Forced, Dissipative Generalizations of Finite-Amplitude Wave-Activity Conservation Relations for Zonal and Nonzonal Basic Flows

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ABSTRACT

The effects of forcing and dissipation are incorporated into finite amplitude, local wave-activity relations for disturbances to zonal and nonzonal flows. The method used is an extension of the momentum-Casimir and energy-Casimir methods that have been applied elsewhere to prove nonlinear stability theorems such as that of Arnol'd, and to generate finite amplitude wave-activity conservation relations for nondissipative flows. The wave activity density and flux, and the source or sink term associated with forcing and dissipation, are all second-order disturbance quantities which, for a large class of flows, may be evaluated in terms of Eulerian quantities.

Explicit forms of the wave-activity relation are given for disturbances to zonally uniform and zonally varying basic states, for two-dimensional flow on a β-plane and for three-dimensional flow on a sphere described by the primitive equations in isentropic coordinates.

1. Introduction

The practice of dividing atmospheric flows into a basic-state part, and a disturbance part associated with “waves” or “eddies” is a familiar one. It may often be shown that the disturbance part satisfies an equation of the form

\[ \frac{\partial A}{\partial t} + \nabla \cdot F = S, \]  

(1.1)

where \( A \) and \( F \) are functions of disturbance quantities, quadratic in the limit of disturbance amplitude, \( \alpha \) say, tending to zero. If \( A \) is taken to be a measure of wave activity then, by analogy with similar equations arising in many fields of physics, the vector \( F \) may be interpreted as representing a flux or transport of that wave activity, perhaps associated with propagation, and \( S \) as the in situ rate of generation or destruction. Such equations have frequently been used to diagnose the propagation of waves on mean flows in the real atmosphere and in numerical models. They seem most useful when \( S \) is negligible, or when \( S \) is not negligible but may be explicitly associated with nonconservative physical processes. It is now widely appreciated that wave energy, for example, satisfies neither condition. If \( A \) is chosen to be wave or eddy energy in the usual sense, then \( S \) would include terms representing conversion from energy in the mean state. These terms are nonzero for conservative motion. It is also widely appreciated that the forms of the terms in (1.1) are in no sense unique. For example, the equation would still be satisfied if an arbitrary vector \( F_1 \) was added to \( F \), and the quantity \( \nabla \cdot F_1 \) was added to \( S \).

When the quantity \( S \) is zero, (1.1) has the form of a conservation relation. When integrated over a fixed region on whose boundaries the normal component of \( F \) vanishes, it may be interpreted as a statement that the wave activity is conserved in the global sense.

Over the last decade or so a number of examples of wave-activity relations have been constructed for which \( S \) vanishes under certain conditions. One is the generalised Eliassen–Palm relation discussed by Andrews and McIntyre (1976, 1978a), which describes disturbances to an Eulerian zonal-mean flow. As shown by Andrews (1987), the quantities \( A, F \) and \( S \) may all be calculated in terms of Eulerian disturbance quantities. The \( S \) contains terms which are either associated with forcing or dissipation, or else with third, or higher, order terms in the wave amplitude. Thus, for small-amplitude, conservative waves \( S \) is effectively zero. For finite-amplitude waves, however, even when the motion is conservative, there appears to be a source or sink of waves associated with the nonlinear terms in the equations. This tends to make interpretation of maps of the flux \( F \), for example, more difficult, and reduces their usefulness.

Wave-activity relations in which the term \( S \) vanishes for all conservative motion, even when the waves have finite amplitude, have been constructed as part of the generalized Lagrangian mean theory of Andrews and

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McIntyre (1978b,c). As discussed by McIntyre (1980), however, there are formidable technical problems in applying this theory to real atmospheric flows, not least because evaluation of the quantities $A$, $F$ and $S$ requires knowledge of particle displacements.

Recently, Killworth and McIntyre (1985) and McIntyre and Shepherd (1987), hereafter referred to as MS87, have constructed quasi-Eulerian conservation relations for disturbances to parallel and nonparallel basic flows that are valid at finite amplitude and that in many circumstances require no knowledge of particle displacements. These results represent finite-amplitude generalizations of the Eliassen–Palm relation and of the conservation relation for small-amplitude disturbances to nonparallel flows found by Andrews (1983a). As discussed in some detail by MS87, the finite-amplitude results may be derived in a systematic manner via the “energy–Casimir” method used by Arnold and others to prove stability theorems for conservative flows. (Comprehensive surveys of the stability theory itself, for various categories of flow, have been given recently by Holm et al. 1985 and Abarbanel et al. 1986.)

The purpose of this paper is to extend these finite-amplitude conservation relations to include the effects of forcing and dissipation. Ad hoc construction of the relevant conserved quantity in the manner of Killworth and McIntyre (1985; section 5), seems at first sight to rely on the material conservation of absolute vorticity, which does not hold in nonconservative systems. Similarly, because the energy–Casimir method is founded on the ideas and methods of Hamiltonian (and therefore conservative) dynamics one might, at first sight, conclude that the method could not be applied to forced or dissipative systems. Nonetheless, the mathematical techniques involved in the construction of the conservation relations using the energy–Casimir method require only the idea of functional variation, and, as will be shown here, may be applied to nonconservative systems to generate wave activity relations. In common with the findings of all other general theories of wave propagation, the presence of forcing and dissipation lead to the appearance of source and sink terms, represented by $S$, in the wave-activity relation (1.1).

The general method by which the wave-activity relations are constructed is illustrated in section 2 in the simplest cases. The method is applied to two-dimensional flow on a $\beta$-plane, firstly with a zonally symmetric basic state, and secondly with a zonally asymmetric basic state. In each case the basic state itself can be subject to forcing and dissipation. The results generalize, in the first case, the conservation relation found by Killworth and McIntyre (1985), and in the second that found by MS87, so representing a finite-amplitude version of the results derived by Andrews (1983a) for nonconservative flows. In section 3 the meteorologically relevant case of three-dimensional flow on a sphere described by the primitive equations is considered. Finite-amplitude wave-activity relations for disturbances to zonal and nonzonal basic flows are presented within the framework of isentropic coordinates.

