On the Median Volume Diameter Approximation for Droplet Collision Efficiency

KAREN J. FINSTAD
Norwegian Hydrotechnical Laboratory, Trondheim, Norway

EDWARD P. LOZOWSKI
Division of Meteorology, The University of Alberta, Edmonton, Canada

LASSE MAKKONEN
Laboratory of Structural Engineering, Technical Research Centre of Finland, Espoo, Finland

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ABSTRACT

In this note, we examine a shortcut for calculating the overall collision efficiency of a droplet spectrum, known as the "median volume diameter" (mvd) approximation. By calculating the overall collision efficiency of a circular cylinder for a variety of natural droplet spectra, first precisely using a spectrum weighting approach, and then as approximated using the mvd, as well as several other representative droplet sizes, we show by comparison that the mvd approximation is a good one, with an average absolute error of about 0.02. While trying to give some mathematical justification for why the mvd approximation works, we show that it can be derived from a single-point numerical integration formula, and that extension of this formula to 2, 3 or 4 points should give correspondingly better approximations. Detailed comparisons confirm that use of the 2-point formula reduces the average error by one-half, while the 3- and 4-point formulae can reduce it even more, depending on the type of spectrum.

1. Introduction

Studies of hail growth and ice accretion require the calculation of the impingement of small water droplets onto larger objects. The overall collision efficiency of the larger object is defined as the fraction of liquid water in the volume swept out by the object that actually collides with it. The collision efficiency depends primarily on the speed of the air flow and on the relative sizes of the collecting object and the water droplets.

Each droplet within a spectrum of sizes has its own collision efficiency. The task of computing the mean collision efficiency for a distribution of droplet sizes is often simplified by substituting for the observed spectrum a monodisperse size spectrum at the mvd of the original distribution. This approximation was first suggested by Langmuir (1944), in his pioneering work on cloud and fog size spectra on Mt. Washington. Since then, it has been widely assumed that the median volume diameter (mvd) gives the best approximation to the full spectrum when calculating collision efficiencies.

However, in recent years some workers have departed from this practice, and have adopted other representative droplet sizes. Prodi et al. (1986) prefer the mean volume droplet diameter (D_{mv}), while McComber (personal communication) uses the mean droplet diameter (D_m). In this note we examine the validity of all three approximations, as applied to the collision efficiencies of infinite circular cylinders.

2. Empirical evaluation of the size distribution parameters

Table 1 lists the sources for 27 different droplet size spectra of varying characteristics. The range of distribution shapes and droplet sizes is illustrated in Fig. 1, where selected examples from Table 1 are presented in a graphical form. We have not attempted here to normalize the spectra, since for the purposes of this paper there is no particular advantage to doing so. The droplet size data were either measured in the field, or in icing wind tunnels, or were estimated from empirical parameterized size distributions. References are also listed in Table 1.

For each spectrum, the mvd, D_{mv}, and D_m were calculated; these are listed in Table 2. The mvd was calculated following the method devised by Lozowski (1978), which is described briefly below.

Assuming a uniform distribution of droplet diameters within each of N size bins of equal width w, then

Corresponding author address: Dr. Karen J. Finstad, Dept. of Geography, Division of Meteorology, University of Alberta, Edmonton T6G 2H4 Canada.

TABLE 1. Sources for the droplet spectra used in this study.

Spectrum	Reference	Source
1	Bain and Gayet (1982)	natural cloud
2	Bain and Gayet (1982)	
3	Bain and Gayet (1982)	
4	Battan and Reitan (1957)	natural cloud
5	Battan and Reitan (1957)	
6	Choularton et al. (1986)	natural cloud
7	Choularton et al. (1986)	
8	University of Alberta, unpublished data	wind-tunnel
9	University of Alberta, unpublished data	
10	University of Alberta, unpublished data	
11	University of Alberta, unpublished data	
12	University of Alberta, unpublished data	
13	Khrgian-Mazin, as in Pruppacher and Klett (1980)	parameterization
14	Khrgian-Mazin, as in Pruppacher and Klett (1980)	
15	Khrgian-Mazin, as in Pruppacher and Klett (1980)	
16	Khrgian-Mazin, as in Pruppacher and Klett (1980)	
17	McComber and Touzot (1981)	wind-tunnel
18	McComber, personal communication	wind-tunnel
19	Makkonen and Stallabrass (1987)	wind-tunnel
20	Makkonen and Stallabrass (1987)	
21	Makkonen and Stallabrass (1987)	
22	Preobrazhenskii (1973)	natural sea-spray
23	Prodi et al. (1986)	wind-tunnel
24	Prodi et al. (1986)	
25	Squires (1958)	natural cloud
26	Squires (1958)	
27	Squires (1958)	

