatment of spatial inhomogeneities as a new form of optical property. This concept is pursued further in section 7 in which this specific form of closure is introduced into a two-stream model to demonstrate, in a gross way, how inhomogeneities affect radiative transfer. The results of the paper are then summarized in section 8.

2. The equation of transfer

The integro-differential equation of radiative transfer for a sourceless medium can be expressed in the form

$$\frac{\partial N(r, \xi)}{\partial z} + \xi_n \cdot \nabla_h N(r, \xi) = -\alpha(r)N(r, \xi)$$

$$+ s(r) \int_{\Omega} p(r, \xi \cdot \xi') N(r, \xi')d\Omega(\xi')$$

(1)

where the dependencies of $N$, $\alpha$, $s$, and $p$ on wavelength are taken to be understood. The notation

$$\xi_n \cdot \nabla_h = \eta \cos \phi \frac{\partial}{\partial x} + \eta \sin \phi \frac{\partial}{\partial y}$$

(2)

is employed and all other symbols are defined in the Appendix of Part I. Introduction of the integro-differential operator (with obvious notational abbreviations)

$$\delta = \xi_n \cdot \nabla_h + \alpha - s \int_{\Omega} p(\xi \cdot \xi')[ ]d\Omega(\xi')$$

transforms (1) to the following equation

$$\left[ \mu \frac{\partial}{\partial z} + \delta \right] N(r, \xi) = 0$$

(3)

for the distribution function $N(r, \xi)$.

Equation (4) and the analyses that follow apply to a sourceless medium. This provides a somewhat uncluttered analyses and simplifies interpretation. Inclusion of source terms and application of these techniques to transfer problems influenced by thermal emission, for example, will be a subject of further study.

3. Group scaling: General analysis

The master equation (4) describes the transfer of radiation on a minute scale. Interest in many atmospheric problems often lies in the properties of the medium averaged over some specified domain (which might represent the field of view of a satellite instrument or the horizontal resolution of a circulation model, for instance). The integration of (4) over the various larger scales of interest, while taking into account the small scale structure of the medium, is beyond the present capabilities of modern computers; it is therefore desirable to perform some form of smoothing over this microscale structure. For example, consider the Reynolds decomposition of the radiance function (omit the $r$ and $\xi$ dependencies hereafter for notational convenience),

$$N = \tilde{N} + N'$$

(5)

into the ensemble average $\tilde{N}$ and a fluctuation $N'$. This fluctuation can be further decomposed

$$N' = N_1 + N''$$

(6)

into a macrogroup $N_1$ and a microgroup $N''$ which might be further decomposed into higher order groups according to

$$N'' = N_{2} + N_{3} + N_{4} + \cdots$$

(7)

These groups can be viewed as representing some overall macrostructure variabilities of the radiance field through $N_1$ and successively smaller scale variabilities through the remaining subgroups. Distinction between the various scales of fluctuations that comprise $N'$ provides for the possibility of some sort of scale demarcation that might arise from, say, other external factors that govern the structure to the optical medium and the radiation incident on its boundaries.

The radiance subgroups are formalized mathematically as sets of Fourier components. As in Part I, the usual transform pairs (in the absence of normalization factors) is

$$N_k = \int N(r, z, \xi) e^{-ik \cdot r} dr$$

$$N(r, z, \xi) = \int N_k e^{ik \cdot r} dk$$

(8)

where fluctuations of a specified spatial scale are characterized by the Fourier frequency $k$. The groups $\tilde{N}$, $N_1$, $N_2 \ldots$ can be identified with the Fourier sets

$$\tilde{N} = N_{k=0} = N_0$$

$$N_1 = \{ N(k), 0 < k \leq k_1 \} = N_1$$

$$N_2 = \{ N(k), k_1 < k \leq k_2 \} = N_2$$

(9)

where $k_1$, $k_2$, etc. define the particular scale demarcation of each group.