2. Two-dimensional flow on a $\beta$-plane

In this section we consider the simplest relevant fluid-dynamical system, two-dimensional flow on a $\beta$-plane described in Cartesian coordinates $(x, y)$. The fluid is taken to be incompressible so that the velocity $\mathbf{u}$ has components $(u, v)$, that may be written in terms of a streamfunction $\psi$, with $u = -\psi_y$ and $v = \psi_x$. The subscripts denote partial derivatives. The equation describing the evolution of the flow is

$$\frac{Dq}{Dt} = D$$

where $D/Dt$ is the material derivative, $\partial/\partial t + \mathbf{u} \cdot \nabla$. The absolute vorticity $q$ is given by

$$q = v_x - u_y + f = \nabla^2 \psi + f,$$

where the Coriolis parameter $f$ is equal to $f_0 + \beta y$, with $\beta$ and $f_0$ both constants. The term $D$ in (2.1), whose form is left arbitrary, represents the net effect on the absolute vorticity field of forcing and dissipation.

In the absence of forcing and dissipation the Eqs. (2.1) and (2.2) have a number of conserved quantities, namely the density of Kelvin’s impulse $yq$, the kinetic energy density $\frac{1}{2} |\nabla \psi|^2$, and an invariant whose density is any function $C(\cdot)$ of the absolute vorticity. These quantities are conserved (in the sense that their integrals over the entire area of the flow are constant. They are also conserved locally in the sense that they satisfy conservation relations which take the form

$$(yq)_t + \nabla \cdot (yq \mathbf{u}) = -\left(\frac{1}{2} v^2 - \frac{1}{2} u^2 + Fu\right)_x + (wv - Fv)_y = 0,$$

where $F$ is a function of $y$, such that $F_y = f$,

$$\left(\frac{1}{2} |\nabla \psi|^2 \right)_t + \nabla \cdot (-q\psi \mathbf{u} - \psi \nabla \psi) = 0,$$

$$(Cq)_t + \nabla \cdot (Cq \mathbf{u}) = 0.$$
The energy-Casimir method used by Arnol’d and others exploits the freedom in (2.5) by choosing that function $C(\cdot)$ for which the first variation in the functional $H = \int \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 + C(q) \right) dxdy$ vanishes for the class of variations natural to the system. Here the permitted class of variations are those that are consistent with an incompressible rearrangement of the field. This is useful from the stability viewpoint because it allows the possibility of showing that the functional representing the difference in $H$ from its value for the undisturbed flow is positive-definite or negative-definite, and thereby deducing a bound on the growth of disturbance quantities. An analogous procedure is clearly possible for an impulse-Casimir invariant $y q + C(q)$.

McIntyre and Shepherd note that the vanishing of the first variation of the functional $H$ indicates that the corresponding local conservation relation, formed by adding (2.4) and (2.5), may be manipulated into a form such that all the terms are manifestly second order in disturbance quantities. This is a consequence of the fact that for one choice of $C(\cdot)$ the quantity

$$\frac{1}{2} |u_0 + u_e|^2 - \frac{1}{2} |u_0|^2 + C(q_0 + q_e) - C(q_0),$$

where $(\_)_0$ denotes the basic-state value of any quantity, and $(\_)_e$, the departure from that value, may be written as the divergence of a flux, plus terms that are second-order in disturbance quantities. The resulting local conservation relation is a finite-amplitude version of that derived by Andrews (1983a). The reader is encouraged to refer to MS87 (section 7) for discussion of the connection with Hamiltonian dynamics and the reason why it is appropriate to call $C(q)$ (the density of) a Casimir.

In the remainder of this paper we shall exploit the fact that the same mathematical operations which lead to this conservation relation may be carried out equally well when the right-hand side of (2.1) is not equal to zero and the dynamics is nonconservative. The extra terms are simply carried along on the right-hand sides of the equations and may be associated with local forcing or dissipation of the relevant wave activity.

a. Zonally symmetric basic states

Following Killworth and McIntyre (1985) we now consider the flow to be disturbed from a basic state which is independent of $x$, but in this case allow the basic flow to vary in time through the influence of forcing and dissipation. This problem provides the simplest possible example of how the wave activity relations in question may be generated. Note that the basic state is not the same as the instantaneous Eulerian zonal mean state, but may be any dynamically consistent $x$-independent flow (in the sense that it satisfies $q_{0e} = D_0$). We continue to denote basic state quantities, which in this case are a function of $y$ only, by $(\_)_0$ and disturbance quantities by $(\_)_e$. Since the flow is incompressible, and may be bounded in the $y$ direction, it is natural to take $v_0 = 0$. This condition may be relaxed, however, if required. The time-dependence of the basic state makes it necessary to allow the function $C$ to be a function of time, as well as $q$. The appropriate generalization of (2.5) is

$$(C(q, t)) - \nabla \cdot (C(q, t) \mathbf{u}) = \left( \frac{\partial C}{\partial \mathbf{q}} \right)(q, t). \quad (2.7)$$

Note that the time derivative on the right-hand side is taken keeping $q$ constant.

When the basic state is zonally symmetric the appropriate conserved quantity may be generated by subtracting (2.3) from (2.5). We choose the member of this family for which

$$-y q_e + C(q_0 + q_e, t) - C(q_0, t) = A(q_0, q_e, t) = O(\alpha^2),$$

where the last expression refers to the limit $\alpha \to 0$, where $\alpha$ is a measure of wave amplitude. The function $C(q, t)$ therefore satisfies the partial differential equation

$$\left( \frac{\partial C}{\partial q} \right)(q_0(y, t), t) = y. \quad (2.8b)$$

If $q_{0p}$ is one-signed then $(\partial C/\partial q)(q, t)$ is a single-valued function of $q$, otherwise it is multiple-valued. The implications of any multiple-valuedness are discussed at some length by MS87 (section 5). Here we shall restrict attention to basic states in which $q_0(y, t)$ is monotonic. It is then possible to define a single-valued function $\Upsilon(q, t)$ such that $\Upsilon(q_0(y, t), t) = y$, and using (2.8b) it may be shown that

$$C(q, t) = \int_0^t \Upsilon(q_0, d\bar{q}). \quad (2.9)$$

The equation for $-y q + C(q, t)$, i.e., (2.3) subtracted from (2.5) with the terms associated with forcing and dissipation included, is satisfied by both the basic flow plus the disturbance, and be the basic flow itself. If the second equation is subtracted from the first we are left with a conservation relation, or rather its dissipative generalization, for the quantity $A$. This takes the form

