Nonlinear Instability of a Forced Baroclinic Rossby Wave

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ABSTRACT

Two-layer, quasi-geostrophic weakly nonlinear and low-order spectral models are developed and used to investigate the instability of forced baroclinic Rossby waves to finite-amplitude perturbations. The results are then applied to the interaction of planetary-scale stationary eddies with synoptic scale transient eddies.

In the weakly nonlinear model, asymptotic series expansions are used in conjunction with the method of multiple time scales. The stability of a forced planetary-scale stationary baroclinic Rossby wave to synoptic-scale perturbations is first examined. The synoptic-scale perturbation modes initially grow exponentially after which they eventually settle into an amplitude vacillation cycle. This vacillation is driven by the linear interference between propagating and stationary synoptic-scale modes with the same zonal and meridional wavenumbers. During this vacillation, the time mean energy of the stationary planetary wave equals its initial value. This indicates that the transient synoptic-scale perturbation has neither an amplifying nor a dissipative influence on the stationary wave. A study of the energetics shows that eddy available potential energy is transferred from the planetary-scale stationary wave to the synoptic-scale perturbation, while eddy kinetic energy is simultaneously transferred in the reverse direction.

The asymptotic series expansions are also used to determine the truncation for a fully nonlinear spectral model. The weakly nonlinear and spectral solutions are compared and are found to agree very well. In addition, by comparing spectral model solutions with and without the higher-order modes of the weakly nonlinear model present, it is found that the evolution of the basic wave and the perturbation are extremely sensitive to the presence of these modes. This suggests that the interaction between planetary-scale stationary eddies with synoptic-scale transient eddies is a nonlinear phenomenon that is very sensitive to the detailed structure of the eddies present.

1. Introduction

The nonlinear transfer of energy between stationary planetary waves and synoptic scale waves has important implications for long-range weather forecasting. Cyclogenesis, which is of primary interest to weather forecasters, is driven by a transfer of energy from the zonal flow to the synoptic-scale waves. Cyclogenesis can proceed more rapidly when energy flows from stationary planetary waves into the growing synoptic scale disturbance. In addition, interactions between the planetary and synoptic scale waves localize the synoptic scale baroclinic activity into preferred storm track regions. This suggests that in a particular region these wave–wave interactions control both the frequency of storm activity and the long term trends of such variables as temperature and precipitation.

Stationary planetary waves initially derive their energy via forcing mechanisms such as topography and asymmetric diabatic heating. If the amplitude of these forced waves exceeds a particular critical value, they can be unstable to synoptic scale perturbations. Through this mechanism, a significant amount of energy can be redistributed from the planetary to the synoptic scale.

There have been several observational studies and general circulation model simulations of the nonlinear interaction between planetary-scale stationary eddies and synoptic-scale transient eddies. Hayashi and Golder (1985) used a GFDL (Geophysical Fluid Dynamics Laboratory) spectral general circulation model to examine the interaction between stationary and transient eddies during the winter. They showed that the planetary-scale stationary eddies are an important energy source for synoptic scale disturbances as about 20% of the synoptic scale wave energy came from the planetary-scale stationary eddies. Their results also indicate that the primary sink of the planetary-scale stationary eddy energy is the above energy transfer from the planetary to the synoptic scale. Using other large numerical models, Opsteegh and Vernaker (1982) and Vallis and Roads (1984) also showed that energy is transferred from the planetary-scale stationary eddies to the transient eddies. The same conclusion was obtained by Youngblut and Sasamori (1980), Lau (1979), Lau and Oort (1982), and Lau and Holopainen (1984) in observational studies of the transient eddies. Sasa-

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mori and Youngblut (1981) examined the linear stability of a stationary baroclinic planetary wave and found it to be unstable to transient planetary-scale perturbations. These results seem to suggest that transient eddies act to dissipate the stationary eddies.

On the other hand, Held et al. (1986) showed that the net effect of the transient eddies may be not to dissipate the stationary eddies, but to amplify them. That is, even though energy is transferred from the planetary to the synoptic scale, the transient eddies can interact among themselves to increase the vertical tilt of the stationary eddies allowing them to gain energy from the vertically sheared zonal flow. It is then possible that the energy derived from the zonal flow may exceed that lost to the transient eddies. Therefore, the net effect of the transient eddies on the stationary eddies remains an open question.

Hayashi and Golder (1985), with a GFDL general circulation model, and Holopainen et al. (1982), in an observational study, examined the energetics of this nonlinear interaction process in much detail. They both showed that available potential energy is transferred from the planetary scale to the synoptic scale, and that a smaller amount of kinetic energy is simultaneously transferred in the reverse direction. These results illustrate that the interaction between planetary-scale stationary eddies and synoptic-scale transient eddies is an important process in the atmosphere.

The purpose of this study is to examine the nonlinear instability of forced baroclinic Rossby waves in order to understand better the interaction of planetary-scale stationary eddies with synoptic-scale transient eddies. In order to investigate the stability of these baroclinic finite-amplitude waves, a weakly nonlinear model is used. This approach involves asymptotic series expansions in conjunction with the method of multiple time scales. The same mathematical technique was used by Deininger (1981) to examine the inviscid weakly nonlinear instability of forced barotropic waves on an infinite beta-plane. This technique will allow us to pursue two important goals in the investigation of the wave–wave interaction. The first is to determine to what extent the perturbation alters the amplitude of the basic wave. The second goal is to calculate the slow weakly nonlinear evolution of the perturbation. We will study in detail the question of whether the perturbation grows to a constant amplitude or undergoes a vacillation cycle. To study these processes, a large nonlinear model is required. These models, which can very well simulate these processes, are very complex and they preclude the possibility of gaining a thorough understanding of the essential physics of the wave–wave interaction. Therefore, the analytical weakly nonlinear approach is an ideal technique to study the basic physics of this problem.

In addition, this same asymptotic expansion technique will be used to determine the truncation level for a fully nonlinear spectral model. Since many spectral models often have been criticized for the arbitrary manner in which the truncation is chosen, this expansion technique, which allows us to logically and consistently deduce the important wave modes that must be retained in a spectral truncation, provides a better approach.

In section 2, a description of the model is given. The weakly nonlinear solution is derived in section 3. In section 4, the spectral model, whose truncation is determined from the weakly nonlinear solution, is presented. A summary and conclusions are discussed in section 5.

2. Model description

The quasi-geostrophic vorticity equations for the two-layer model, with boundary forcing and Ekman layers at the upper and lower boundaries, can be written in dimensionless form (Pedlosky, 1979) as

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi) + J(\theta, \nabla^2\theta) + \beta\frac{\partial\psi}{\partial x} = -r(\nabla^2\psi - \nabla^2\psi_f) \quad (2.1)$$

$$\frac{\partial}{\partial t}(\nabla^2 - 2F)\theta + J(\psi, \nabla^2\theta) + J(\theta, \nabla^2\psi) - 2FJ(\psi, \theta) + \beta\frac{\partial\theta}{\partial x} = -r(\nabla^2\theta - \nabla^2\theta_f), \quad (2.2)$$

where ψ and θ are the barotropic and baroclinic streamfunctions, respectively, defined as

$$\psi = \frac{\psi_1 + \psi_2}{2} \quad (2.3)$$

$$\theta = \frac{\psi_1 - \psi_2}{2}, \quad (2.4)$$

where the subscript "1" denotes the upper layer and "2" the lower layer, and $\nabla^2(\psi_f \pm \theta_f)$ represents the vorticity at the upper and lower boundaries, respectively.

The zonal and meridional coordinates, x and y, are scaled by the channel width L. An advective time L/U is used to scale time, where U is a characteristic horizontal wind speed. The Jacobian operator J is with respect to x and y. The other dimensionless parameters are

$$\beta = \frac{\beta_0 L^2}{U}$$

$$F = f_0^2 L^2 \bigg/ \left[\left(\frac{\rho_1 - \rho_2}{\rho_2}\right)\frac{gD}{2}\right]$$

$$r = \frac{2(\nu f_0)^{1/2} L}{DU}, \quad (2.5)$$

where β is the ratio of the planetary vorticity gradient to the relative vorticity gradient, the internal Froude number F is the ratio of the square of the horizontal length scale to the square of the Rossby radius of deformation, and the dissipation parameter r is the ratio of the dissipative time scale to the adjective time scale. Here, f_0 is the Coriolis parameter, β_0 the dimensional planetary vorticity gradient, ρ_1 and ρ_2 the densities of the upper and lower layers, respectively, g the gravitational acceleration, ν the kinematic viscosity, and D the dimensional depth of the fluid.