4. Hierarchy and closure

The individual Fourier components that comprise these sets are governed by the transformed radiative transfer equation which was derived in some detail in Part I. Equation (30) of Part I may be written in the form

$$\left[ \mu \frac{\partial}{\partial z} + \delta \right] \tilde{N} = 0$$

(10)

where $\tilde{N}$ represents all sets of Fourier components,

$$\tilde{N} = [N_0, N_1, N_2, \cdots]$$

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and the operator

$$\mathcal{H} = \bar{\alpha} + \bar{\beta} - \int \mathcal{H} \frac{\partial}{\partial z} \right] d\Omega$$

(11)

follows directly from the derivation of (30) of Part I where \(\bar{\alpha}, \bar{\beta}\) and \(\mathcal{H}\) are matrix operators. It follows from (9) and (10) that the governing equation for the \(k = 0\) group is

$$\left[ \mu \frac{\partial}{\partial z} + \mathcal{H}_{0,0} \right] \mathcal{N}_0 = -\mathcal{H}_{0,>} \mathcal{N}_>, \tag{12}$$

where

$$\mathcal{H}_{0,0} = \alpha_{0,0} + \beta_{0,0} - \int S_{0,0} \frac{\partial}{\partial z} \right] d\Omega \tag{13}$$

is the operator for the ensemble average radiance field and

$$\mathcal{H}_{0,>} \mathcal{N}_> = \sum_{k=0}^{\infty} \mathcal{H}_{0,k} \mathcal{N}_k \tag{14}$$

represents the influence of all smaller scales on the ensemble averaged transport. A sequence of equations thus follows

$$\left[ \mu \frac{\partial}{\partial z} + \mathcal{H}_{1,k} \right] \mathcal{N}_1 = -\mathcal{H}_{0,0} \mathcal{N}_0 - \mathcal{H}_{0,>} \mathcal{N}_> \tag{15}$$

for each group. The two terms that appear on the right-hand side of (15) represent the coupling of the \(k\) group to those groups of a characteristically larger \(k'\) and smaller \(k''\) scale defined respectively as

$$\mathcal{H}_{0,<} \mathcal{N}_< = \sum_{k=0}^{k'<k} \mathcal{H}_{0,k} \mathcal{N}_k, \tag{16}$$

$$\mathcal{H}_{0,>} \mathcal{N}_> = \sum_{k=k'}^{\infty} \mathcal{H}_{0,k} \mathcal{N}_k. \tag{17}$$

The sequence (15) continues for all \(k\) until it is ended by some form of truncation. Truncation is only appropriate when the sequence converges such that the coupling term (17) either vanishes completely or is small compared to the remaining terms of the transport equation. For this case, it may be possible to parameterize the smaller scale effects in terms of the larger scale transport properties, thus providing a closed hierarchy of equations. Parameterizations of this type will be referred to as closure.

It is apparent from the computations presented in Part I that the convergence of (15), which is a necessary condition for the introduction of closure, is likely to be a natural consequence of multiple scattering that acts to filter out the smaller scale contributions. Thus it is anticipated that the small scale coupling operator \(\mathcal{H}_{0,>} \mathcal{N}_>\) can be replaced at some chosen level (i.e., \(k\) scale) in the hierarchy with an appropriately chosen function of \(\mathcal{N}_k\). While specific forms of closure are discussed in the following section, the general concept of closure is presented here and, for the sake of discussion, the first level in the hierarchy is chosen as the dominant scale of interest. The closure function is denoted by \(\vartheta(\mathcal{N}_0)\) and its introduction into (12) yields

$$\left[ \mu \frac{\partial}{\partial z} + \mathcal{H}_{0,0} \right] \mathcal{N}_0 = \vartheta(\mathcal{N}_0). \tag{18}$$

The closure function might assume many different forms of varying degrees of complexity. For instance, consider a special example of the form

$$\vartheta(\mathcal{N}_0) = \vartheta^* \mathcal{N}_0 \tag{19}$$

in which case (18) becomes

$$\left[ \mu \frac{\partial}{\partial z} + \mathcal{L}_{0,0} \right] \mathcal{N}_0 = 0 \tag{20}$$

where \(\mathcal{L} = \mathcal{H} - \vartheta^*\). Thus (20) describes the transfer of the averaged radiance field in which the influence of smaller spatial fluctuations (i.e., fluctuations on a scale smaller than that over which the average is performed) define a new operator \(\mathcal{L}\) and thus are treated as optical properties of the host medium. Because of the simplicity and intuitive appeal of this formulation, closure of the type (19) is discussed more fully below. In the meantime, however, two unanswered questions pertinent to the use of closure immediately arise. What is an appropriate form of \(\vartheta(\mathcal{N}_0)\) and at what level in the hierarchy should closure be introduced? It is anticipated that the answer to these queries will be a subject of future debate and research.