$$\begin{align*}
\frac{\partial A}{\partial t} + \frac{\partial}{\partial x} \left( \frac{1}{2} v_e^2 - \frac{1}{2} u_e^2 + (u_0 + u_e)A \right) &+ \frac{\partial}{\partial y} \{ -u_e v_e + v_e A \} \\
&= \{ \Upsilon(q_0 + q_e) - \Upsilon(q_0) \} D_e \\
&+ D_0 \int_0^u (q_e - \bar{q}) \frac{\partial^2 \Upsilon}{\partial q^2} (q_0 + \bar{q}, t) d\bar{q} \\
&+ \int_0^u \frac{\partial^2 \Upsilon}{\partial q^2} (q_0 + \bar{q}, t)(q_e - \bar{q}) d\bar{q}
\end{align*} \quad (2.10a)$$
and represents the forced, dissipative time-dependent generalization of Eq. (5.17) in Killworth and McIntyre (1985). Note that it follows from (2.9) that

\[ A(q_0, q_e, t) = \int_0^{q_e} Y(q_0 + q, t) - Y(q_0, t) dq \]

\[ = \int_0^{q_e} (q_e - q) \left( \frac{\partial Y}{\partial q} \right)(q_0 + q, t) dq. \]

(2.10b)

The first form is closest to that used by Killworth and McIntyre, the second makes it a little clearer that A is indeed second order in the limit \( q_e \rightarrow 0 \).

Bearing in mind the remarks made earlier concerning the close relation between Kelvin’s impulse and momentum, it should be noted that (2.10a) and (2.10b) may also be constructed by adding the momentum equation and (2.5). Details of the calculation are set out in the Appendix. A similar method is used in the next section to construct the corresponding wave activity relation for the primitive equations.

The wave-activity relation (2.10a) is quite different from the “diffusive” generalization of the conservation relation given by Killworth and McIntyre [1985, Eq. (7.19)]. The latter relies on the definition of a redistribution function \( P(x, t; y_\ast) \) which gives \( q(x, t) \) when convolved with the basic-state vorticity profile \( q_0(y_\ast) \). Considerable freedom is allowed in choosing \( P \), enough that it is relatively easy to generate functional forms for \( P \) which have the required properties. For almost all these choices, however, the corresponding wave activity \( \langle A \rangle \) is not a second-order disturbance quantity. The possibility remains that there is just one redistribution function which makes \( \langle A \rangle \) second order, but as yet proof that this is the case, let alone an algorithm for calculating the required \( P \), has remained elusive.

Note that the function \( Y(q, t) \), and therefore \( C(q, t) \) is defined only for values of \( q \) lying in the range found in the basic state at time \( t \). The requirement that \( A \) be second order in wave amplitude is applied in the limit \( q_e \rightarrow 0 \), i.e., \( q \rightarrow q_0 \), and therefore makes no constraint on the behavior of \( Y(q, t) \) for values of \( q \) which are not close to a value attained by \( q_0 \). In the conservative case there is no problem (at least provided that the basic flow is an accessible state of the whole system) since \( q \) is materially conserved, and no value lying outside the range can arise. This is not true in the presence of forcing or dissipation. It is clear, however, that (2.10a) remains valid for any choice of \( Y(q, t) \) outside the range and, from (2.10b), that the terms in (2.10a) are \( O(\alpha^2) \) in the limit \( \alpha \rightarrow 0 \), provided that \( (\partial Y/\partial q)(q, t) \) is bounded at the endpoints of the range.

b. Zonally asymmetric basic states

We now consider a basic state which is constant in time but zonally varying, with \( (u, v) = (u_0(x, y), v_0(x, y)) \). Substitution into the vorticity equation (2.1) gives

\[ J(\psi_0, q_0) = D_0 \]

(2.11)

where \( J \) is the Jacobian with respect to \( x \) and \( y \). Note that the sum of forcing and dissipation in the basic state does not vanish unless streamlines and absolute vorticity contours are coincident.

We also generalize Eq. (2.5) to that for functions \( C(q, \mu) \), where \( \mu \) is some scalar function of position which serves as a downstream coordinate. The role of \( \mu \) in the steady, zonally asymmetric case is analogous to that of \( t \) in the unsteady, zonally symmetric case. It is straightforward to show that

\[ C(q, \mu) + \{ uC(q, \mu) \}_x + \{ vC(q, \mu) \}_y \]

\[ = \left( \frac{\partial C}{\partial q} \right) D + \left( \frac{\partial C}{\partial \mu} \right) u \cdot \nabla \mu. \]

(2.12)

In this case we generate a wave activity relation by combining the energy equation with (2.12) rather than with (2.5). The introduction of \( \mu \) is necessary to allow for forced basic states in which \( \psi_0 \) is not a function of \( q_0 \).

We may write

\[ C(q_0 + q_e, \mu) - C(q_0, \mu) + \frac{1}{2} (u_0 + u_e)^2 \]

\[ + \frac{1}{2} (v_0 + v_e)^2 - \frac{1}{2} u_0^2 - \frac{1}{2} v_0^2 \]

\[ = u_0 \cdot u_e + q_e \left( \frac{\partial C}{\partial q} \right)(q_0, \mu) + \frac{1}{2} |u_e|^2 \]

\[ + A(q_e, q_0, \mu) \]

\[ = \left( u_0 - k \times \nabla \left( \frac{\partial C}{\partial q} \right)(q_0, \mu) \right) \cdot u_e \]

\[ - \nabla \cdot \left( k \times u_e \left( \frac{\partial C}{\partial q} \right)(q_0, \mu) \right) + \frac{1}{2} |u_e|^2 \]

\[ + A(q_e, q_0, \mu), \]

(2.13)

where

\[ A(q_e, q_0, \mu) = \int_0^{q_e} (q_e - q) \frac{\partial^2 C}{\partial q^2} (q_0 + q, \mu) dq. \]

(2.14)

If we choose the function \( C(q, \mu) \) such that

\[ \psi_0 = \frac{\partial C}{\partial q} (q_0, \mu) \]

(2.15a)

then the first term on the right-hand side of (2.13) disappears, leaving a second-order wave quantity and the divergence of a flux. An equivalent equation for \( C \), being
follows from (2.11).

The dependence on \( \mu \) is required because in general \( \psi_0 \) varies along contours of \( q_0 \). When the basic state is unforced \( \psi_0 \) is a function of \( q_0 \) and the \( \mu \) dependence may be dropped, recovering the result of MS87.