v_i, the volume of water contained in the ith bin is given by

$$v_i = \frac{n_i}{24}\pi w^3(i^4 - (i-1)^4) \quad (1)$$

where n_i is the number of drops in the ith bin. If V is the total volume of all droplets in all bins, we can define a cumulative fractional volume u_k such that

$$u_k = \frac{1}{V}\sum_{i=1}^{k} v_i. \quad (2)$$

Next we find a particular u_k such that $u_{k-1} < 0.5$ and $u_k > 0.5$. The mvd corresponds to a cumulative fractional volume of exactly 0.5, and can be found by interpolating between $(u_{k-1})w$ and $(u_k)w$. Thus:

$$\text{mvd} = w\left(\frac{(0.5 - u_{k-1})}{(u_k - u_{k-1})} \times [k^4 - (k-1)^4] + (k-1)^4\right)^{0.25}. \quad (3)$$

Where the droplets are distributed within nonuniform size bins, (1) and (3) must be suitably modified.

Using the same definitions for v_i and V, and letting N be the total number of droplets in all bins, the mean volume droplet diameter (following Prodi et al. 1986) is:

$$D_{\text{mv}} = \left(\frac{6}{\pi}\frac{V}{N}\right)^{1/3}. \quad (4)$$

The mean droplet diameter is defined as

$$D_{\text{m}} = \frac{\sum_i n_i D_i}{N}, \quad (5)$$

where D_i is the mean droplet diameter for the ith bin.

In order to test the relative merits of the three approximations, the overall collision efficiencies have been calculated for the mvd (E_{mvd}), mean volume droplet (E_{mv}), and mean droplet (E_{m}) of each spectrum in Table 1. For each spectrum, the weighted average collision efficiency (E_{spec}) has also been calculated using the fractional volume (i.e, v_i/V) of each bin as the weighting factor for the collision efficiency of the mvd of that bin. See Table 2 for a listing of the E_{spec} values.

All of the collision efficiencies were calculated for the same conditions, i.e., droplets impinging on an in-

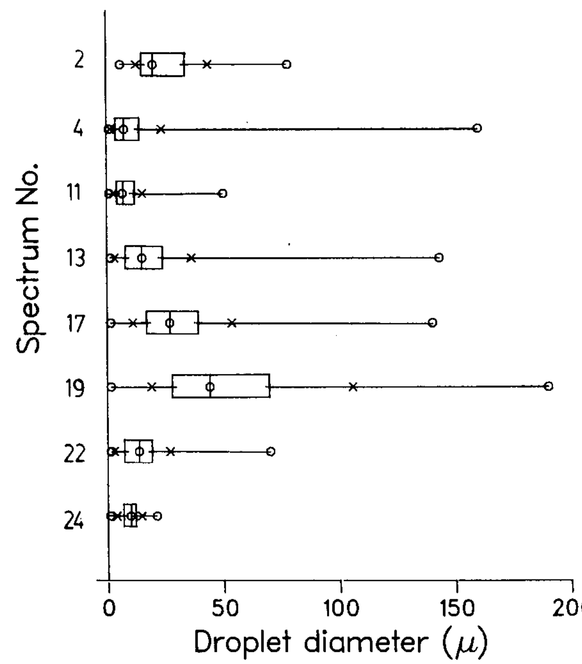

FIG. 1. A selection of droplet size spectra from Table 1. The droplet distributions are schematically represented by box plots: the central barred circle of each box represents the median diameter, the box ends represent the 25th and 75th percentile diameters, the crosses represent the 10th and 90th percentile diameters, and the outer open circles represent the minimum and maximum diameters in the spectrum.

TABLE 2. Special droplet sizes and spectrum weighted collision efficiency E_{spec} for the spectra listed in Table 1.

Spectrum	mvd	D_m	D_{mv}	E_{spec}
1	8.8	5.6	6.8	.024
2	13.3	9.5	11.0	.080
3	17.5	10.3	13.1	.175
4	28.5	14.6	19.7	.404
5	52.9	19.9	28.4	.593
6	15.0	10.3	12.0	.110
7	14.5	9.7	11.4	.101
8	24.5	15.3	18.4	.300
9	26.0	14.0	18.1	.346
10	34.8	17.6	23.2	.458
11	127.7	54.3	75.6	.850
12	112.5	35.8	59.4	.834
13	56.9	29.9	39.2	.664
14	93.6	49.9	64.8	.806
15	126.2	69.2	88.6	.864
16	165.9	96.1	119.9	.904
17	75.7	18.8	31.9	.698
18	111.0	14.5	33.6	.800
19	16.5	8.2	11.3	.190
20	14.4	7.9	10.3	.120
21	13.4	6.7	8.9	.103
22	77.6	11.6	24.8	.716
23	34.1	24.1	28.2	.493
24	46.2	25.2	32.2	.591
25	81.4	41.2	54.7	.770
26	43.0	23.0	30.9	.572
27	34.9	24.3	28.3	.469

finite circular cylinder of diameter 0.034 m (a typical overhead conductor diameter), embedded in a potential cross-flow of air at 10 m s^{-1} and -10°C. Calculations have also been made for the same spectra and different cylinder sizes (0.01 to 0.1 m) and wind speeds (10 to 20 m s^{-1}), but they are not included here since the results were similar to those for the conditions given above. We have used the analytical approximation formulae for collision efficiency given by Finstad et al. (1988), but it should be noted that this approximation cannot be used to calculate collision efficiencies below 0.01. Comparisons between E_{mvd}, E_{mv}, E_m, and E_{spec} are shown in Fig. 2.