5. An example of closure: Cloud amount

It will now be demonstrated that the existing use of cloud amount in the context of radiative transfer constitutes a form of closure and, as such, questions such as those posed above must be addressed. The starting point for this discussion is (12), written in the form

$$\mu \frac{\partial \bar{N}}{\partial z} = -\bar{\alpha} \bar{N} + \int_S \bar{S} \bar{N} d\Omega - \alpha' \bar{N}' + \int_S \bar{S}' \bar{N}' d\Omega \tag{21}$$

where the notation of (5) is invoked and the equivalence of \(\bar{N}\) to \(\mathcal{N}_0\) and of \(\alpha'\) to the collective groups \(\mathcal{N}_1, \mathcal{N}_2, \ldots\) is noted. Note also that the last two terms on the right-hand side of (21) are the equivalent of (14) in this notation.

For illustration, consider the distribution of optical properties shown in Fig. 1 as representing the optical medium. Let a volume fraction \(H\) of the optical medium be composed of \(\alpha_1\) and \(S_1\) properties and let the remaining fraction \((1 - H)\) possess properties \(\alpha_2\) and \(S_2\). If we assume these properties to be uniformly distributed in the vertical, then \(H\) is the fractional area covered by type 1 "cloud" properties while type 2
properties are taken to represent clear sky optical properties. With the further decomposition of these properties as

\[ \alpha_1 = \bar{\alpha}_1 + \alpha'_1 \]
\[ \alpha_2 = \bar{\alpha}_2 + \alpha'_2 \]  

(22)

the ensemble averaged optical properties take the form

\[ \bar{\alpha} = H\bar{\alpha}_1 + (1 - H)\bar{\alpha}_2. \]  

(23)

One of the assumptions central to this closure scheme states that the ensemble averaged radiance field is also related to \( H \) according to

\[ \bar{N} = H\bar{N}_1 + (1 - H)\bar{N}_2, \]  

(24)

which is an assumption that is often criticized (Stephens 1984) and is one that is discussed in more detail below.

With the above decomposition of the optical properties and the radiance field, the covariance terms become

\[ \bar{\alpha}'N = H(\Delta\alpha_1\Delta N_1 + \alpha'_1\bar{N}_1') \]
\[ + (1 - H)(\Delta\alpha_2\Delta N_2 + \alpha'_2\bar{N}_2') \]  

(25)

\[ S'N = H(\Delta S_1\Delta N_1 + \bar{S}_1'\bar{N}_1') \]
\[ + (1 - H)(\Delta S_2\Delta N_2 + \bar{S}_2'\bar{N}_2') \]  

(26)

where

\[ \Delta\alpha_1 = \bar{\alpha}_1 - \bar{\alpha} \]
\[ \Delta\alpha_2 = \bar{\alpha}_2 - \bar{\alpha} \]  

(27)

The following closure

\[ \bar{\alpha}'_1\bar{N}_1' = \bar{\alpha}'_2\bar{N}_2' = \bar{S}'_1\bar{N}_1' = \bar{S}'_2\bar{N}_2' = 0, \]  

(28)

is introduced into (25) and (26) and permits the reduction of (21) to

\[ \mu \frac{\partial [H\bar{N}_1 + (1 - H)\bar{N}_2]}{\partial z} = -H\bar{\alpha}_1\bar{N}_1 - (1 - H)\bar{\alpha}_2\bar{N}_2 \]
\[ + H\int_{\Omega} S_1\bar{N}_1 d\Omega + (1 - H)\int_{\Omega} S_2\bar{N}_2 d\Omega \]  

(29)

which further separates into two uncoupled equations

\[ \mu \frac{\partial \bar{N}_1}{\partial z} = -\bar{\alpha}_1\bar{N}_1 + \int_{\Omega} \bar{S}_1\bar{N}_1 d\Omega \]
\[ \mu \frac{\partial \bar{N}_2}{\partial z} = -\bar{\alpha}_2\bar{N}_2 + \int_{\Omega} \bar{S}_2\bar{N}_2 d\Omega. \]  

(30)

These two equations are the equivalent plane-parallel transfer equations for each component of the distribution. They are solved separately for the cloud radiances \( \bar{N}_1 \) and clear sky radiances \( \bar{N}_2 \); the ensemble average is obtained from the synthesis formula (24). This approach has become almost universal in treating radiative transfer through a partially cloudy atmosphere. It has always been assumed that the method is based solely on the assumption inherent to (24). The identification of the closure assumption (28), which requires uncorrelated fluctuation fields, has apparently not been stated before.

a. Inherent versus apparent cloud amount

The assumptions implicit to (24) are now considered. To facilitate this discussion assume that the optical property \( \alpha \) is distributed within the domain according to some general distribution function \( P_\alpha(\alpha) \). The quantity

\[ H_\alpha = \int_{\alpha_0}^{\infty} P_\alpha(\alpha) d\alpha \]  