The required wave activity relation now follows upon adding (2.4) and (2.12) with forcing and dissipation included, and then subtracting the basic-state value of each term. The divergence on the right-hand side of (2.14), which involves a vector which is \( O(\alpha) \), then appears inside the time-derivative. Reversing the order in which time-derivative and divergence are taken allows this vector to be included in the flux term. The modified flux may then be shown to be \( O(\alpha^2) \). Further manipulation, including the addition to the flux of the identically nondivergent term \( \nabla \times \frac{1}{2} \psi_e^2 q_0 k \) (where \( k \)

\[
\frac{\partial^2 C}{\partial q \partial \mu} = \frac{D_0}{J(\mu, q_0)} \tag{2.15b}
\]

other vertical coordinate. In practice the various terms in the wave activity relations in pressure coordinates are found to be made up of long and complicated expressions. It seems that for the primitive equations, conservation laws and expressions of wave, mean-flow interaction take their simplest form when the equations are written in isentropic coordinates, as has become evident from the work of Andrews (1983b, 1987) and Tung (1986), for example. We therefore choose to use isentropic coordinates in presenting wave activity relations for three-dimensional hydrostatic flow on a sphere.

We describe the flow in terms of coordinates \( (\lambda, \phi, \theta) \), representing longitude, latitude and potential temperature, respectively. The horizontal part of the velocity \( u \) is taken have components \( (u, v, 0) \). The horizontal momentum equations may then be written in the form

\[
a \cos \phi u_t + \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 + M \right)_\lambda \tag{3.1a}
\]

\[
-a v_v \cos \phi - a \cos \phi (F - \dot{\theta} v_\theta) = 0
\]

\[
au_t + \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 + M \right)_\theta + au_\theta - a (G - \dot{\theta} u_\theta) = 0 \tag{3.1b}
\]

where \( a \) is the radius of the earth, \( M \) is the Montgomery potential, \( \dot{\theta} \) is the diabatic heating rate, \( F \) and \( G \) are the body force per unit mass exerted on the fluid. Here, \( \dot{z} \) is the vertical component of the absolute vorticity, defined by

\[
\dot{z} = 2 \Omega \sin \phi + \frac{1}{a \cos \phi} \left[ v_\lambda - (u \cos \phi)_\theta \right],
\]

where \( \Omega \) is the Earth’s rotation rate. We note for future reference that

\[
M = H(p, \theta) + g z
\]

where \( p \) is the pressure, \( H \) the enthalpy (a function of pressure and potential temperature), \( g \) the gravitational acceleration and \( z \) the geometric (and geopotential) height. For a perfect gas \( H \) takes the form \( H(p) \), where \( \Pi(p) \) is the Exner function \( c_p(p/p_b)^{\gamma} \), \( p_b \) being some reference pressure.

The equation expressing conservation of mass is

\[
(\sigma \cos \phi)_t + (\sigma u)_\lambda + (\sigma v \cos \phi)_\phi + (\sigma \theta a \cos \phi)_\theta = 0, \tag{3.2}
\]

where \( \sigma \) is a density, defined such that \( \sigma dV \) is the mass inside the volume element \( dV \), where \( dV = \cos \phi d\lambda d\phi d\theta \). Note that \( \sigma = \rho \delta z / \delta \theta \), where \( \rho \) is the density in geometrical coordinates. Together with the hydrostatic equation, \( g^{-1} \rho_0 = -\alpha \), or equivalently \( M_\theta = (\partial H / \partial \theta) \), (3.1a–3.2) form a complete set of equations for the flow.
Equations (3.1a) and (3.2) may be combined to derive an equation which expresses conservation of angular momentum about the rotation axis, being

\[
(\sigma a \cos^2 \phi [u + a \Omega \cos \phi]), + (\sigma u \cos \phi [u + a \Omega \cos \phi] + \tau \cos \phi)_\lambda + (\sigma v \cos^2 \phi [u + a \Omega \cos \phi])_\phi \\
+ (\sigma \theta a \cos \phi [u + a \Omega \cos \phi] - a g^{-1} p \lambda) \cos \phi)_\theta \\
= \sigma F a \cos^2 \phi, \tag{3.3}
\]

where

\[
\tau(p, \theta) = g^{-1} \int_0^p \hat{\rho} \left( \frac{\partial^2 \mathcal{H}}{\partial \hat{\rho} \partial \theta} \right)(\hat{\rho}, \theta) d\hat{\rho}. \tag{3.4}
\]

For a perfect gas the right-hand side reduces to the expression \( g^{-1} c_p p (p/p_0)^{\kappa/(\kappa + 1)} \). Note that the \( \theta \)-derivative inside the integral is to be taken keeping \( p \), (rather than \( \lambda \) and \( \phi \)) constant.

An energy equation may be formed by multiplying (3.1a) by \( u \), (3.1b) by \( v \), (3.2) by \( \frac{1}{2}(u^2 + v^2) \), and then adding the results. After some manipulation it may be shown that

\[
\left( \sigma a \cos \phi \left[ \frac{1}{2} u^2 + \frac{1}{2} v^2 + H \right] \right)_t \\
+ \left( \sigma u \left[ \frac{1}{2} u^2 + \frac{1}{2} v^2 + H \right] \right)_\lambda \\
+ \left( \sigma v \cos \phi \left[ \frac{1}{2} u^2 + \frac{1}{2} v^2 + H \right] \right)_\phi \\
+ \left( \sigma \theta a \cos \phi \left[ \frac{1}{2} u^2 + \frac{1}{2} v^2 + H \right] \right) + (\sigma vg z)_\lambda \\
+ (\sigma vg z \cos \phi) \phi + (\sigma \theta g z - z p) \sigma a \cos \phi \\
= \hat{\theta} a \sigma \cos \phi \left( \frac{\partial H}{\partial \theta} \right) + \sigma a \cos \phi (u F + v G). \tag{3.5}
\]

We may interpret the quantity \( \sigma [\frac{1}{2} u^2 + \frac{1}{2} v^2 + H] \) as an energy density. In the absence of diabatic heating and body forces (3.5) is a conservation equation for this energy.