The fluid is bounded at $y = 0$ and $y = 1$ by rigid walls. Since the velocity perpendicular to the walls must vanish, it is necessary that

$$\frac{\partial \psi}{\partial x} = \frac{\partial \theta}{\partial x} = 0 \quad \text{at} \quad y = 0, 1, \tag{2.6}$$

and Phillips (1954) showed that the condition

$$\frac{\partial}{\partial t} \int_0^\lambda \frac{\partial \psi_n}{\partial y}\, dx = 0 \quad \text{at} \quad y = 0, 1 \tag{2.7}$$

must be satisfied for both layers. The periodic boundary conditions require that

$$\begin{aligned} \psi(x + \lambda, y, t) &= \psi(x, y, t) \\ \theta(x + \lambda, y, t) &= \theta(x, y, t), \end{aligned} \tag{2.8}$$

where λ is the period in the zonal direction. Several of the above model parameters are given constant values that will remain unchanged throughout this study. These are

$$L_D = \left(\frac{(\rho_1 - \rho_2)}{\rho_2} \frac{gD}{2f_0^2}\right) = 1000 \text{ km}$$

$$L = 5000 \text{ km}$$

$$U = 10 \text{ m s}^{-1}$$

$$\beta_0 = 1.6 \times 10^{-11} \text{ (m s)}^{-1}$$

$$\frac{L}{U} = 5.8 \text{ days}$$

$$F = 25$$

$$\beta = 40.$$

Observational and general circulation model results have both indicated that available potential energy is transferred from the planetary-scale stationary eddies to the synoptic-scale transient eddies whereas kinetic energy is transferred in the reverse direction. To study this wave–wave interaction, we will examine the nonlinear instability of a forced planetary-scale stationary basic wave to synoptic-scale perturbations. As we will discuss later, no zonal flow is present in this basic state. In addition, because both barotropic and baroclinic energy conversions play important roles in this wave–wave interaction, the wind field associated with the basic wave should include both horizontal and vertical shear. If the amplitude of the basic wave exceeds a critical value, then the forced planetary wave will be either barotropically and/or baroclinically unstable and some of its energy can be transferred to the synoptic scale.

If we define

$$\nabla^2 \psi_f = 0 \tag{2.9}$$

$$\nabla^2 \theta_f = -(k^2 + \pi^2)(F_T e^{ikx}) \sin \pi y + \text{c.c.}, \tag{2.10}$$

where c.c. denotes a complex conjugate, then (2.1) and (2.2) admit a solution of the form

$$\psi = 0 \tag{2.11}$$

$$\theta = D_T(e^{ikx} + e^{-ikx}) \sin \pi y, \tag{2.12}$$

where

$$\left(ik\beta - \frac{r}{2}(k^2 + \pi^2)\right) D_T = \frac{r}{2}(k^2 + \pi^2) F_T. \tag{2.13}$$

If D_T is chosen to be real, F_T must be complex. This solution, which consists of a baroclinic stationary wave with no zonal flow present, will represent the basic flow whose stability we wish to examine.

It is important to comment on the choice of this type of externally imposed forcing. In the atmosphere, planetary-scale quasi-stationary eddies are driven by a complex process that involves topography and asymmetric diabatic heating together in a vertically-sheared zonal flow. Therefore, to elucidate the essential physics of the wave–wave interaction, it is far simpler to apply the forcing in (2.10) and not to include explicitly topography, diabatic heating, or a zonal flow.

3. Weakly nonlinear theory

In this section, we first seek a solution to (2.1) and (2.2) of the form

$$\psi = \psi'(x, y, t) \tag{3.1}$$

$$\theta = D_T(e^{ikx} + e^{-ikx}) \sin \pi y + \theta'(x, y, t) \tag{3.2}$$

in order to examine the linear stability of the basic wave. The zonal wavenumber k is assigned a value of 2.0. This corresponds to a wavenumber 2 planetary wave at 45°N in the atmosphere. In addition, we choose $r = 0.5$, which corresponds to a dissipative time scale of 10 days. (A similar linear stability analysis of a free barotropic Rossby wave was first performed by Lorenz, 1972.)

The linearized form of (2.1) and (2.2) admit of an infinite series solution of the form

$$\psi' = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} C_{M,m,n} e^{imkx} \sin(n\pi y) + \sum_{l=1}^{\infty} B_{M,l} \cos l\pi y \tag{3.3}$$

$$\theta' = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} C_{T,m,n} e^{imkx} \sin(n\pi y) + \sum_{l=1}^{\infty} B_{T,l} \cos l\pi y, \tag{3.4}$$

which satisfies the boundary conditions. To obtain a manageable solution, the series must be truncated. In order to gain a qualitative understanding of the stability properties of the basic wave, we will choose a very simple truncation:

$$\psi' = C_{M,m,n}e^{imkx}\sin n\pi y + \text{c.c.} \quad (3.5)$$

$$\theta' = C_{T,m',n'}e^{im'kx}\sin n'\pi y + \text{c.c.} \quad (3.6)$$

where

$$|m - m'| = |n - n'| = 1. \quad (3.7)$$

The indices m, n, m', n' are assigned the values $m = 2$, $n = 2$, $m' = 3$ and $n' = 1$. These values correspond to the most unstable pair of modes as determined from a linear stability analysis of the basic wave.

The form of the perturbation is obviously very simple. The inclusion of more perturbation modes would preclude the possibility of obtaining an analytical solution and would therefore require the use of numerical techniques. The main advantage of this analytical solution, as we will see later, is that it can yield considerable insight into the essential properties of the nonlinear interaction of stationary planetary waves with synoptic-scale transient eddies.

The other important reason for selecting this truncation is that the asymptotic series approach will be used to determine the truncation for a fully nonlinear low-order spectral model. Although the results will apply directly only to a model with just a few spectral components, this solution will be beneficial as it will suggest a technique for choosing the truncation for a much larger spectral model.

The equations governing the stability of the basic wave to the perturbation are

$$-iw\frac{(4k^2 + 4\pi^2)C_{M,2,2}}{2} = \left[ik\beta - \frac{r}{2}(4k^2 + 4\pi^2)\right] \times C_{M,2,2} + i8\pi k^3 D_T C_{T,3,1} \quad (3.8)$$

$$-iw\frac{(9k^2 + \pi^2 + 2F)C_{T,3,1}}{2} = (i3k\pi(k^2 + \pi^2) - 2Fik\pi)D_T C_{M,2,2} + \left(i\frac{3}{2}k\beta - \frac{r}{2}(9k^2 + \pi^2)\right)C_{T,3,1}, \quad (3.9)$$

where $C_{m,2,2}$ and $C_{T,3,1}$ are both proportional to e^{-iwt}. The critical basic wave amplitude D_{T_c} occurs at the value of $D_T = 0.13$. If we write

$$D_T = D_{T_c} + \Delta, \quad (3.10)$$

where $\Delta/D_{T_c} \ll 1$, it can be shown that one of the roots of w satisfies

$$\text{Im}(w)\alpha\Delta. \quad (3.11)$$

Therefore, the perturbation growth rate is $O(|\Delta|)$.

We discussed previously the linear stability properties of the basic wave. But, linear theory only tells us about the initial exponential growth of the perturbation, and not about the behavior of the perturbation at finite-amplitude where nonlinear interactions play an important role. Basic questions still remain such as whether or not the perturbation equilibrates at a constant amplitude or whether it undergoes a vacillation cycle. It is also important to determine how the growing perturbation affects the basic wave. The answers to such questions require the study of nonlinear processes.

In our model, we will study this nonlinear problem by using an analytical approach. To do so, the basic wave amplitude must only slightly deviate from its neutrally stable value. When this condition is satisfied, the fully nonlinear system of partial differential equations can be replaced by an infinite sequence of systems of linear partial differential equations. Although these systems of equations are still quite complicated, it is now possible to obtain approximate analytical solutions.