(31)

might be considered as a “cloud” amount defined with respect to some threshold value \( \alpha_0 \). This quantity is referred to here as the inherent cloud amount and is an intrinsic property of the medium. In a similar fashion, consider the radiance distribution \( P_N(N) \) which defines the quantity

\[ H_N = \int_{N_0}^{\infty} P_N(N) dN \]  

(32)

with respect to some threshold (clear sky) radiance \( N_0 \). The apparent cloud amount, \( H_N \), is a property of the radiance field and as such is not a fundamental property of the medium. The following equality

\[ H_\alpha = H_N \]  

(33)

is thus implicit to (24). The validity of this assumption is questionable and intuitive examples can be offered to contradict it (viewing clouds from space at some oblique angle for instance). The relevance and signif-
ificance of such an assumption to existing research programs like the International Satellite Cloud Climatology Program (ISCCP) is made obvious by reference to Figs. 2 and 3. Figures 2a and 2b display the 12Z visible (vis) and infrared (ir) METEOSAT images for 2 July 1983. The two areas highlighted on these images are regions of convective cloud over the tropical African continent and the stratocumulus cloud region west of the same continent. Figures 3a and 3b represent the relevant $P_N^{vis}$ and $P_N^{ir}$ distributions for each of these regions. Estimates of the threshold (clear sky) radiances $N_0$ for each region are also shown, although not without ambiguity (e.g., Mathews and Rosow 1987). The actual relationship between the satellite derived $H_N^{vis}$ or $H_N^{ir}$ climatologies to that of the intrinsic property $H_N$ remains unknown and there is currently no theoretical way of establishing it. The development of an appropriate theory that addresses this issue will continue to be a subject of great challenge for future research.

b. Correlated fluctuations

The second assumption, stated in the form of (28), seems intuitively and indeed fundamentally incorrect. Figure 4a provides comparative time series of reflected 0.59 $\mu$m radiation, cloud liquid water and droplet concentrations as determined from the FSSP particle spectrometer, and the volume extinction coefficient as deduced from Mie theory and the FSSP distributions. The measurements shown were obtained from aircraft penetrations of a stratocumulus cloud and further discussion of these observations is provided in the study of Stephens and Platt (1987). Figure 4b presents the correlation between the perturbation reflected radiance and the perturbation volume extinction coefficient derived from the measured droplet distributions. These correlations present observational evidence that further refutes the assumption of (28). The perturbation radiances were obtained along the level in the cloud at which the droplet size distributions (and thus optical properties) were measured. The lack of a perfect correlation is to be expected as the reflected radiance is influenced by the cloud properties below the level of measurement to some extent and thus affected by the vertical incoherence of the fluctuations.

While (28) is generally incorrect, there is an important exception for which the assumption of uncorrelated fluctuations seems valid. This occurs for the case of infrared radiances emitted from optically thick clouds which radiate as blackbodies. These radiances are determined by the appropriate radiating or blackbody temperature. Provided that the fluctuations of this temperature remain small (we might interpret this as small horizontal variations of atmospheric temperature) then $N' = 0$ and the closure relationships (28) are valid. Thus it is reasonable to deduce the ensemble averaged radiance from a simple weight of clear and cloud radiances according to (25) under these circumstances. This situation is usually assumed to apply to the IR radiative transfer through clouds. Figure 5, taken from the work of Stephens and Platt (1987), shows an example of the reflected visible radiance as a function of the upwelling 10–12 $\mu$m radiances obtained from measurements along a level flight path above a stratocumulus cloud layer. The measured upwelling IR radiance shows a large fluctuation with values ranging from clear sky surface values and stretched out along an inverted J shaped curve radiates associated with the determined cloud top temperature. Clearly the notion of small fluctuations of IR radiance are not supported by this type of analysis and the assumption that $N' = 0$ is generally applicable to the IR must be questioned.