In considering three-dimensional flow we shall exploit the fact that in the conservative case any function, \( C(P, \theta) \) say, of potential temperature and potential vorticity \( P = \xi/\sigma \) is constant following the fluid motion. We may derive an equation for the potential vorticity, by taking \( \partial C/\partial P \) (3.1b) \( - \partial / \partial \Theta \) (3.1a). If we multiply \( \partial C/\partial P \) and use the continuity equation (3.2) we obtain

\[
(\sigma a C(P, \theta) \cos \phi)_t + (\sigma u C(P, \theta))_\lambda \\
+ (\sigma v C(P, \theta) \cos \phi)_\phi \\
= \left( \frac{\partial C}{\partial P} \right)_\lambda \left[ (G - \hat{\theta} v_\phi)_\lambda - (F \cos \phi - \hat{\theta} u_\phi \cos \phi)_\phi \right] \\
+ a \cos \phi (\sigma \theta)_\phi \left[ P \left( \frac{\partial C}{\partial P} \right)_\lambda - C \right]. \tag{3.6}
\]

In the case \( C(P, \phi) = P \) this reduces to (2.4) of Haynes and McIntyre (1987); the equation may then be written in the form of a conservation relation even in the presence of diabatic heating and body forces.

We now proceed to calculate wave activity relations for flows described by these equations.

a. Zonally symmetric basic state

We consider a basic state in which the various flow quantities, denoted as before by \( \left( \cdot \right)_0 \), are independent of \( \lambda \), but may be functions of time. Our goal is to generate an equation

\[
(\sigma a \cos \phi)_t + F^{(\lambda)} + (F^{(\phi)} \cos \phi)_\phi \\
+ (F^{(\theta)} a \cos \phi)_\theta = a S \cos \phi,
\]

taking the required form of a wave-activity relation in spherical coordinates, with each of the terms being second-order in disturbance quantities. The method followed is entirely analogous to that set out in the Appendix for the incompressible case. We consider the sum of the momentum equation (3.3), and the equation for the Casimir density (3.6), choosing that function \( C(P, \theta, t) \) which makes the difference between \( \sigma \cos \phi \{ \cos \phi (u + a \Omega \cos \phi) + C(P, \theta, t) \} \) and its value in the basic state equal to the divergence of a flux plus a second-order wave quantity. Small variations in the value of this quantity, may be written, at leading-order in wave amplitude, as

\[
\sigma_0 \cos \phi [\cos \phi (u_0 + a \Omega \cos \phi) + C_0] \\
+ u_0 \sigma_0 \cos^2 \phi + \sigma_0 \left( \frac{\partial C}{\partial P} \right)_0 P_0 \cos \phi
\]

\[
= \sigma_0 \cos \phi \left[ \cos \phi (u_0 + a \Omega \cos \phi) + C_0 - P_0 \left( \frac{\partial C}{\partial P} \right)_0 \right] \\
+ u_0 \cos \phi \left[ \sigma_0 \cos \phi + a^{-1} \left( \frac{\partial C}{\partial P} \right)_0 P_0 \right] \\
+ \left[ a^{-1} \left( \frac{\partial C}{\partial P} \right)_0 v_\epsilon \right]_\lambda - \left[ a^{-1} \left( \frac{\partial C}{\partial P} \right)_0 u_\epsilon \cos \phi \right]_\phi. \tag{3.7}
\]

where the equality may be shown to hold using the result that \( \sigma_0 P_0 a \cos \phi = -\sigma_0 P_0 a \cos \phi + v_\epsilon \left( \sigma_0 \cos \phi \right) \phi + O(\alpha^2) \), and the notations \( C_0 \) and \( (\partial C/\partial P)_0 \), for example, are used for the functions \( C(P, \theta, t) \) and \( (\partial C/\partial P)(P, \theta, t) \), evaluated at \( P = P_0 \). It is therefore required that

\[
C_0 - P_0 \left( \frac{\partial C}{\partial P} \right)_0 + \cos \phi (u_0 + a \Omega \cos \phi) = 0 \tag{3.8}
\]
and

\[ a\sigma_0 \cos \phi + \left( \frac{\partial C}{\partial P} \right)_{0 \phi} = 0. \]  

(3.9)

The two conditions (3.8) and (3.9) may be shown to be equivalent by differentiating (3.8) with respect to \( \phi \), which leads to (3.9) multiplied by \( P_0 \). Integrating (3.9) with respect to \( \phi \) gives the result

\[ \left( \frac{\partial C}{\partial P} \right)_{0 \phi} = -a \int_{0}^{\phi} \sigma_0(\phi, \theta, t) \cos \phi \, d\phi, \]  

(3.10)

analogous to (2.8b) for the case of two-dimensional flow discussed in section 2. The right-hand side is a monotonic function of \( \phi \), and is proportional to the mass contained in an isentropic layer in the basic flow, between the equator and latitude \( \phi \). It is clear that \( C \) will be a single-valued function of \( P \) providing that \( P_{0 \phi} \) is one-signed, again in close analogy with the result for two-dimensional flow. We may then define the function \( m(P, \theta, t) \) such that \( -m(P_0(\phi, \theta), \theta, t) \) is equal to the right-hand side of (3.10), and integrate (3.10) to give

\[ C(P, \theta) = -\int_{0}^{P} m(\tilde{P}, \theta, t) \, d\tilde{P}. \]  

(3.11)

Returning to (3.7) it may be seen that a wave-activity relation may be generated by adding (3.3), (3.6) suitably modified for a time dependent function \( C(P, \theta, t) \), \( -\partial / \partial \lambda \) of \( a^{-1}(\partial C/\partial P)_{0 \phi} \) multiplied by (3.1b) and \( \partial / \partial \phi \) of \( a^{-1}(\partial C/\partial P)_{0 \theta} \) multiplied by (3.1a), and then subtracting the basic-state value of each term. Note that the second and third terms of (3.1b) and (3.1a), contribute to \( F^{(h)} \) and \( F^{(\phi)} \) respectively. The first terms of these equations, when combined in the specified manner, cancel the time derivative of the flux divergence appearing on the right-hand side of (3.7), and the right-hand sides contribute to \( S \). After some algebra we obtain