We begin the derivation of the weakly nonlinear solution by substituting

$$\psi = \psi'(x, y, t, T) \quad (3.12)$$

$$\theta = (D_{T_c} + \Delta)(e^{ikx} + e^{-ikx})\sin\pi y + \theta'(x, y, t, T) \quad (3.13)$$

into (2.1) and (2.2). Since the perturbation growth rate is proportional to Δ, it is convenient to introduce the time variable

$$T = |\Delta|t. \quad (3.14)$$

Here, T will be regarded as the slow time variable, and t the fast variable. The perturbation equations become

$$\left(\frac{\partial}{\partial t} + |\Delta|\frac{\partial}{\partial T}\right)\nabla^2\psi' + J(\psi', \nabla^2\psi') + J(\theta', \nabla^2\theta') + J(\bar{\theta}, \nabla^2\theta') + J(\theta', \nabla^2\bar{\theta}) + \beta\frac{\partial\psi'}{\partial x} + r\nabla^2\psi' = 0 \quad (3.15\text{a})$$

$$\left(\frac{\partial}{\partial t} + |\Delta|\frac{\partial}{\partial T}\right)(\nabla^2 - 2F)\theta' + J(\psi', \nabla^2\theta') + J(\theta', \nabla^2\psi') + J(\psi', \nabla^2\bar{\theta}) + J(\bar{\theta}, \nabla^2\psi') - 2FJ(\psi', \theta') - 2FJ(\psi', \bar{\theta}) + \beta\frac{\partial\theta'}{\partial x} + r\nabla^2\theta' = 0, \quad (3.15\text{b})$$

where

$$\bar{\theta} = (D_{T_c} + \Delta)(e^{ikx} + e^{-ikx})\sin\pi y + \theta'(x, y, t, T). \quad (3.16)$$

We assume a solution of the form

$$\psi'(x, y, t, T) = |\Delta|^{1/2}\psi^{(1)} + |\Delta|\psi^{(2)} + |\Delta|^{3/2}\psi^{(3)} + \cdots$$

$$\theta'(x, y, t, T) = |\Delta|^{1/2}\theta^{(1)} + |\Delta|\theta^{(2)} + |\Delta|^{3/2}\theta^{(3)} + \cdots.$$

The justification for expanding in powers of $|\Delta|^{1/2}$ can be found in Pedlosky (1970). Briefly, the change in the basic wave, due to the perturbation heat and momentum fluxes, is expected to be $O(|\Delta|)$. Since the fluxes are proportional to the square of the wave perturbation amplitude, the amplitude of the perturbation should be $O(|\Delta|^{1/2})$.

The resulting $O(|\Delta|^{1/2})$ system is

$$\frac{\partial}{\partial t}\nabla^2\psi^{(1)} + J(\theta^{(1)}, \nabla^2\bar{\theta}_c) + J(\bar{\theta}_c, \nabla^2\theta^{(1)}) + \beta\frac{\partial\psi^{(1)}}{\partial x} + r\nabla^2\psi^{(1)} = 0 \quad (3.17)$$

$$\frac{\partial}{\partial t}(\nabla^2 - 2F)\theta^{(1)} + J(\psi^{(1)}, \nabla^2\bar{\theta}_c) + J(\bar{\theta}_c, \nabla^2\psi^{(1)}) - 2FJ(\psi^{(1)}, \bar{\theta}_c) + \beta\frac{\partial\theta^{(1)}}{\partial x} + r\nabla^2\theta^{(1)} = 0, \quad (3.18)$$

where

$$\bar{\theta}_c = D_{T_c}(e^{ikx} + e^{-ikx})\sin\pi y. \quad (3.19)$$

These equations are identical to the linear stability equations of (3.5) and (3.6), so we will take the same unstable pair of modes as the solution. After noting that ψ' and θ' are also functions of T, we can write the solution to (3.17) and (3.18) as

$$\psi^{(1)} = \frac{\gamma}{2}A(T)(e^{i(2kx-wt)} + e^{-i(2kx-wt)})\sin 2\pi y \quad (3.20)$$

$$\theta^{(1)} = \frac{A(T)}{2}(e^{i(3kx-wt)} + e^{-i(3kx-wt)})\sin\pi y, \quad (3.21)$$

where

$$\gamma = -8\pi k^3 \Big/ \left[\frac{w(4k^2 + 4\pi^2)}{2} + k\beta + \frac{ir}{2}(4k^2 + 4\pi^2)\right]. \quad (3.22)$$

At this stage of the problem, except for the slowly-varying amplitude $A(T)$ which so far is arbitrary, the $O(|\Delta|^{1/2})$ solution is known.

The $O(|\Delta|)$ system of equations, which we will consider next, are

$$\frac{\partial}{\partial t}\nabla^2\psi^{(2)} + J(\theta^{(2)}, \nabla^2\bar{\theta}_c) + J(\bar{\theta}_c, \nabla^2\theta^{(2)}) + \beta\frac{\partial\psi^{(2)}}{\partial x} + r\nabla^2\psi^{(2)} = 0 \quad (3.23)$$

$$\frac{\partial}{\partial t}(\nabla^2 - 2F)\theta^{(2)} + J(\psi^{(2)}, \nabla^2\bar{\theta}_c) + J(\bar{\theta}_c, \nabla^2\psi^{(2)}) - 2FJ(\psi^{(2)}, \bar{\theta}_c) + \beta\frac{\partial\theta^{(2)}}{\partial x} + r\nabla^2\theta^{(2)} = -J(\psi^{(1)}, \nabla^2\theta^{(1)}) - J(\theta^{(1)}, \nabla^2\psi^{(1)}) + 2FJ(\psi^{(1)}, \theta^{(1)}). \quad (3.24)$$

The homogeneous solution has the same form as in the $O(|\Delta|^{1/2})$ system and is thus set equal to zero.

We will next consider the forcing terms on the right side of (3.23) and (3.24). It can easily be shown that the remaining terms on the right side of (3.24) can be rewritten as

$$J(\psi^{(1)}, \nabla^2\theta^{(1)}) + J(\theta^{(1)}, \nabla^2\psi^{(1)}) - 2FJ(\psi^{(1)}, \theta^{(1)}) = -\frac{(5k^2 - 3\pi^2 + 2F)}{4} ik\pi\gamma|A(T)|^2(e^{ikx} - e^{-ikx}) \times (4\sin 3\pi y - 2\sin\pi y) + \frac{(5k^2 - 3\pi^2 + 2F)}{4} \times ik\pi\gamma|A(T)|^2(e^{i(5kx-2wt)} - e^{-i(5kx-2wt)}) \times (4\sin\pi y - 2\sin 3\pi y). \quad (3.25)$$

These terms force an additional four wave modes. Also, because of the interaction between these modes and the basic wave, the particular solution is in fact an infinite series. Thus, as in the $O(|\Delta|^{1/2})$ solution, a severe truncation must be made. A very simple truncation is one which includes the four wave modes on the right side of (3.25). The interaction of these four modes with the basic wave also forces the lower order modes. Therefore, the highly truncated particular solution is of the form

$$\psi_{p2}^{(2)} = C_{M,2,2}e^{i2kx}\sin 2\pi y + \text{c.c.} \quad (3.26)$$

$$\theta_{p2}^{(2)} = C_{T,1,1}^{(2)}e^{ikx}\sin\pi y + C_{T,1,3}^{(2)}e^{ikx}\sin 3\pi y + C_{T,3,1}^{(2)}e^{i3kx}\sin\pi y + C_{T,5,1}^{(2)}e^{i(5kx-2wt)}\sin\pi y + C_{T,5,3}^{(2)}e^{i(5kx-2wt)}\sin 3\pi y + \text{c.c.} \quad (3.27)$$

The above truncation is obviously still very simple. But, it is important to stress that we are not trying to determine accurately the $O(|\Delta|)$ solution. We are mainly interested in studying how the higher order modes affect the evolution of the lower order modes. As we will see later, this truncation does in fact help to illustrate the fundamental role that these higher order modes play.

After determining the particular solution, the complete solution to $O(|\Delta|)$ can be written as

$$\psi = |\Delta|^{1/2}\frac{\gamma}{2}A(T)(e^{i(2kx-wt)} + e^{-i(2kx-wt)})\sin 2\pi y + |\Delta|\alpha_{M,2,2}|A(T)|^2(e^{i2kx} + e^{-i2kx})\sin 2\pi y \quad (3.28)$$

$$\begin{aligned}\theta = {} & \left(D_{T_c} + \Delta + |\Delta|\frac{(5k^2 - 3\pi^2 + 2F)k\pi\gamma(k\beta + ir(k^2 + \pi^2))}{2(k^2\beta^2 + r^2(k^2 + \pi^2)^2)}|A(T)|^2\right)(e^{ikx} + e^{-ikx})\sin\pi y \\ & + |\Delta|^{1/2}\frac{A(T)}{2}(e^{i(3kx-wt)} + e^{-i(3kx-wt)})\sin\pi y + |\Delta|\alpha_{T,1,3}|A(T)|^2(e^{ikx} + e^{-ikx})\sin 3\pi y \\ & + |\Delta|\alpha_{T,3,1}|A(T)|^2(e^{i3kx} + e^{-i3kx})\sin\pi y + |\Delta|\alpha_{T,5,1}|A(T)|^2(e^{i(5kx-2wt)} + e^{-i(5kx-2wt)})\sin\pi y \\ & + |\Delta|\alpha_{T,5,3}|A(T)|^2(e^{i(5kx-2wt)} + e^{-i(5kx-2wt)})\sin 3\pi y, \end{aligned} \quad (3.29)$$

where the α are defined in appendix A.

