Development of Boundary Layer Rolls from Dynamic Instabilities

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ABSTRACT

The development of atmospheric boundary layer rolls from the inflection point and parallel instabilities is examined analytically using several three-dimensional linear models of flow in a neutral, rotational fluid. These models are formulated so that either arbitrary or observed background wind profiles can be examined easily to see which roll modes would likely occur. Necessary and sufficient conditions for the development of particular atmospheric modes are determined. These conditions are expressed as polynomials in the critical (eddy) Reynolds number $Re_\ast$ and depend on Fourier coefficients of a given height-dependent background wind profile. The preferred values of orientation angle $\theta$ and aspect ratio $A$, which describe the expected roll geometry, are assumed to be those that produce the smallest values of $Re_\ast$.

The ability of the models to successfully reproduce the modes arising from the inflection point and parallel instability mechanisms is tested by using idealized wind profiles to approximate the mean wind in the boundary layer. For the Ekman profile, the preferred values of $\theta$ and $A$ are found to agree with those given by larger models, indicating that the simpler analytical models are incorporating the crucial information contained in the wind profile.

Finally, a direct comparison with observations of atmospheric boundary layer rolls is given using a mean wind profile obtained during the 1981 West German KonTur experiment. For this case, the preferred values of $\theta$, $A$ and $Re_\ast$ associated with the pure inflection point instability mechanism agree well with their observed values. It is this easy, direct comparison between the model results and observations that is the significant contribution of this analytical modeling approach.

1. Introduction

Boundary layer rolls are usually observed when long parallel bands of cumulus clouds form. Although most of the rolls produce individual cumulus clouds lined up like a string of pearls, rolls can also produce continuous cloud bands (Kuettnet, 1959, 1971). These observations have led to the identification of certain characteristics, or features, that are common to most boundary layer rolls. In general, the rolls are aligned nearly parallel to the mean wind direction in the boundary layer (Brown, 1980). A characteristic roll wavelength $L$ of approximately one to eight times the depth $z_T$ of the boundary layer is also observed (Kelly, 1984). Two dynamic instability mechanisms are generally associated with the development of boundary layer rolls: the parallel instability and the inflection point instability (Brown, 1980).

The parallel instability was first studied by Lilly (1966). Through the Coriolis terms, roll circulations are able to extract energy from the large-scale shear parallel to the roll axis, as shown by Kaylor and Faller (1972) and as reviewed in the energetics analysis given in section 2. Lilly's initial analysis suggests that the modes originating from the parallel instability are meaningful only when the Reynolds number, $Re_\ast$, which is a measure of the wind stress upon the fluid, is small. This result has led most investigators to believe that this mechanism is of secondary importance, since the magnitude of the eddy Reynolds number is generally large in the atmosphere (Brown, 1980). However, some investigators believe it has significant effects on the development of boundary layer rolls (e.g., Etling, 1971; Gammelsrød, 1975).

The inflection point instability can be viewed as a generalized Kelvin–Helmholtz type instability (Brown, 1980). When an inflection point exists in the large-scale mean wind profile, the roll circulations may extract energy from the large-scale mean wind shear perpendicular to the roll axis. An inflection point in the velocity profile is a necessary, but not sufficient condition for this instability mechanism to exist (Rayleigh, 1880; Drazin and Howard, 1966). However, one way of obtaining both necessary and sufficient conditions for the existence of an instability mechanism is by the application of bifurcation and stability theorems in an analysis of a mathematical model (e.g., Shirer, 1987).

Three approaches have generally been used to study
the development of boundary layer rolls. One approach is to calculate the production terms in the roll kinetic energy budget by using tower, Doppler radar, or aircraft measurements that have been made in the presence of roll circulations (LeMone, 1973, 1976; Rabin et al., 1982; Brümmer, 1985). The magnitudes of the production terms are then compared with one another to determine which term is dominant. The instability mechanism associated with the dominant production term is then assumed to explain the development of roll circulations. A second approach is to approximate the mean wind in the boundary layer by an idealized profile and then to develop a linear model for study of the instabilities able to produce boundary layer circulations (Brown, 1972; Lilly, 1966; Asai, 1970; Asai and Nakasui, 1973). A third approach that links a model directly to the observations is to formulate the model so that observed, preferably pre-roll, mean wind profiles could be used in the stability analysis and then the model results could be compared with the observations of the rolls (Shirer, 1986; Shirer and Brümmer, 1986). This last approach is the one we choose here for study of the modes developing from the individual, as well as the mixed, dynamic mechanisms of inflection point and parallel instabilities.

In the present study, we examine three low-order models of three-dimensional flow in a neutral atmosphere. For the atmosphere, an eddy Reynolds number is the appropriate forcing parameter for the study of the inflection point and parallel instabilities. Necessary and sufficient conditions for instability in each model are determined by performing a linear analysis of the motionless solution to the equations (Gelaro, 1987a). These conditions are expressed as polynomials in the critical eddy Reynolds number Re, and depend on the Fourier coefficients of an arbitrary height-dependent mean wind profile, as well as on an orientation angle θ and an aspect ratio A. The preferred values of θ and A are assumed to be those that produce the smallest values of Re.