6. A second example of closure: New optical properties

In relation to (21) it was stated that introduction of closure in the form of (20) produces a new transfer equation with new types of optical properties. To provide an example of such a closure consider the distribution functions $P_\alpha(0), P_\xi(0)$ and $P_N(N)$ with standard deviations $\sigma_\alpha, \sigma_\xi$ and $\sigma_N$ respectively. Let $r_{\alpha N}$ and $r_{SN}$ represent the correlation of $\alpha$ and $\xi$ with $N$. The covariances are thus

$$\overline{\alpha'N} = \sigma_\alpha \sigma_N r_{\alpha N}$$
$$\overline{\xi'N} = \sigma_\xi \sigma_N r_{SN}$$

and the deviations are

$$f_{\alpha} = \sigma_{\alpha}/\overline{\alpha}$$
$$f_{\xi} = \sigma_{\xi}/\overline{\xi}$$
$$f_{N} = \sigma_{N}/\overline{N}$$

Suppose that these fractional deviations are well-behaved. The covariances then become

$$\overline{\alpha'N} = f_{\alpha}f_{\xi}r_{\alpha N}\overline{\alpha'N} = C_{\alpha N}\overline{\alpha'N}$$
$$\overline{\xi'N} = f_{\xi}f_{N}r_{SN}\overline{\xi'N} = C_{SN}\overline{\xi'N}$$

and their incorporation into (21) produces

$$\mu \frac{\partial \overline{N}}{\partial z} = - (1 + C_{\alpha N}) \overline{\alpha'N} + (1 + C_{SN}) \int \overline{\xi'N} d\Omega.$$  

With the introduction of the following scalings

$$\tilde{\alpha} = (1 + C_{\alpha N})\overline{\alpha}$$
$$\tilde{\xi} = \left[1 + C_{SN}\right]^{1/2} \overline{\xi}$$

and recalling the definition of the scattering function $S$ in Part I, (37) reduces to

$$\mu \frac{\partial \overline{N}}{\partial z} = - \tilde{\alpha} \overline{N} + \tilde{\xi} \int p \overline{N} d\Omega.$$
Fig. 2. The 12Z visible (top) and IR (bottom) METEOSAT images for 2 July 1983. The images are part of the ISCCP CX dataset with an associated resolution of approximately 8 km. Two areas of interest are enclosed within the rectangular border and labeled S, (a) region of stratocumulus, and C, (b) a region of convective cloud.
Thus by virtue of (37) and of the scalings (38), the fluctuations quantities are incorporated in the radiative transfer equations as new optical properties of the medium.

Observational data to test this type of closure are lacking and the assumptions built into (39) may be overly simple. For instance, the correlations $r_{SN}$ and $r_{xN}$ can be expected to vary with an angle such that $r > 0$ for $0 \leq \mu \leq 1$ (reflected radiances) and $r < 0$ for transmitted radiances. This overall dependence is more or less confirmed in Fig. 6, in which the results illustrated are from the theoretical calculations reported in Part I for a Gaussian medium. Based on these calculations, the correlation $r_{xN}$ is shown to be a function of $\mu$. The radiances used to derive these correlations apply to the upper and lower boundaries of the medium. Other information relevant to the calculations are listed in the figure caption. The dotted line shown on Fig. 6 is the gross relationship used in the next section.

7. Closure in a two stream model: A simple parameterization

The concept of closure of the form (20) and exemplified by (36) is now introduced into the two-stream formulation of the radiative transfer equation. This is carried out here for two main reasons. The first is a heuristic one because a two-stream model developed in this manner provides meaningful insight into
the way spatial inhomogeneities influence radiative transfer processes. The second reason is motivated by the more pragmatic desire to account for the effects of heterogeneity in a simple parametric form using existing radiative transfer schemes.

a. The two flow equations for heterogeneous media

The starting point for discussion is the radiative transfer equation (21), which is conveniently written as

\[
\frac{\partial}{\partial z} \tilde{N}(z, \xi) = -\tilde{\alpha}(z)\tilde{N}(z, \xi) + \frac{s(z)}{2} \int_{\xi} p(z, \xi \cdot \xi) N(z, \xi) d\Omega(\xi) \\
\times N(z, \xi) d\Omega(\xi) - \alpha'(z)N'(z, \xi) \\
+ \frac{s'(z)}{2} \int_{\xi} p(z, \xi \cdot \xi) N'(z, \xi) d\Omega(\xi). \tag{40}
\]

The usual two-stream procedure is to integrate (21) over the up- and downward hemispheres to obtain an equation for the upwelling and downwelling irradiances. These quantities are defined here as

\[
F^+(z) = \int_{0}^{2\pi} \int_{0}^{\pi} N(z, \mu, \phi) \mu \, d\mu \, d\phi
\]

\[
F^-(z) = \int_{0}^{2\pi} \int_{0}^{\pi} N(z, \mu, \phi) \mu \, d\mu \, d\phi. \tag{41}
\]

Before proceeding with the integration of (40) to provide the relevant two flow transfer equation, the fluctuation terms of (40) need further consideration. To this end (36) is invoked producing

\[
\alpha'(z)N'(z, \xi) = C_{aN}(\xi) \tilde{\alpha}N(z, \xi)
\]

\[
\frac{s'(z)}{2} \int_{\xi} p(z, \xi \cdot \xi) N'(z, \xi) d\Omega(\xi)
\]

\[
= \frac{s(z)}{2} \int_{\xi} p(z, \xi \cdot \xi) N(z, \xi) C_{aN}(z, \xi) d\Omega(\xi). \tag{42}
\]

FIG. 4a. Time series of reflected 0.59 μm radiances, liquid water content droplet concentration and volume extinction coefficient obtained from aircraft measurements reported in Stephens and Platt (1987).