The $O(|\Delta|^{3/2})$ system of equations, which will be used to close the problem are

$$\frac{\partial}{\partial t}\nabla^2\psi^{(3)} + J(\theta^{(3)}, \nabla^2\bar{\theta}_c) + J(\bar{\theta}_c, \nabla^2\theta^{(3)}) + \beta\frac{\partial\psi^{(3)}}{\partial x} + r\nabla^2\psi^{(3)} = -\left\{\frac{\partial}{\partial T}\nabla^2\psi^{(1)} + J(\psi^{(1)}, \nabla^2\psi^{(2)}) + J(\psi^{(2)}, \nabla^2\psi^{(1)}) + J(\theta^{(1)}, \nabla^2\theta^{(2)}) + J(\theta^{(2)}, \nabla^2\theta^{(1)}) + J(\theta^{(1)}, \nabla^2\bar{\theta}_\Delta) + J(\bar{\theta}_\Delta, \nabla^2\theta^{(1)})\right\} \quad (3.30)$$

$$\frac{\partial}{\partial t}(\nabla^2 - 2F)\theta^{(3)} + J(\psi^{(3)}, \nabla^2\bar{\theta}_c) + J(\bar{\theta}_c, \nabla^2\psi^{(3)}) - 2FJ(\psi^{(3)}, \bar{\theta}_c) + \beta\frac{\partial\theta^{(3)}}{\partial x} + r\nabla^2\theta^{(3)} = -\left\{\frac{\partial}{\partial T}(\nabla^2 - 2F)\theta^{(1)} + J(\psi^{(1)}, \nabla^2\theta^{(2)}) + J(\psi^{(2)}, \nabla^2\theta^{(1)}) + J(\theta^{(1)}, \nabla^2\psi^{(2)}) + J(\theta^{(2)}, \nabla^2\psi^{(1)}) + J(\psi^{(1)}, \nabla^2\bar{\theta}_\Delta) + J(\bar{\theta}_\Delta, \nabla^2\psi^{(1)}) - 2FJ(\psi^{(1)}, \theta^{(2)}) - 2FJ(\psi^{(2)}, \theta^{(1)}) - 2FJ(\psi^{(1)}, \bar{\theta}_\Delta)\right\}, \quad (3.31)$$

where

$$\bar{\theta}_\Delta = (e^{ikx} + e^{-ikx})\sin\pi y. \quad (3.32)$$

As in the $O(|\Delta|)$ system of equations, the homogeneous solution is set equal to zero. To find the evolutionary equation, we only consider those forcing terms that are of the form of the lower order modes. But, these forcing terms are resonant, and if they are not removed, the solution will grow linearly in time and eventually invalidate the original series expansion. Therefore, the resonant forcing must be removed.

The first step in this procedure is to multiply (3.30) by $e^{-i2kx}\sin 2\pi y$ and (3.31) by $e^{-i3kx}\sin\pi y$ and integrate over the domain. The requirement that the resulting two equations be linearly dependent enables us to determine the evolutionary equation, which is of the form

$$\frac{dA}{dT} + \left(g_1|A|^2 - g_2\frac{\Delta}{|\Delta|}\right)A = 0 \quad (3.33)$$

where g_1 and g_2 are both complex coefficients and are defined in appendix B.

If (3.33) is multiplied by A^* (the complex conjugate of A) and summed with the complex conjugate of (3.33) multiplied by A, (3.33) can be rewritten as

$$\frac{d|A|^2}{dT} + (g_1 + g_1^*)|A|^4 - (g_2 + g_2^*)\frac{\Delta}{|\Delta|}|A|^2 = 0. \quad (3.34)$$

This is a classical equation first derived by Landau (1944) in his study of turbulent flows, and it is commonly referred to as the Landau equation. The coefficient of $|A|^4$ is called the Landau constant, and the sign of this constant which is positive in this case, determines the type of finite-amplitude behavior the perturbation undergoes. After noting that

$$\frac{d|A|^2}{dT} = -|A|^4\frac{d|A|^{-2}}{dT}, \quad (3.35)$$

(3.34) can be written in linear form as

$$\frac{d|A|^{-2}}{dT} + (g_2 + g_2^*)\frac{\Delta}{|\Delta|}|A|^{-2} - (g_1 + g_1^*) = 0, \quad (3.36)$$

which has the solution

$$|A|^2 = A_0^2 \Bigg/ \left[\frac{\Delta}{|\Delta|}\frac{(g_1 + g_1^*)}{(g_2 + g_2^*)}A_0^2 + \left(1 - \frac{\Delta}{|\Delta|}\times\frac{(g_1 + g_1^*)}{(g_2 + g_2^*)}A_0^2\right)e^{-(g_2+g_2^*)\frac{\Delta}{|\Delta|}T}\right], \quad (3.37)$$

where $A_0 = |A(T = 0)|$; $|A|^2$ increases monotonically with T and approaches a constant value

$$|A(T = \infty)|^2 = \frac{\Delta}{|\Delta|}\frac{\mathrm{Re}(g_2)}{\mathrm{Re}(g_1)} \quad (3.38)$$

as $T \rightarrow \infty$, which is independent of the initial conditions.

The evolution of $|A(T)|$ is shown for $k = 2.0$ in Fig. 1. The amplitude of wavenumbers $2k$ and $3k$ are plotted in Fig. 2. Both wave modes initially grow after which they settle into amplitude vacillation cycles. Since $|A(T)|$ becomes constant at large T, it may seem surprising that vacillation cycles occur. But, as can be seen in (3.28) for wavenumber $2k$ and in (3.29) for wavenumber $3k$, there is an interference effect between the propagating and stationary components which accounts for this vacillation. The period of the vacillation cycle is just that of the propagating wave (approximately 12 days).

This result is exactly analogous to the amplitude vacillations observed in the rotating annulus experiments with topography of Li et al. (1986). They find, in agreement with the present study, that the frequency of propagation and the frequency of vacillation are the same for all traveling waves. Therefore, in the laboratory annulus experiments with topography, as long as the baroclinic waves can draw a significant amount of energy from the topographically forced stationary wave, large amplitude vacillations can occur via the above interference mechanism.

The results of this section can be nicely summarized with the aid of a bifurcation diagram (Fig. 3). If we define the amplitude of the perturbation as

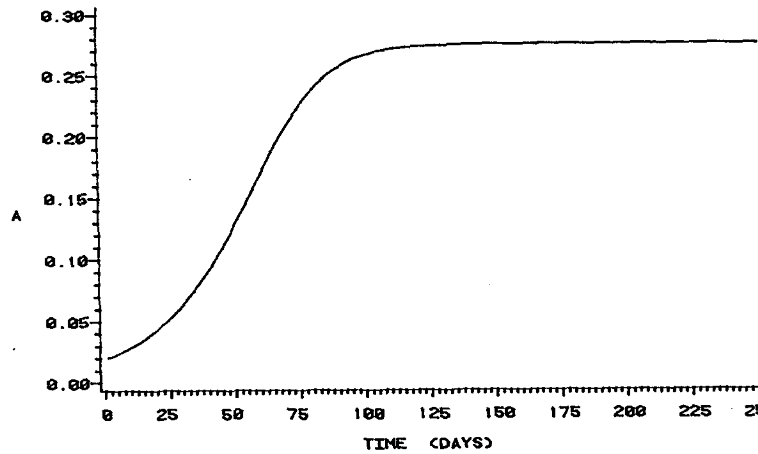

FIG. 1. The amplitude A as a function of time $\Delta = 0.013$, $A(0) = 0.02$, $r = 0.5$.

$$Z = |\Delta|^{1/2}A, \tag{3.39}$$

(3.33) becomes

$$\frac{dZ}{dt} + (g_1|Z|^2 - g_2\Delta)Z = 0. \tag{3.40}$$

We can use (3.40) to first determine the stability of the basic wave $Z = 0$ solution, respectively. The linearized form of (3.40) is

$$\frac{dZ'}{dt} - g_2\Delta Z' = 0, \tag{3.41}$$

which clearly indicates instability for $\Delta > 0$.