\[
\frac{\partial A}{\partial t} a \cos \phi + \frac{\partial}{\partial \lambda} \left[ uA + \frac{1}{2} \sigma_0(u_e^2 - v_e^2) \cos \phi \right. \\
+ a \cos \phi \tilde{r}(p_e, p_0, \theta) + \frac{\partial}{\partial \phi} (vA \cos \phi) \\
+ \sigma_0 u_e \sigma_0 \cos^2 \phi + \frac{\partial}{\partial \theta} (-g^{-1}p_e M_e \cos \phi) \\
= \sigma_0 (F - \dot{\theta} u_e) + a \cos^2 \phi - (\sigma \dot{\theta})_e \sigma_0 u_e a \cos^2 \phi \\
- m_e[(G - \dot{v}_e) \lambda - (F \cos \phi - \dot{u}_e \cos \phi)] \\
+ \sigma_0 P_e^{-1} \left[ \frac{\partial^2 C}{\partial P^2} \right]_0 \left[ (F_0 \cos \phi - u_0 \dot{\theta}_0 \cos \phi) \right] \\
- a \cos \phi P_0 \sigma_0 (\sigma \dot{\theta}_0) - (\sigma \dot{\theta})_e a \cos \phi \\
\times \left[ \frac{\partial^2 C}{\partial P^2} \right]_0 \left[ (F_0 \cos \phi - u_0 \dot{\theta}_0 \cos \phi) \right] \\
- \left[ \frac{\partial^2 C}{\partial P^2} \right]_0 \left[ (F_0 \cos \phi - u_0 \dot{\theta}_0 \cos \phi) \right] \\
+ \frac{\partial^2 C}{\partial P^2} \left[ (P_0 + \tilde{P}) \cos \phi \right].
\]

(3.12a)

where

\[ A = \sigma_0 u_e \cos \phi - a \int_{0}^{P_e} (P_e - \tilde{P}) \frac{\partial}{\partial \tilde{P}} \left[ m(\tilde{P}, \theta, t) d\tilde{P} \right]. \]  

(3.12b)

and

\[ \tilde{r}(p_e, p_0, \theta) = \int_{0}^{P_e} g^{-1} \frac{\partial^2 H}{\partial \tilde{P} \partial \tilde{\theta}} (p_0 + \tilde{P}, \theta) \, d\tilde{P}. \]  

(3.12c)

The corresponding results for flow on a beta plane that is steady, uniform and conservative, with \( F = G = \dot{\theta} = 0 \) and \( v_0 = 0 \), have been independently confirmed by D. G. Andrews (personal communication).

**b. Zonally asymmetric basic state**

We now proceed to consider disturbances to a forced, zonally asymmetric flow. The required wave activity is generated, as in section 2b, by adding the equations, in this case (3.5) and a modified form of (3.6), for energy and a suitable function of \( P, \theta \), and a quantity \( \mu(\lambda, \phi, \theta) \) which must vary along contours of \( P_0 \). We consider the disturbance values of the quantity, \( \sigma_0 \cos \phi \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 + H + C(P, \theta, \mu) \right) \), which at leading order in \( \alpha \) may be written in the form

\[
\sigma_0 \cos \phi \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 + H + C_0 \right) \\
+ p_e \sigma_0 \cos \phi \left( \frac{\partial H}{\partial P} \right)_0 + \cos \phi \sigma_0 (u_0 u_e + v_0 v_e) \\
+ \sigma_0 \cos \phi P_e \left( \frac{\partial C}{\partial P} \right)_0 \\
= p_e \cos \phi g^{-1} \left[ \frac{1}{2} u^2 + \frac{1}{2} v^2 + H + C_0 \right] \\
- P_0 \left( \frac{\partial C}{\partial P} \right)_0 + gz \sigma_0 u_e \cos \phi \left( \sigma_0 u_0 \right) \\
+ a^{-1} \left( \frac{\partial C}{\partial P} \right)_0 - v_0 \sigma_0 v_0 \cos \phi - a^{-1} \left( \frac{\partial C}{\partial P} \right)_0 \\
+ p_g^{-1} \cos \phi \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 + H + gz + C_0 \right)
\]
\[
- P_0 \left( \frac{\partial C}{\partial P} \right) \phi - \left[ u_c a^{-1} \cos \phi \left( \frac{\partial C}{\partial P} \right) \phi \right] + \left[ v_c a^{-1} \left( \frac{\partial C}{\partial P} \right) \phi \right] + O(\alpha^2). 
\]

The right-hand side follows using the hydrostatic relation in the form \( P_0 = -g \sigma \), the relation \( (\partial H/\partial p) = \rho^{-1} \) and the disturbance form of the formula for \( P_0 \), to substitute for \( P_0 \) in terms of \( u_c, v_c \) and \( \sigma \). If the right-hand side of (3.13) is to be written as a divergence, the coefficients of \( u_c, v_c \) and \( u_c \) must vanish. It is clear that this is possible only if the mass-flow along isentropic surfaces in the basic state is nondiagonal. This follows immediately from (3.2) when there is no diabatic heating in the basic state. Otherwise we must divide the basic-state horizontal velocity \((u_0, v_0, 0)\) into two parts denoted by \((\_)_{\text{rot}}\) and \((\_)_{\text{div}}\), the mass-flow vector associated with the latter being irrotational, such that

\[
\begin{align*}
\sigma_0 u_0 \phi_{\text{rot}} + (\sigma_0 v_0 \cos \phi) \phi_{\text{rot}} &= 0, \\
\sigma_0 u_0 \phi_{\text{div}} + (\sigma_0 v_0 \cos \phi) \phi_{\text{div}} &= 0.
\end{align*}
\]

This separation is unique, and the ‘dissipative’ part of the horizontal velocity vanishes when \( \theta_0 = 0 \). We may define a streamfunction, \( \Psi_0 \), for the nondiagonal part of the horizontal mass flow, such that

\[
\begin{align*}
\sigma_0 u_0 \phi_{\text{rot}} &= -a^{-1} \Psi_0, \\
\sigma_0 v_0 \cos \phi &= a^{-1} \Psi_0.
\end{align*}
\]

and take

\[
C(P, \theta, \mu) = \int_0^P \Psi_0(P_0, \theta, \mu) dP_0.
\]

It follows from the basic-state forms of (3.1a) and (3.1b) that the coefficient of \( P_0 \) on the right-hand side of (3.13) vanishes and that the coefficients of \( u_c \) and \( v_c \) may be replaced by \( \sigma_0 u_0 \cos \phi \) and \( \sigma_0 v_0 \cos \phi \), respectively. In the manipulation to derive the wave activity relation, these latter terms lead to an extra contribution to \( S \), just as the terms inside the divergence are incorporated into the expression for the flux.