FIG. 4b. An example of the correlation of \( \alpha' \) and \( N' \) as determined from the observations shown in Fig. 4a. The open and closed circles refer to two different traverses in the same cloud and at the same level. The data shown on Fig. 4a correspond to one of these traverses.
The localized reflection and transmission functions are defined as

\[ t^\pm = \tilde{t} \pm t' \]
\[ r^\pm = \tilde{r} \pm r' \]  

(46)

where

\[ \tilde{t} = 7 \frac{\tilde{\omega}_0}{4} (4 + 3g) \]
\[ \tilde{r} = \frac{\tilde{\omega}_0}{4} (4 - 3g) - \frac{1}{4} \]  

(47)

are the functions that are appropriate to the averaged (i.e., plane parallel medium) and are therefore the parameters usually associated with the Eddington approximation. The scattering albedo used in (47) is defined by the averaged properties

\[ \tilde{\omega}_0 = \frac{\tilde{s}}{\tilde{\alpha}}. \]

The fluctuation properties are

\[ t' = \tilde{C}_s \tilde{\omega}_0 \frac{1}{2} (1 + g) - \tilde{C}_a \]
\[ r' = \tilde{C}_s \tilde{\omega}_0 \frac{1}{2} (1 - g). \]  

(48)

Therefore the effects of spatial heterogeneity on the radiative transfer appear in the form of fluctuations of the reflection and transmission properties of the medium. The closure parameters that define these fluctuation properties depend directly on the magnitude

\[ r_{\alpha N} \]

(49)

The correlation coefficient \( r_{\alpha N} \) as a function of the cosine of zenith angle \( \alpha \) as determined from the theoretical calculations described in Part I. The cloud is Gaussian and \( \langle \alpha \rangle = 2.8, \tilde{\omega}_0 = 0.95, \]
\[ L_x = 10, \Delta = 1. \]  

The radiances are azimuthally averaged (i.e. \( m = 0 \)).
of the heterogeneities as characterized by the variances \( \sigma_\alpha \) and \( \sigma_\tau \). In this way, media with the same averaged optical properties but characteristically different spatial structure possess different radiative properties and (45) therefore accommodate the behavior of the inhomogeneous medium noted in Part I. Note also that for media with uncorrelated fluctuations, \( \bar{\mathcal{C}}_\alpha = \bar{\mathcal{C}}_\tau = 0 \), (45) reverts to the equivalent plane parallel form.

b. General solution

The solution of (45) can be written as

\[
F^\pm(\tau) = m_+ e_+ (\pm) e^{A^+ \tau} + m_- e_- (\pm) e^{A^- \tau} \tag{49}
\]

where \( m_+ \) and \( m_- \) are constants defined from appropriate boundary conditions. The roots are defined as

\[
\Lambda_\pm = -\tau' \pm [\tilde{T}^2 - \tilde{r}^2 + \rho^2]^{1/2} \tag{50}
\]

which revert to the roots of the standard Eddington problem as \( \tau', \rho' \to 0 \). In (49), the functions \( e_\pm (\pm) \) are defined as

\[
e_+ (\pm) = 1 \pm \frac{\tau^\pm}{\Lambda_+} \tag{51}
\]

\[
e_- (\pm) = 1 \pm \frac{\tau^\pm}{\Lambda_-} \tag{51}
\]

and the reflection limit for a semi-infinite layer follows from these functions as

\[
\mathcal{R}_\infty = e_- (+)/e_- (-). \tag{52}
\]

c. Net flux profile

It will now be demonstrated how the flux fields derived above explain the results of Part I and might be used to deduce further properties of the heterogeneous optical medium. Toward this end, consider the net downward irradiance

\[
F_{net} = F^+ - F^- \tag{53}
\]

and the vertical divergence of net flux

\[
\frac{dF_{net}}{dz} = -2\tilde{\alpha}(1 - \tilde{\omega}_0)(F^+ + F^-)
\]

\[
+ \tilde{\alpha}\bar{C}_\alpha \left( 1 - \frac{\bar{C}_\tau}{\bar{C}_\alpha} \tilde{\omega}_0 \right) F_{net}. \tag{54}
\]