We will next consider the stability of the vacillating solution, which is, from (3.38),

$$|Z(t = \infty)|^2 = \frac{\mathrm{Re}(g_2)}{\mathrm{Re}(g_1)}\Delta. \tag{3.42}$$

The stability of this solution is more conveniently determined from (3.36) written in the form

$$\frac{d|Z|^{-2}}{dt} + (g_2 + g_2^*)\Delta|Z|^{-2} - (g_1 + g_1^*) = 0. \tag{3.43}$$

If we let

$$|Z|^{-2} = |Z(t = \infty)|^{-2} + Z', \tag{3.44}$$

the linear version of (3.43) becomes

$$\frac{dZ'}{dt} + (g_2 + g_2^*)\Delta Z' = 0. \tag{3.45}$$

Because $(g_2 + g_2^*) > 0$, these amplitude vacillations are always stable.

The final step is to redo the linear stability analysis of the basic wave with all the perturbation modes (both the lower and the higher order modes as well as a purely zonal perturbation mode) present. This was done, and it was found that the stability properties were only slightly altered.

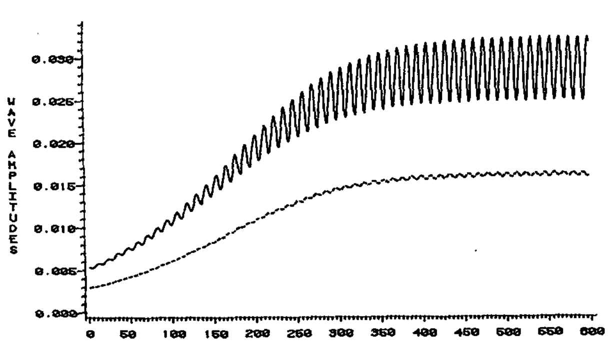

FIG. 2. The amplitudes of the lower-order perturbation modes as a function of time. Solid line: zonal wavenumber $2k$; dashed line: zonal wavenumber $3k$. $\Delta = 0.013$, $A(0) = 0.05$, $r = 0.5$.

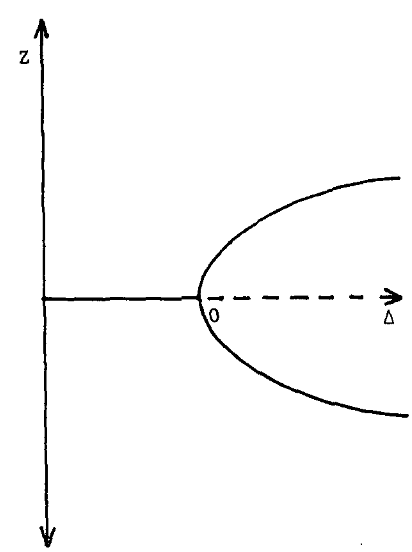

FIG. 3. The bifurcation diagram. $Z = |\Delta|^{1/2}A$ is shown schematically as a function of Δ. Solid line: linearly stable; dashed line: linearly unstable.

The total perturbation energy (kinetic plus available potential energy) is defined as:

$$\mathrm{PE} = \frac{1}{2}\iint\left\{\left(\frac{\partial\psi'}{\partial x}\right)^2 + \left(\frac{\partial\psi'}{\partial y}\right)^2 + \left(\frac{\partial\theta'}{\partial x}\right)^2 + \left(\frac{\partial\theta'}{\partial y}\right)^2 + 2F(\theta')^2\right\}dxdy, \quad (3.46)$$

where ψ' and θ' are the perturbation quantities in (3.1) and (3.2). If we substitute the expansion

$$\psi' = |\Delta|^{1/2}\psi^{(1)} + |\Delta|\psi^{(2)} + |\Delta|^{3/2}\psi^{(3)} + \cdots \quad (3.47)$$

$$\theta' = |\Delta|^{1/2}\theta^{(1)} + |\Delta|\theta^{(2)} + |\Delta|^{3/2}\theta^{(3)} + \cdots \quad (3.48)$$

into (3.46) and differentiate with respect to time, the energy equation to $O(|\Delta|^2)$ becomes

$$\left(\frac{\partial}{\partial t} + |\Delta|\frac{\partial}{\partial T}\right)\mathrm{PE} = -|\Delta|(\text{Term I}) - |\Delta|^{3/2}(\text{Term II}) - |\Delta|^2(\text{Term III}) \quad (3.49a)$$

where

$$\text{Term I} = \iint\left(\frac{\partial U_T}{\partial y} + \frac{\partial V_T}{\partial x}\right)(U_M^{(1)}V_T^{(1)} + U_T^{(1)}V_M^{(1)}) \times dxdy + 2\iint\frac{\partial U_T}{\partial x}U_M^{(1)}U_T^{(1)}\,dxdy + 2\iint\frac{\partial V_T}{\partial y}V_M^{(1)}V_T^{(1)}dxdy + 2F\iint\frac{\partial\bar\theta}{\partial y} \times V_M^{(1)}\theta^{(1)}dxdy + 2F\iint\frac{\partial\bar\theta}{\partial x}U_M^{(1)}\theta^{(1)}dxdy + r\iint\{|\nabla\psi^{(1)}|^2 + |\nabla\theta^{(1)}|^2\}dxdy \quad (3.49b)$$

$$\text{Term II} = \iint\left(\frac{\partial U_T}{\partial y} + \frac{\partial V_T}{\partial x}\right)(V_T^{(1)}U_M^{(2)} + U_T^{(1)}V_M^{(2)} + V_T^{(2)}U_M^{(1)} + U_T^{(2)}V_M^{(1)})dxdy + 2\iint\frac{\partial U_T}{\partial x}(U_M^{(2)}U_T^{(1)} + U_M^{(1)}U_T^{(2)})dxdy + 2\iint\frac{\partial V_T}{\partial y}(V_M^{(2)}V_T^{(1)} + V_M^{(1)}V_T^{(2)})dxdy + 2F\iint\frac{\partial\bar\theta}{\partial y}(V_M^{(1)}\theta^{(2)} + V_M^{(2)}\theta^{(1)})dxdy + 2F\iint\frac{\partial\bar\theta}{\partial x}(U_M^{(1)}\theta^{(2)} + U_M^{(2)}\theta^{(1)})dxdy + 2r\iint\{\nabla\psi^{(1)}\cdot\nabla\psi^{(2)} + \nabla\theta^{(1)}\cdot\nabla\theta^{(2)}\}dxdy \quad (3.49c)$$

$$\text{Term III} = \iint\left(\frac{\partial U_T}{\partial y} + \frac{\partial V_T}{\partial x}\right)(U_M^{(3)}V_T^{(1)} + V_M^{(3)}U_T^{(1)})dxdy + 2\iint\frac{\partial U_T}{\partial x} \times U_M^{(3)}U_T^{(1)}dxdy + 2\iint\frac{\partial V_T}{\partial y} \times V_M^{(3)}V_T^{(1)}dxdy + 2F\iint\frac{\partial\bar\theta}{\partial y} \times V_M^{(3)}\theta^{(1)}dxdy + 2F\iint\frac{\partial\theta}{\partial x} \times U_M^{(3)}\theta^{(1)}dxdy + r\iint\nabla\psi^{(1)}\cdot\nabla\psi^{(3)}dxdy. \quad (3.49d)$$

The definitions

$$U_T = -\frac{\partial\bar\theta}{\partial y}, \quad V_T = \frac{\partial\bar\theta}{\partial x} \quad (3.50)$$

for the basic wave winds, and

$$U_M^{(i)} = -\frac{\partial\psi^{(i)}}{\partial y}, \quad V_M^{(i)} = \frac{\partial\psi^{(i)}}{\partial x},$$

$$U_T^{(i)} = -\frac{\partial\theta^{(i)}}{\partial y}, \quad V_T^{(i)} = \frac{\partial\theta^{(i)}}{\partial x} \quad (3.51)$$

for the perturbation wind field are made, where M denotes barotropic, T baroclinic, and $i = 1, 2, 3$. It should be noted that $\theta^{(3)}$ which can be shown to be arbitrary, has been set equal to zero.

Equations (3.49b), (3.49c), and (3.49d) each have several types of energy conversion terms present. There are four types of barotropic energy conversions that involve a variety of zonal and meridional fluxes of momentum both parallel and perpendicular to the basic wave wind field. The two types of baroclinic energy conversions include both zonal and meridional heat fluxes. It can be shown, at each order, that all six energy conversions are of the same magnitude and are therefore equally important.