We now proceed to generate the wave activity relation, by adding (3.5), (3.6) modified to include \( \mu \)-dependence and with \( C \) specified by (3.15), \((\partial/\partial \alpha) \) of (3.1b) multiplied by \( a^{-1}(\partial C/\partial P) \), \((\partial/\partial \phi) \) of (3.1a) multiplied by \( a^{-1}(\partial C/\partial P) \cos \phi \), (3.1b) multiplied by \( -\sigma_0 v_0 \cos \phi \) and (3.1a) multiplied by \( -\sigma_0 u_0 \cos \phi \), and then subtracting the basic-state value of each term. As before, terms from the left-hand sides of (3.1a) and (3.1b) which are inside \( \lambda \)-derivatives or \( \phi \)-derivatives, should be included in the flux. The resulting equation is of the required form; the flux components \( F^{(\lambda)} \) and \( F^{(\phi)} \) do not depend on time-derivatives while the component \( F^{(\theta)} \) depends on \( P_0 \). Note that the term inside the \( \theta \)-derivative in (3.5) depends on \( P_0 \). This form, however, has one drawback in that \( F^{(\lambda)} \) includes a term \( \sigma_0 u_0 M_0 \) and \( F^{(\phi)} \) a term \( \sigma_0 v_0 M_0 \). These tend to dominate in the quasi-geostrophic limit, but the dominant terms cancel when the divergence is calculated. A better-conditioned form may be generated as follows. Denoting the Bernoulli function, \( \frac{1}{2} u^2 + \frac{1}{2} v^2 + M \) by \( B \), we substitute from (3.1b) for \( \sigma_0 u_0 B_0 \), which appears in the expression for \( F^{(\lambda)} \), and from (3.1a) for the corresponding part of \( F^{(\phi)} \). Thus we use

\[
\sigma_0 u_0 B_0 = P_0^{-1} B_0 [ (G - \theta v_0) \phi - v_0 - u_0 \phi - a^{-1} B_0].
\]

and take

\[
\begin{align*}
\sigma_0 u_0 B_0 \cos \phi &= P_0^{-1} B_0 [ (G - \theta v_0) \phi - v_0 - u_0 \phi - a^{-1} B_0], \\
\sigma_0 v_0 B_0 \cos \phi &= P_0^{-1} B_0 [ (G - \theta v_0) \phi - v_0 - u_0 \phi - a^{-1} B_0].
\end{align*}
\]

After making these substitutions, transferring the forcing and dissipation terms to the right-hand side, and adding the nondiagonal contribution \((\frac{1}{2}(a_0 P_0^{-1} B_0^2) + \frac{1}{2}(a_0 P_0^{-1} B_0^2) \cos \phi, 0)\) to the flux, we are left with a wave activity relation which is well-conditioned in the quasi-geostrophic limit, and takes the form
\[
+ \left\{ \left( \frac{\partial C}{\partial \mu} \right)(P, \mu, \theta) - \left( \frac{\partial C}{\partial \mu} \right)(P_0, \mu, \theta) \right\} (\sigma u) \cdot \nabla \mu + (\sigma \dot{\theta})_{\text{div}} \cos \phi \int_0^{P_F} \frac{\partial^2 C}{\partial P^2} (P_0 + \tilde{P}, \mu, \theta)
\]
\[
+ (P_0 + \tilde{P}) \frac{\partial^2 C}{\partial P^2} (P_0 + \tilde{P}, \mu, \theta) \right\} (P_e - \tilde{P})d\tilde{P} + \cos \phi P_e \left[ \sigma \nu \text{div} (\sigma u) e - \sigma \nu \text{div} (\sigma v) e \right],
\]

(3.18a)

where

\[
A = \sigma \left[ \frac{1}{2} u_e^2 + \frac{1}{2} v_e^2 + \int_0^{P_F} (P_e - \tilde{P}) \frac{\partial^2 C}{\partial p^2} \right] \times (P_0 + \tilde{P}, \theta)d\tilde{P} + \int_0^{P_F} (p_e - \tilde{P}) \frac{\partial^2 H}{\partial p} (p_0 + \tilde{P}, \theta)d\tilde{P}
\]
\[
+ \sigma \epsilon (u_e u_0 + v_e v_0) + \frac{1}{2} g^{-1} p_e^2 \left( \frac{\partial H}{\partial p} \right)_e
\]

(3.18b)

\[
h(p_e, p_0, \theta) = \int_0^{P_F} (p_e - \tilde{P}) \frac{\partial^2 H}{\partial p^2} (p_0 + \tilde{P}, \theta)d\tilde{P}.
\]

(3.18c)

An equation of this form does not seem to have been given before, even in the conservative case when the right-hand side is zero, \( u_0 \text{div} = v_0 \text{div} = 0 \) and the \( \mu \) dependence may be dropped. The resulting conservation relation bears obvious similarities to (3.7) (for two-dimensional flow) and (B1–B2C) (for threedimensional quasi-geostrophic flow) in MS87. The conservative version of (3.18) is equal to the latter in the quasi-geostrophic limit. In the case where the basic state is at rest, so that \( C \) is identically zero, the expression for \( A \) is equal to the kinetic energy density plus a quantity which may be regarded as the available potential energy density, being the counterpart in isentropic coordinates to the quantity derived by Andrews (1981) in geometric coordinates. Andrews’ expression for the available potential energy may itself be derived by applying the energy–Casimir method in geometrical coordinates to a basic state at rest. The Casimir is in this case a function of potential temperature only.

4. Discussion

It has been shown how certain finite-amplitude conservation relations may be generalized to apply in forced and dissipative systems. The extra terms associated with forcing and dissipation have all been put on the right-hand side of the conservation equation and none included in the flux. There are arguments that in some circumstances it may be appropriate to modify the flux. For example, in the case of disturbances to steady zonally-varying flows it is possible to choose a form of the flux which satisfies the group-velocity property (e.g., Andrews 1983b). Even then, the term on the right-hand side includes contributions associated with the forcing of the basic state, making it difficult to justify that this approach must be the best one. The conservation law for potential vorticity given by Haynes and McIntyre (1987) makes an interesting contrast. In that case the forced and dissipative terms may always be incorporated into the flux, by virtue of the way in which the potential vorticity is constructed mathematically.