The first term on the right-hand side of (54) is that obtained from the usual Eddington approximation while the second term results from the present treatment of inhomogeneities. Since the net flux divergence is related to the radiative heating rate through the relation

\[
Q = \frac{1}{\rho C_p} \frac{dF_{net}}{dz} \tag{55}
\]

where the fluctuation heating term is

\[
Q' = -\frac{1}{\rho C_p} \tilde{\alpha}\bar{C}_\alpha \left( 1 - \frac{\bar{C}_\tau}{\bar{C}_\alpha} \tilde{\omega}_0 \right) F_{net}. \tag{57}
\]

To gain some insight into how the closure parameters might affect the profile of net flux and heating rate, consider the first term of (54) and integrate between depths 0 and \( z \). This integration produces

\[
F_{net}(z) - F_{net}(0) = \int_0^z \frac{dF_{net}}{dz'} dz'
\]

\[
= -\tilde{\alpha}(1 - \tilde{\omega}_0) \int_0^z U dz' \leq 0 \tag{58}
\]

where \( U \) is the scalar irradiance

\[
U = 2(F^+ + F^-)
\]

which is positive by definition. Thus for all \( z \),

\[
F_{net}(z) \leq F_{net}(0)
\]

which bounds the net downward irradiance to a maximum value at the top of the medium. This term therefore produces a heating throughout the layer. The second term of (54), by contrast, admits solutions of net flux that grow exponentially with \( z \) when the fraction \( f = \bar{C}_\tau \tilde{\omega}_0 / \bar{C}_\alpha > 1 \) and solutions that decay exponentially from cloud top when \( f = \bar{C}_\tau \tilde{\omega}_0 / \bar{C}_\alpha < 1 \). Obviously a profile of net flux that increases downward from cloud top cannot be permitted throughout the entire depth of the medium as this would violate energy conservation. Thus it is likely that the conditions for which \( f \) departs from unity will be concentrated mainly at the boundaries of the medium and that \( \bar{C}_\alpha \) and \( \bar{C}_\tau \) are likely to be small, deep in its interior. This seems reasonable because the radiance field deep in the medium tends to be isotropic and perhaps uncorrelated with variations in optical properties.

d. Reflection, transmission and absorption by a heterogeneous layer

It is a relatively straightforward task to use the two-layer model to derive the reflectance, transmission and absorption of finitely deep, horizontally inhomogeneous layers. Consider the conditions

\[
F^+(\tau = 0) = F_0
\]

\[
F^-(\tau^*) = 0
\]

at the upper and lower boundaries of a layer of averaged optical thickness \( \tau^* \). These conditions, when substituted into (49), yield

\[
m_+ = -F_0 e_+ (+) e^{A^+ \tau}/\Delta(\tau^*)
\]

\[
m_- = F_0 e_+ (+) e^{A^- \tau}/\Delta(\tau^*) \tag{59}
\]
where
\[ \Delta(\tau^*) = e_+ (+) e_-( -) e^{A_+ \tau^*} - e_+ (+) e^{A_+ \tau^*} - e_- (-) e^{A_- \tau^*}. \]  
\[ \text{(60)} \]

With these relations, the reflectance of the layer follows as
\[ R = \frac{F^- (0)}{F^+ (0)} = \frac{e_+ (+) e_- (-)}{\Delta(\tau^*)} \left[ e^{A_+ \tau^*} - e^{A_- \tau^*} \right]. \]
\[ \text{(61a)} \]
and the transmittance as
\[ T = \frac{F^- (\tau^*)}{F^+ (0)} = \frac{e^{(A_+ + A_-) \tau^*}}{\Delta(\tau^*)} \times [e_+ (+) e_- (-) - e_+ (+) e_+ (-)]. \]
\[ \text{(61b)} \]

These formulae are used to derive the reflection and transmission of irradiance through an optical medium of varying degrees of heterogeneity. The results of the calculations are presented in Figs. 7 and 8 for the specified values of \( \tilde{C}_s \) and \( \tilde{C}_a \). Observations that might be used to determine these fluctuation properties are lacking in detail. The estimated magnitude of the parameters used in these calculations was based on a combination of some of the observations reported in Stephens and Platt (1987) and on the calculations reported in Part I.