The $O(|\Delta|)$ energy conversion terms in (3.49b) add to zero since the asymptotic series expansion is about a neutrally stable solution. Physically, this fact does nothing more than illustrate that at neutral stability, the perturbation gains the same amount of energy from the basic wave as it loses to dissipation. Nevertheless, to determine the total energy transferred between the basic wave and perturbation, we will retain these terms.

The $O(|\Delta|^{3/2})$ energy conversions are associated with the interference process. The perturbation energy undergoes cycles of growth and decay depending upon the phase difference between the propagating and stationary wave components. This process, which causes the perturbation amplitude to vacillate, does not represent the incipient growth of the disturbance. That is accomplished by the slow $O(|\Delta|^2)$ terms. These $O(|\Delta|^2)$ energy conversions, whose summation must initially be positive, can be shown to add to zero as t approaches infinity. Therefore, when studying the energetics of the final vacillation state, we will only consider the energy conversions to $O(|\Delta|^{3/2})$.

Similarly, for the basic wave, a lengthy calculation yields to lowest order

$$\frac{\partial}{\partial t}(\mathrm{BWE} + \mathrm{PE}) = -|\Delta|^{3/2} \iint r(\nabla^2\bar{\theta} - \nabla^2\theta_f)\theta^{(3)}_{1,1}\,dxdy - 2r|\Delta|^{3/2} \iint \{\nabla\psi^{(1)}\cdot\nabla\psi^{(2)} + \nabla\theta^{(1)}\cdot\nabla\theta^{(2)}\}dxdy, \tag{3.52}$$

where PE is defined in (3.46) and the basic wave energy is defined as

$$\mathrm{BWE} = \frac{1}{2}\iint \left\{\left(\frac{\partial\bar{\theta}_T}{\partial x}\right)^2 + \left(\frac{\partial\bar{\theta}_T}{\partial y}\right)^2 + 2F(\bar{\theta}_T)^2\right\}dxdy \tag{3.53}$$

where

$$\bar{\theta}_T = (D_T + |\Delta|\theta^{(2)}_{1,1} + |\Delta|^{3/2}\theta^{(3)}_{1,1} + \cdots) \times (e^{ikx} + e^{-ikx})\sin\pi y \tag{3.54}$$

and $\theta^{(2)}_{1,1}$ and $\theta^{(3)}_{1,1}$ are higher order corrections to the amplitude of the basic wave. Thus, the temporal variation of the total energy of the system depends on two effects. First, there is an external energy source or sink that varies with the phase difference between the $O(|\Delta|^{3/2})$ correction to the basic wave and the forcing. Second, the total energy undergoes a fluctuation that depends upon the phase relationship between the stationary and propagating components of wavenumbers $2k$ and $3k$.

The barotropic and baroclinic energy conversions and the energy lost to dissipation are shown in Fig. 4. The $O(|\Delta|)$ energy conversions and $O(|\Delta|)$ dissipation, which exactly balance each other, undergo a slow steady increase which eventually levels off at a constant value. Superimposed on the $O(|\Delta|)$ conversions are the fluctuating $O(|\Delta|^{3/2})$ conversions which are responsible for the amplitude vacillation cycle seen in Fig. 2. The slow conversions, which are of $O(|\Delta|^2)$ and cause the initial slow growth of the perturbation, are not shown in Fig. 4.

The $O(|\Delta|)$ baroclinic and barotropic energy conversions are very important. It is these energy conversions that dominate the energetics of the interaction between the stationary planetary wave and the synoptic scale perturbation. As we discussed in the Introduction, observational and GCM studies have shown three basic properties of this wave–wave interaction: 1) Available potential energy is transferred from the planetary-scale stationary eddies to the synoptic-scale transient eddies, 2) kinetic energy is transferred from the synoptic scale back to the stationary waves, 3) the available potential energy transfer is larger than that of the kinetic energy transfer (the available potential energy transfer and the kinetic energy transfer are represented by the baroclinic and barotropic energy conversions in Fig. 4, respectively). Therefore, the energetics of the wave–wave interaction seen in this simple weakly nonlinear model qualitatively agrees with that observed in the atmosphere.

4. Spectral model solutions

In this section, the weakly nonlinear solution will be used to specify the truncation for a low-order spectral model. The motivation for this method comes from (3.34), which can be written as

$$\frac{d|A|^2}{dT} + \left[(g_{1,1} + g_{1,3} + g_{5,1} + g_{5,3}) \times |A|^2 - g_2\frac{\Delta}{|\Delta|}\right]|A|^2 = 0, \tag{4.1}$$

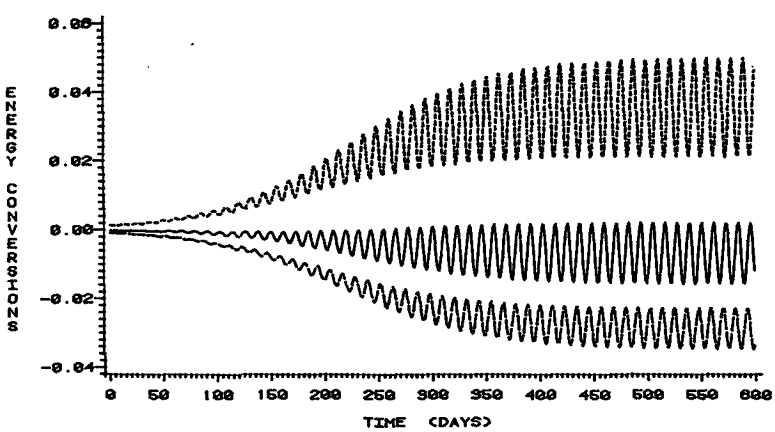

FIG. 4. The weakly nonlinear energy conversions as functions of time. Solid line: energy transfer by barotropic processes from the basic wave to the perturbation; short dashed line: energy transfer by baroclinic processes from the basic wave to the perturbation; long dashed line: the energy the perturbation loses to dissipation. $\Delta = 0.013$, $A(0) = 0.05$, $r = 0.5$.

where

$$(g_1 + g_1^\dagger) = g_{1,1} + g_{1,3} + g_{5,1} + g_{5,3}. \qquad (4.2)$$

Each $g_{m,n}|A|^2A(T)$ term in (3.34) represents the wave–wave interaction of one of the higher-order modes with the two lower-order modes. Therefore, (4.1) shows that the finite-amplitude evolution of the larger lower-order modes depends upon the smaller higher-order modes.

The relative importance of each of the higher-order modes is determined by the magnitude of the $g_{m,n}$, which are

$$|g_{1,1}| = 5.56$$
$$|g_{1,2}| = 46.90$$
$$|g_{5,1}| = 17.58$$
$$|g_{5,3}| = 5.90.$$

Since $g_{1,3}$ is the largest of the $g_{m,n}$, the (1,3) mode is the most important higher-order mode in determining the slow evolution of the lower-order modes. The spectral expansion is chosen to consist of the lower- and higher-order modes of the weakly nonlinear solution, which are

$$\psi = C_{M,2,2}e^{i2kx}\sin 2\pi y + \text{c.c.} \qquad (4.3)$$

$$\theta = (D_{T_c} + \Delta + C_{T,1,1})e^{ikx}\sin \pi y + C_{T,1,3}e^{ikx}\sin 3\pi y + C_{T,3,1}e^{i3kx}\sin \pi y + C_{T,5,1}e^{i5kx}\sin \pi y + C_{T,5,3}e^{i5kx}\sin 3\pi y + \text{c.c.} \qquad (4.4)$$

In an analogous way, this method can be used to suggest a truncation level for a much larger spectral model. The lower-order modes are first determined from a complete linear stability analysis. The higher-order modes then arise from all possible combinations of wave–wave interactions amongst the lower-order modes. The truncation level then consists of the lower and the higher-order modes.

We will now test the accuracy of the weakly nonlinear solutions by comparing them with the spectral model results. The spectral solutions are obtained by numerically integrating the spectral system with a sixth-order Runge-Kutta scheme. The amplitudes associated with wavenumbers $2k$ and $3k$ are plotted in Fig. 5. The same initial conditions, supercriticalilty factor Δ, basic wavenumber k, and value of r are used as in the weakly nonlinear solutions.