A comparison of Fig. 2 with Fig. 1 of Makkonen (1984) reveals a disagreement in the sign of $E_{spec} - E_{mvd}$. The disagreement is due to a misinterpretation of the original droplet spectrum diagram in Bain and Gayet (1982). Figure 2 reflects the correct interpretation.

The median volume diameter is seen to perform better than either the mean volume diameter or the mean diameter in all cases. Over the range of spectra considered here, the mvd gives the best approximation to the spectrum weighted collision efficiency, with an average absolute error of 0.020. The D_m approximation gives an average error of -0.229 although one should note that it may be much larger for some individual cases. The D_{mv} approximation improves the average absolute error to -0.124, but this is still not comparable

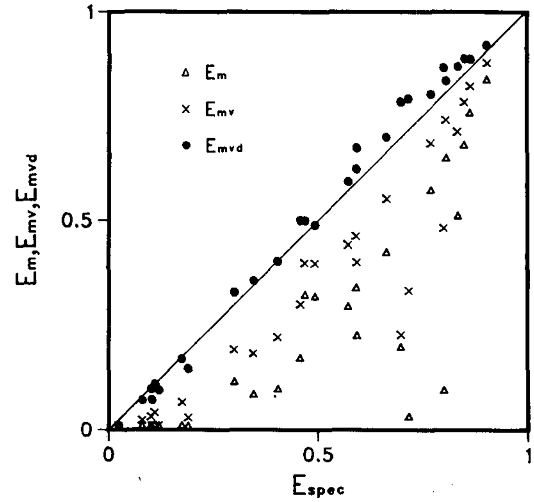

FIG. 2. Comparison of overall collision efficiencies for the mean, mean volume, and median volume drops (E_m, E_{mv} and E_{mvd}, respectively) vs the spectral weighted collision efficiency (E_{spec}). The solid line indicates a 1:1 slope.

to the performance of the mvd approximation. Why should this be so?

3. Mathematical justification for the mvd approximation

Although it has been recognized for over 40 years in the icing community that use of the median volume diameter to estimate the overall collision efficiency for a cloud droplet spectrum usually gives good results, the reasons for this generally good agreement have been obscure. In this paper, we have confirmed this widespread belief, but we felt that we ought to try to give some justification for it. In so doing, we hope to offer some insight into where the approach may fail, and to give some idea as to how the approximation may be improved without having to calculate the spectrum-weighted collision efficiency in detail.

Let $g(D)$ be the volumetric density function of the droplet spectrum, so that $g(D)dD$ is the fraction of the total droplet volume (or mass) contained in droplets of size D to $D + dD$. If $E(D)$ is the overall collision efficiency for droplets of size D, then the exact spectrum-weighted mean overall collision efficiency for a droplet distribution with diameters less than or equal to D_{max} is

$$E_{spec} = \int_0^{D_{max}} g(D)E(D)dD. \qquad (6)$$

Let $f(D)$ be the volumetric distribution function defined so that

$$f(D) = \int_0^D g(D)dD. \qquad (7)$$

Then in terms of f, (6) may be rewritten as

$$E_{\rm spec} = \int_0^1 E(f)df. \tag{8}$$

Now let D^* be a single droplet size which yields a collision efficiency equal to $E_{\rm spec}$, that is

$$E_{\rm spec} = E(D^*) \tag{9}$$

When $E_{\rm spec}$ is already known, it is always possible to find D^*. What we would like to know is whether there is a rule for finding D^*, a priori. One reason for wishing to find such a rule is to yield an algorithm for finding $E_{\rm spec}$, which is faster than numerically integrating (6) over the entire spectrum, keeping in mind that $E(D)$ must be determined for each size category D in the integration.

Another reason is that the detailed droplet spectrum may be unmeasured or at least unavailable, but that certain spectral statistics such as the mean diameter, median volume diameter, etc., may be available or otherwise determinable from the measurements. For some types of measurements, such as those made with a rotating multicylinder, for example, D^* itself is one of the quantities determined. In this case, it is useful to know the relationship between D^* and the mvd or other droplet parameters. Such knowledge could also be applied in the derivation of quantities other than collision efficiency; liquid water content for example.