Figure 7 presents the domain averaged reflection (albedo) as a function of the layer averaged optical thickness. This diagram is purposely presented in a manner similar to that of Fig. 7 of Part I. The values of the closure parameters \( \tilde{C}_s \) and \( \tilde{C}_a \) reported on the diagram are consistent with the values derived from the calculations of Part I and the calculations reported in Fig. 6 of this paper. The two-stream results derived for the equivalent plane parallel layer are represented by the points labeled with \( \tilde{C}_a = 0 \) and \( \tilde{C}_s = 0 \). The effect of the inhomogeneities is to reduce the average albedo of the layer. Note that the results obtained for heterogeneous cloud layers using the simple closure scheme are very similar to those obtained from the more sophisticated theory of Part I.

The albedo, transmittance and absorptance of layers of averaged optical thickness \( \langle \tau \rangle = 1, 10 \) and 50 are shown in Fig. 8 as a function of the ratio \( \tilde{C}_s / \tilde{C}_a \). The absorptions of heterogeneous cloud layers characterized by values of this ratio that exceed unity are significantly larger than the absorptions of equivalent plane parallel cloud layers. Conversely, the absorption decreases as the value of the ratio decreases from unity which is expected from (54). Whereas the absorption either increases or decreases from the value deduced for the equivalent homogeneous medium, the albedo was lower than the plane parallel value for most values of this ratio.

A comment on the ratio \( \tilde{C}_s / \tilde{C}_a \) and the conditions under which it might differ from unity is required.

Consider (34) written in the expanded form
\[ \alpha^{\prime} N^\prime = (\alpha^\prime + s^\prime) N^\prime = \sigma_a \sigma_N r_{aN} + \sigma_s \sigma_N r_{aN}, \]
where \( a \) and \( s \) are respectively the volume absorption and scattering coefficients and it is also assumed that the variability of the scattering function \( S \) is determined solely by the variability of \( s \). Consider some local fluctuation in the microphysical properties of the cloud that give rise to a local increase in \( a \). It is also expected that this fluctuation results in an increase in \( s \) since both are correlated. Consider the local fluctuation in reflected radiation from cloud top. It is to be expected that the correlation between this reflected radiation and the absorption coefficient (i.e., \( r_{aN} \)) is negative since a locally enhanced absorption will result in a darker cloud. On the other hand, the correlation \( r_{aN} \) is expected to be positive as an enhanced scattering cross section produces more reflection. Based on these arguments, whether the cloud appears darker or brighter locally depends on whether the absorption darkening or the scattering brightening effects dominate. Nevertheless, under the conditions described it can be qualitatively expected that
\[ \sigma_a \sigma_N r_{aN} + \sigma_s \sigma_N r_{aN} < \sigma_s \sigma_N r_{aN}, \]
which would provide for the possibility that \( \tilde{C}_s / \tilde{C}_a > 1 \). The arguments provided here are sketchy and are based on intuition. They do indicate that it is reasonable to expect values of the ratio \( \tilde{C}_s / \tilde{C}_a \) other than unity. These arguments require a more quantitative assessment and should be tested more thoroughly in future analyses of in situ radiation and cloud microphysical observations.
8. Summary and conclusions

This paper presents a formulation of the radiative transfer equation that allows for the distinction between the various groups of spatial scales that comprise the radiance field. The formulation provides a convenient means for studying the effects of spatial inhomogeneity and scale interaction on the radiative transfer. The notions of scale hierarchy and closure in the radiative transfer equation are introduced. It was demonstrated that the customary treatment of partial cloudiness using cloud amount as a weighting parameter requires two specific assumptions. The first of these deals with the equivalence of the two forms of cloud amount, which are introduced here as apparent and inherent cloud amount. It is shown that the second and previously unacknowledged assumption constitutes a form of closure and requires that the fluctuations of radiance and of optical depth (which is related to cloud liquid water) be uncorrelated. It is noted that the closure assumption has validity for circumstances related to IR transfer through opaque clouds. Another simple example of closure is described which allows for the treatment of spatial inhomogeneities as a new form of optical property. This concept is pursued further in the context of a two-stream model to demonstrate, in a gross way, the effects of inhomogeneities on radiative transfer. Comparisons with the more formal calculations of Part I are presented. The simple model is able to predict radiative properties that depend on the characteristic structure of the inhomogeneities. It demonstrates the general feature that heterogeneities tend to make the atmosphere more transparent (and less reflective) than that of an equivalent uniform atmosphere. The simple model also indicates that the absorption within an inhomogeneous atmosphere depends in a complex way on the nature of the inhomogeneities.

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