Although the spectral and weakly nonlinear solutions are similar, they do not agree exactly. The amplitudes of the vacillations differ, and the average perturbation amplitudes are not the same. At first, this may appear to be puzzling, since one intuitively expects that the weakly nonlinear and spectral solutions would converge as Δ approaches zero. But, as we will see in the following discussion, this expectation is incorrect. It was shown earlier that the evolution of the lower-order modes depends crucially upon the presence of the stationary (1,3) mode. (A wave with zonal wavenumber m and meridional wavenumber n is referred to as the (m,n) mode.) However, a more complete linear stability analysis of the basic wave would show that the (1,3) mode does make a small contribution to the $O(|\Delta|^{1/2})$ growing perturbation. This mode also propagates since it must have the same frequency as the lower-order (2,2) and (3,1) modes. The present weakly nonlinear solution does not take into account the interaction of the propagating portion of the (1,3) mode with the lower-order modes. Therefore, when Δ is small, we expect similar but not identical solutions for the weakly nonlinear and spectral models.

We will now examine the sensitivity of the spectral model solutions to the presence of the higher-order modes by comparing the spectral solutions with and without the higher-order modes present. The complete spectral solutions, with all the higher-order modes present, are first numerically integrated from an initial state represented by the unstable basic wave. The re-

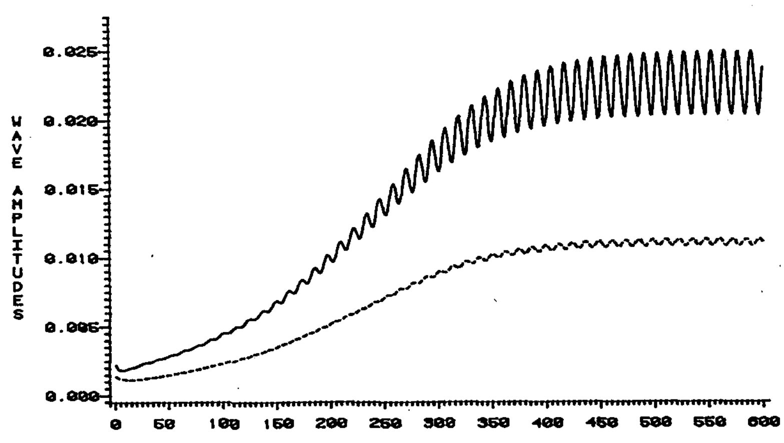

FIG. 5. The amplitudes of the lower-order perturbation modes as a function of time. Solid line: zonal wavenumber $2k$; dashed line: zonal wavenumber $3k$. The same initial conditions are used as in Fig. 2.

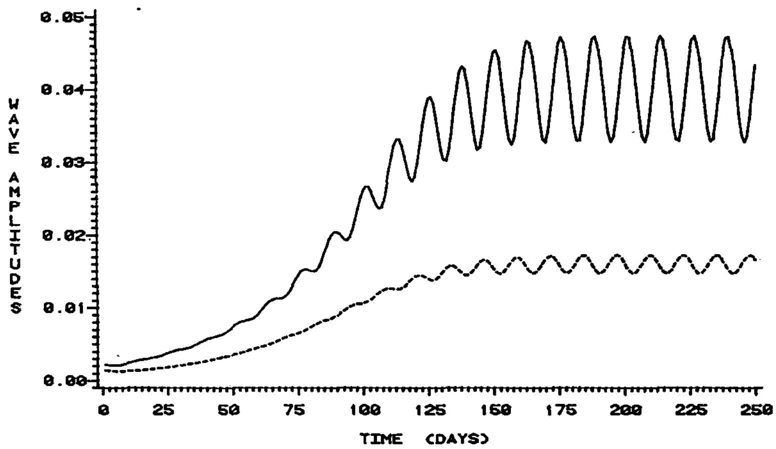

FIG. 6. As in Fig. 5 except $\Delta = 0.032$.

sulting amplitudes of wavenumbers $2k$ and $3k$ are shown as functions of time in Fig. 6. The behavior of this solution is quite similar to that in Fig. 5 where Δ is smaller.

In contrast, when the higher-order modes (i.e., $C_{T,1,3}$, $C_{T,5,1}$ and $C_{T,5,3}$) are set equal to zero, the solution is drastically different (Fig. 7). The amplitudes of the perturbation modes are about four times larger and the amplitude vacillation has completely disappeared. An explanation for the difference in perturbation amplitudes will be presented later. To account for the absence of the vacillation cycle, it is important to remember that these amplitudes vacillate because of the interference between the stationary and propagating components of the same wave mode. But, the stationary components of the (2,2) and (3,1) modes arise from the interaction of the (1,3) mode with the basic wave. Therefore, when $C_{T,1,3}$ equals zero, the (2,2) and (3,1) modes no longer have stationary components and so a vacillation cycle cannot occur.

The total influence of the transient eddy perturbation on the stationary basic wave is studied by examining the temporal evolution of the basic wave. It is found, for the complete spectral model, that the basic wave also undergoes an amplitude vacillation. During this vacillation, the time-averaged basic wave energy is only slightly different from its initial value. Therefore, on average, the basic wave gains the same amount of energy from the forcing as it loses to the perturbation. This indicates that the transient eddy perturbation has neither a dissipative nor an amplifying influence on the stationary basic wave. On the other hand, when the higher order modes are set equal to zero, the basic wave energy is reduced by 34 percent. Therefore, this perturbation has a dissipative influence on the basic wave. These results indicate that the exclusion of the higher-order modes can even lead to a misinterpretation of the net effect of the perturbation on the basic wave.

We will now examine the role these higher-order modes play. One may expect that it is the direct energy transfer from the lower to the higher-order modes that

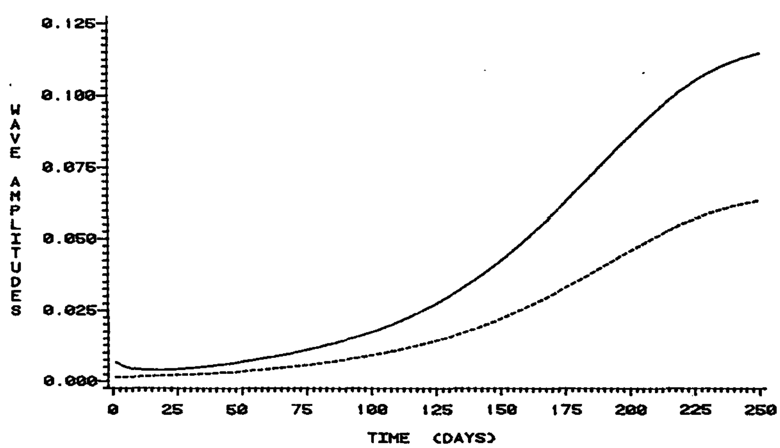

FIG. 7. As in Fig. 6 except the higher-order modes are set equal to zero.

accounts for the differences in the spectral model solutions with and without the higher-order modes. However, this process is in fact very minor. In order to understand why the higher-order modes affect the amplitude of the lower-order modes, it is necessary to examine the solution of the weakly nonlinear system to $O(|\Delta|^{3/2})$. The propagating component of wavenumbers $2k$ and $3k$ is

$$\psi'_{\text{prop}} = \left(|\Delta|^{1/2}\frac{\gamma}{2}A(T) + |\Delta|^{3/2}C^{(3)}_{M,2,2}(T)\right) \times e^{i(2kx-wt)}\sin 2\pi y + \text{c.c.} \quad (4.5)$$

$$\theta'_{\text{prop}} = |\Delta|^{1/2}\frac{A(T)}{2}e^{i(3kx-wt)}\sin \pi y + \text{c.c.} \quad (4.6)$$

It can be shown that $C^{(3)}_{M,2,2}$ is proportional to the sum of two terms: 1) a linear term in $A(T)$ which represents the initial linear growth of the perturbation when the basic wave in unstable, 2) nonlinear terms of the form $|A|^2A(T)$ that result from the interaction between the lower-order and higher-order modes. Both of these effects control the phase relation between the amplitude of the barotropic stream function $|\Delta|^{1/2}\gamma/2A(T) + |\Delta|^{3/2}C^{(3)}_{M,2,2}(T)$ in (4.5) and the amplitude of the baroclinic stream function $|\Delta|^{1/2}A(T)/2$ in (4.6).

The importance of this process can be seen from the $O(|\Delta|^2)$ energy conversion terms in (3.49d) where it is shown that the growth rate of the perturbation depends upon the phase relation between the slowly varying amplitudes of the barotropic and baroclinic stream functions. The small $O(|\Delta|^{3/2})$ wave amplitude $C^{(3)}_{M,2,2}$, which represents a correction to the phase difference between the amplitudes of the barotropic and baroclinic modes, determines whether the perturbation grows or decays. The linear contribution to $C^{(3)}_{M,2,2}$ causes the phase difference to increase beyond that at neutral stability in order to release the energy of the basic wave. In contrast, the nonlinear part of $C^{(3)}_{M,2,2}$ reduces the phase difference set up by the instability of the basic wave.