The "mvd approximation" essentially states that the median volume diameter may be used to estimate D^*. In the following discussion we will use the notation D_x for the xth percentile droplet diameter where D_x is defined so that $f(D_x) = x/100$ where $0 < x < 100$. In this notation mvd = D_{50}. Why should $D_{50} = D^*$?

We can begin to answer this question by numerically integrating (8) using the well-known method of Gauss-Legendre quadrature. In order to do this, we must first transform the integration interval to $(-1, 1)$. Let

$$y = 2f - 1. \tag{10}$$

Then (8) becomes

$$E_{\rm spec} = \int_{-1}^1 0.5E\left(\frac{y+1}{2}\right)dy \tag{11}$$

which has the n-point Gauss-Legendre quadrature approximation:

$$E_{\rm spec} = 0.5\sum_{i=1}^{n} w_i E\left(\frac{x_i+1}{2}\right). \tag{12}$$

For n from 1 to 4, the weighting factors, w_i, and the zeroes of the Legendre polynomials, x_i, are as listed in Table 3 (Abramowitz and Stegun 1964). This gives the following approximation formulae:

single-point

$$E_{\rm spec} \approx E(D_{50}) \tag{13}$$

TABLE 3. Constants for the Gauss-Legendre integration formulae.

n	x_i	w_i
1	0.0	2.0
2	±0.57735	1.0
3	0.0, ±0.77459	0.88888, 0.55555
4	±0.33998, ±0.86113	0.65214, 0.34785

two-point

$$E_{\rm spec} \approx 0.5(E(D_{79}) + E(D_{21})) \tag{14}$$

three-point

$$E_{\rm spec} \approx 0.444E(D_{50}) + 0.278(E(D_{89}) + E(D_{11})) \tag{15}$$

four-point

$$E_{\rm spec} \approx 0.326(E(D_{67}) + E(D_{33})) + 0.174(E(D_{93}) + E(D_7)). \tag{16}$$

Thus we see that $D_{50} = D^*$ because $E(D_{50})$ is the first-order approximation to $E_{\rm spec}$, and that the mvd is in this sense the optimum choice for approximating the collision efficiency of the full spectrum with that of a single droplet. Furthermore, by extending the Gauss-Legendre formula to 2, 3 or 4 points, one can proceed in a rational fashion to develop new and more accurate approximation formulae requiring only slightly more computational effort than it takes to calculate the mvd approximation.

Approximations to $E_{\rm spec}$ calculated according to these additional formulae are shown in Fig. 3. Use of the two point-formula reduces the average error for this set of spectra by 50 percent over the mvd approximation, to 0.009. Indeed, the error is halved each time another point is added to the approximation formula. Thus the three-point approximation gives an average error of 0.005, and the 4-point gives 0.002. However, the degree of improvement varies somewhat with the type of spectrum, being most marked for the more complicated spectra such as number 17 (McComber and Touzot 1981), which has a double-peaked distribution.

4. Conclusions

The median volume diameter has been shown to perform significantly better than either the mean volume diameter or the mean diameter, as a single droplet size approximation to the collision efficiency of an entire spectrum. This has been shown empirically for a wide variety of droplet spectra impinging on an infinitely long circular cylinder, and under a number of different conditions, although results for only one set of conditions are presented here.

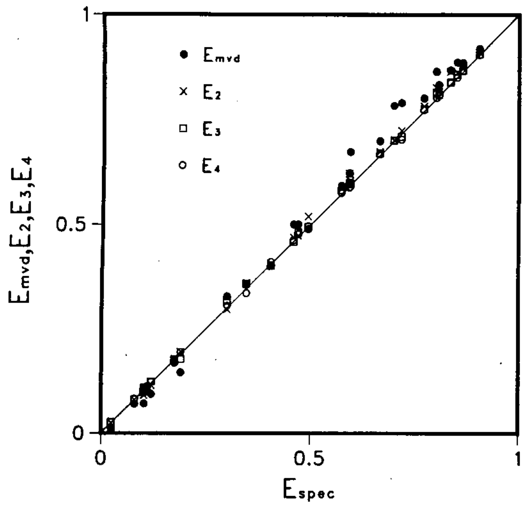

FIG. 3. Comparison of overall collision efficiencies for the mvd, 2-point, 3-point and 4-point (E_{mvd}, E_2, E_3 and E_4, respectively) approximations vs the spectral weighted collision efficiency (E_{spec}). The solid line indicates a 1:1 slope.

The mvd approximation has also been justified on mathematical grounds, by applying Gauss-Legendre quadrature to the integral defining the spectrum-weighted mean collision efficiency. The latter procedure has also yielded improved approximation methods which utilise two, three, or four droplet size parameterizations in lieu of the full spectrum.

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