In the absence of further evidence it seems best to leave the wave activity relations in the forms derived here. The most appropriate split between \( F \) and \( S \) may only be decided in the light of practical experience of using these relations as diagnostics of wave-propagation. The same may also be said of the choice of basic state. In the zonally varying case it may be found most appropriate to take the time–mean state (which is almost certainly forced) as the basic state, or perhaps an unforced state which is as close as possible to the time–mean state, or perhaps neither of these. It is unlikely that there is any single best choice for all conceivable applications.

As indicated earlier, it is important that the various functions associated with the Casimirs must be single-valued. Otherwise information on which branch of the function is appropriate must be carried along in the calculation. In the conservative case, discussed by MS87, this information is equivalent to ‘colouring’ different parts of the fluid initially and then later requiring knowledge of the ‘colour’ of a particular fluid particle. In the nonconservative case the ‘colour’ of the fluid particle may change in time and satisfies a complicated evolution equation. The single-valuedness is therefore an important practical requirement if the various quantities appearing in the wave-activity relations are to be evaluated. In the two-dimensional case discussed in section 2, it is ensured if there is a one-to-one correspondence between pairs \((q_0, x)\) or \((q_0, \mu)\) in the zonally asymmetric case, and points in space. Similar conditions are required on isentropic surfaces in the three-dimensional case. If the particular basic state selected does not satisfy this condition, then the only practical solution may be to choose another one which does.

It remains to be seen whether these new wave-activity relations are as useful in the diagnosis of wave generation, propagation and breaking, as their simpler predecessors have been. Perhaps most interesting are the possibilities for the study of disturbances to zonally asymmetric basic states, such as the time-averaged flow in the troposphere and the stratosphere. Up to the present, work in this area, such as that of Hoskins et al. (1983) and Plumb (1986) has tended to exploit certain approximate conservation relations, valid on basic states with slow zonal variation. These studies, however.
ever, have attempted not only to quantify the wave propagation, but also to express its relation with the wave-induced forcing of the mean state. Considerable progress has thus been made towards a qualitative picture of the forcing of the time-mean state by the transient eddies. The unity with wave, mean-flow interaction theory is an attractive feature of the generalized Lagrangian mean formulation, as well as the small-amplitude properties of the Eliassen–Palm flux and its divergence. Plumb (1986) has indicated that it may be recovered within an Eulerian formulation for the zonally asymmetric case, again at small amplitude, with some restrictions inherent in the consideration of time-averaged states and their forcing. A succinct expression of the connection has so far proved inaccessible at finite amplitude within the current quasi-Eulerian approach. Attempting to recover a full description of the two-way wave, mean-flow interaction problem must therefore be a priority for future work.

Notwithstanding the above difficulties, there is no obstacle in principle to diagnosing the various terms in (3.18a), for example, from time series of atmospheric data. Let us suppose that we choose to take the time–mean atmospheric state as our basic state. We then have the functions $P_0$ and $\Psi_0$ on each isentropic surface. We also need to define the function $\mu$, needed to label different points along an isentropic contour of $P_0$. There is a certain amount of freedom in this choice, the only requirement being that contours of $\mu$ and $P_0$ are nowhere parallel on any isentropic surface. (Andrews 1983a, in the two-dimensional case, takes $\mu$ such that the two sets of contours are everywhere perpendicular.) For the function $\Psi_0(P, \mu)$ to be single-valued it is also required that, on any isentropic surface, each pair $(P_0, \mu)$ corresponds to a unique point in latitude and longitude. It may be that this is sometimes allowed by a suitable choice of $\mu$. Otherwise it may be possible to keep $\Psi_0$ single-valued by confining interest to a particular range of latitudes. In any case, once $P_0$ and $\mu$ are fixed, $\Psi_0$ may be determined from (3.16). The way is then clear to split all physical quantities into basic state and disturbance parts therefore to calculate the various terms in (3.18a). As noted by one of the referees it might seem a little far-fetched to think of the calculating the source/sink terms. Nonetheless, the result established in this paper remains, that these terms are associated with explicitly nonconservative physical processes. The outstanding problem is to find a choice of basic state that can also play the role of mean state in as mathematically economical a way as possible. It seems almost inevitable that this choice will not be the Eulerian mean state.

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APPENDIX

Derivation of (2.10a) from the Momentum Equations

For brevity we consider the conservative case only. The momentum equations in the $x$ and $y$ directions are

$$u_t + \left(\frac{1}{2} u^2 + \frac{1}{2} v^2\right)_x - qv = -p_x \quad (A1a)$$

$$v_t + \left(\frac{1}{2} u^2 + \frac{1}{2} v^2\right)_y + qu = -p_y \quad (A1b)$$

where $p$ is the pressure divided by the density. The latter is assumed constant. Equation (A1) may be written in the form of a conservation relation for the momentum (per unit mass) as

$$u_t + (u^2 - Fu + p)_x + (uv - Fv)_y = 0. \quad (A2)$$

We choose that function $C$ for which the disturbance part of $u + C(q)$ may be written as a divergence, plus a term which is explicitly second-order in wave amplitude. We have that

$$(u + C(q))_e = u_e + qC'(q_0) + O(\alpha^2)$$

$$= -(u_e C(q_0))_y + (v_e C(q_0))_x$$

$$+ u_e[1 + C'(q_0)q_0] + A(q_e; q_0), \quad (A3)$$

and deduce, as in section 2, but with time dependence of the basic state suppressed, that $C'(q_0) = -y$ and that $A(q_e; q_0)$ has the form given in (2.10b).

We now add (A3) to (2.5) and take the disturbance part, giving

$$A_1 - (u_e C(q_0))_{xt} + (v_e C(q_0))_{xt} + [u(u - F) + p]$$

$$+ uC(q)]_{xy} + [v(u - F) + vC(q)]_{xy} = 0. \quad (A4)$$

In this equation the flux terms, i.e., the last two terms, are $O(\alpha)$. To obtain expressions that are explicitly $O(\alpha^2)$ it is necessary to swap the order of the time and space derivatives in the second and third terms, which are themselves $O(\alpha)$, and include the term $v_e C(q_0)$ in the $x$-component of the flux and the term $-u_e C(q_0)$ in the $y$-component. Substitution for $u_e$ and $v_e$ from (A1a) and (A1b) allows both components to be reduced to forms which are explicitly $O(\alpha^2)$, and indeed identical to those on the left-hand side of (2.10a).

REFERENCES