These results therefore show that the small higher-order modes primarily act to reduce the phase difference between the barotropic and baroclinic stream functions and hence the growth rate of the perturbation. It is for this reason that the perturbation amplitudes are significantly reduced when the higher-order modes are set equal to zero.

5. Summary and conclusions

The purpose of this study was to examine the nonlinear instability of forced baroclinic Rossby waves in order to better understand the interaction of planetary-scale stationary eddies with synoptic-scale transient eddies. This stability problem was investigated with both weakly nonlinear and low-order spectral models. The truncation for the spectral model is chosen from both the lower and higher-order modes of the asymptotic series expansion associated with the weakly nonlinear solution.

In the weakly nonlinear model, synoptic-scale amplitude vacillation cycles are found that are driven by the linear interference between propagating lower-order and stationary higher-order perturbation modes with the same zonal and meridional wavenumbers. Therefore, the period of the vacillation is just that of the propagating perturbation modes. At first, the presence of these stationary synoptic-scale perturbation modes may seem surprising. But, it was shown that the propagating lower-order perturbation modes can interact with each other to force particular higher-order stationary modes. These stationary modes can, in turn, interact with the stationary basic wave to force an entire spectrum of higher-order stationary modes. This mechanism for driving an amplitude vacillation cycle is regarded as being secondary to the sequence of baroclinic growth, barotropic decay seen both in numerical models and observational studies by Simmons and Hoskins (1978) and Randel and Stanford (1985), respectively. These amplitude vacillations, in contrast to the cycles found in this study, are driven by instabilities of the vertically sheared zonal flow.

The question of the total effect of the synoptic-scale perturbation upon the stationary basic wave is also addressed. It is found that the time-mean basic wave energy equals its initial value. This indicates that the transient eddy perturbation neither amplifies nor dissipates the stationary basic wave. It is difficult to relate these results to the atmosphere because of the crude external forcing (i.e., no zonal flow or topography), but, nevertheless, they still illustrate the importance of the stationary planetary wave forcing when studying the net effect of the transient eddies.

The energetics of the weakly nonlinear solution were examined and compared with that seen in observational analyses and GCM experiments. These studies indicate three basic properties of the interaction between planetary-scale stationary eddies and synoptic-scale transient eddies: 1) eddy available potential energy is transferred from the planetary-scale stationary eddies to the synoptic-scale transient eddies, 2) eddy kinetic energy is transferred in the opposite direction from the synoptic-scale transient eddies upscale to the planetary-scale stationary eddies, 3) the eddy available potential energy transfer dominates the transfer of eddy kinetic energy. These three properties are all found in the weakly nonlinear solution. Therefore, this simple weakly nonlinear model does indeed capture much of the fundamental behavior of wave–wave interactions seen in the atmosphere.

The weakly nonlinear and spectral model solutions are found to be qualitatively similar. A comparison of the spectral solutions with and without the higher-order modes present shows that the evolution of the synoptic-scale perturbation is sensitive to the presence of the higher-order modes. This suggests that the wave–wave interaction is very sensitive to the detailed structure of the perturbation. Without these higher-order modes, too much energy is transferred from the planetary scale to the synoptic scale. This is because the higher-order modes control the nonlinear growth of the lower-order modes. In this particular case, the higher-order modes reduce the phase relation between the amplitude of the lower-order modes. This process decreases the growth rate of the lower-order modes. These results also illustrate the dramatic affect that small amplitude components can have on a spectral model solution.

The primary limitation of this model is that a vertically sheared zonal flow has been excluded. Because the vertically sheared zonal flow is an important energy source for synoptic-scale disturbances, the presence of the zonal flow is expected to alter the phase relation among the various growing synoptic modes. This should significantly affect the amount of energy transferred from the planetary scale to the synoptic scale. In addition, if topography is also included, a more realistic forcing of the stationary wave could be obtained.

Another important factor worth considering is the severe truncation used for the lower-order perturbation modes. A two-mode truncation is chosen in order to enable an analytic solution to be obtained. But, to more accurately determine the stability properties of the stationary basic wave, it is necessary to include more modes in the truncation. Therefore, the dependency of the solution on both a vertically sheared zonal flow and the truncation level should be studied in future research.

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APPENDIX A

Expressions for $\alpha_{m,n}$

$$\alpha_{T,3,1} = \frac{(5k^2 - 3\pi^2 + 2F)\gamma}{\frac{g}{16\pi^2 D_{T_c}}\left(\frac{\beta}{\pi} - \frac{r(k^2 + 9\pi^2)}{ik\pi}\right) + \left(\frac{3\beta}{\pi} - \frac{r(9k^2 + \pi^2)}{ik\pi}\right)} \tag{A1}$$

$$\alpha_{T,1,3} = \frac{-\alpha_{T,3,1}g}{16\pi^2 D_{T_c}} \tag{A2}$$

$$\alpha_{M,2,2} = -\frac{\alpha_{T,3,1}\left(\frac{3\beta}{\pi} - \frac{r(9k^2 + \pi^2)}{ik\pi}\right)}{(6k^2 + 6\pi^2 - 4F)D_{T_c}} \tag{A3}$$

$$\alpha_{T,5,1} = -\frac{(5k^2 - 3\pi^2 + 2F)k\pi\gamma}{2w(25k^2 + \pi^2 + 2F) - 5\beta k - ir(25k^2 + \pi^2)} \tag{A4}$$

$$\alpha_{T,5,3} = \frac{(5k^2 - 3\pi^2 + 2F)k\pi\gamma}{2[2w(25k^2 + 9\pi^2 + 2F) - 5\beta k - ir(25k^2 + 9\pi^2)]} \tag{A5}$$

where

$$g = 16k^2 D_{T_c} - \frac{\left(\frac{2\beta}{\pi} - \frac{r(4k^2 + 4\pi^2)}{ik\pi}\right)\left(\frac{3\beta}{\pi} - \frac{r(9k^2 + \pi^2)}{ik\pi}\right)}{(6k^2 + 6\pi^2 - 4F)D_{T_c}}. \tag{A6}$$

APPENDIX B

Coefficients in the Evolutionary Equation

$$\begin{aligned}
&\left(\mu_M \frac{(9k^2 + \pi^2 + 2F)}{2} - \frac{\mu_T(4k^2 + 4\pi^2)\gamma}{2}\right)g_1 \\
&= ik\pi\alpha_{T,1,3}(16\mu_T(-k^2 + \pi^2) - 2\mu_M\gamma \\
&\quad \times (-3k^2 + 5\pi^2 + 2F)) + ik\pi\alpha_{T,5,1}(32\mu_T k^2 \\
&\quad - 2\mu_M\pi\gamma(-21k^2 + 3\pi^2 - 2F)) + ik\pi\alpha_{T,5,3} \\
&\quad \times (-8\mu_T(2k^2 + \pi^2) - \mu_M\gamma(21k^2 + 5\pi^2 + 2F)) \\
&\quad + i\gamma\mu_T 4k^4\pi^2(5k^2 - 3\pi^2 + 2F) \\
&\quad \times (k\beta + ir(k^2 + \pi^2)) - i\mu_M \frac{(3k^2 + 3\pi^2 - 2F)}{2} \\
&\quad \times \frac{(5k^2 - 3\pi^2 - 2F)k\pi\gamma(k\beta + ir(k^2 + \pi^2))}{2(k^2\beta^2 + r^2(k^2 + \pi^2)^2)}
\end{aligned} \tag{B1}$$

$$\begin{aligned}
&\left(\mu_M \frac{(9k^2 + \pi^2 + 2F)}{2} - \mu_T \frac{(4k^2 + 4\pi^2)\gamma}{2}\right)g_2 \\
&= -ik\pi(8\mu_T k^2 - \mu_M\gamma(3k^2 + 3\pi^2 - 2F))
\end{aligned} \tag{B2}$$

where

$$\mu_M = \frac{w(4k^2 + 4\pi^2)}{2} + k\beta + i\frac{r}{2}(4k^2 + 4\pi^2) \tag{B3}$$

$$\mu_T = (3k\pi(k^2 + \pi^2) - 2Fk\pi)D_{T_c}. \tag{B4}$$

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