One limitation in applying a linear spectral model is that the results are valid only for slightly supercritical values of the forcing parameter. However, the fact that smooth transitions, or bifurcations, between nonlinear solutions are determined by an analysis of the appropriately linearized equations (e.g., Iooss and Joseph, 1980; Gelaro, 1987b) suggests that a linear model is the best initial one to use for study of these dynamic instability mechanisms. Indeed, the results given by the linear spectral models should be equivalent to the results given by numerical studies in which the fastest growing perturbations are examined (Lilly, 1966; Asai and Nakasuji, 1973), since both methods essentially involve determination of the most active harmonics (Lauferweiler, 1987) and are valid only for slightly supercritical values of the eddy Reynolds number. A comparison between the results of these two modeling approaches is one way to test the ability of a truncated spectral model to successfully reproduce the modes arising from the inflection point and parallel instability mechanisms.

In the following section, we develop the low-order models used in this study. We discuss in detail only the two-wavenumber model for the pure inflection point instability case, since the methodology for the development of the other model is identical.

2. Model development

Here we develop two linear spectral models to study the formation of boundary layer rolls. One model, having two vertical wavenumbers, is developed to study the case of the inflection point instability, and we label it M2. The other model, having two vertical wavenumbers and nonzero Coriolis terms, is formulated to study the case of mixed inflection point and parallel instabilities, and we denote it as M2F.

We begin with the three-dimensional shallow Boussinesq system for a rotating fluid in a neutral atmosphere. We consider the circulations to be perturbations superimposed upon a hydrostatic, horizontally moving flow. Consequently, the Boussinesq system is expressed as a system of perturbation equations. Since the basic current Vb(z) has larger vertical and temporal scales than those of the roll circulation, we do not constrain the basic current to be a solution to the Boussinesq equations. Moreover, we assume that the basic state is time-independent and so investigate only the method by which the mean wind shear affects the initial development of the roll circulations.

We assume that the domain is finite in the vertical and is both infinite and cyclically continuous in the horizontal. The characteristic lengths in the horizontal are represented by Lx in the eastward direction x, and Ly in the northward direction y, and the domain top is represented by zτ, where 0 < x ≤ Lx, 0 < y ≤ Ly, and 0 < z ≤ zτ. We assume that the flow is 2π-periodic in the horizontal directions, and that the upper and lower boundaries are rigid and stress-free.

a. The dimensionless forms

In order to study the solutions to the three-dimensional Boussinesq system, it is wise to first cast the problem in dimensionless form. This technique can be used to reveal parameters that are of interest and to study their contributions more readily. As in Shirer (1986) we begin with a system written in the eastward and northward coordinates x and y, and then rotate it into roll coordinates, where one axis is parallel to the rolls and the other axis is perpendicular to the rolls. We define an aspect ratio

\[ A = 2\pi f/L, \]  

(2.1)

where \( L \) is a function of \( L_x \) and \( L_y \), is the horizontal wavelength of the rolls (Shirer, 1986). We also define the dimensionless Coriolis parameter
\[ f^* = f z_T / \nu n^2, \]  
(2.2)
in which \( f \) is the Coriolis parameter and \( \nu \) is the eddy viscosity, and the eddy Reynolds number,
\[ \text{Re} = |V(z_T)| z_T / \nu, \]  
(2.3)
in which \( |V(z_T)| \) is the wind speed at the top of the domain.

Using appropriate scalings of the variables (Stensrud, 1985), we obtain a dimensionless Boussinesq system that is identical to the one in Shirer (1986) for a neutral atmosphere. This system is
\[ \begin{align*}
\partial u^*/\partial t^* + ReV_T^* \cdot \nabla u^* + Reu^* \partial U^*/\partial z^* \\
- f^* A^{-1} u^* + \partial \varphi^*/\partial x^* - A^{-1} \nabla u^* = 0, \\
(2.4)
\partial v^*/\partial t^* + ReV_T^* \cdot \nabla v^* + Rev^* \partial V^*/\partial z^* \\
+ f^* A^{-1} v^* + \partial \varphi^*/\partial y^* - A^{-1} \nabla v^* = 0, \\
(2.5)
\partial w^*/\partial t^* + ReV_T^* \cdot \nabla w^* + A^{-2} \partial \psi^*/\partial z^* \\
- A^{-1} \nabla w^* = 0, \\
(2.6)
\nabla \cdot \mathbf{v}^* = 0, \\
(2.7)
in which we have defined
\[ \nabla = i \partial / \partial x^* + j \partial / \partial y^* + k \partial / \partial z^*, \]  
(2.8)
\[ \nabla^2 = A^2 \partial^2 / \partial x^* + A^2 \partial^2 / \partial y^* + \partial^2 / \partial z^2. \]  
(2.9)

b. The spectral equations

Since this study is restricted to circulations that occur in a cyclic, rectangular domain, the horizontal and vertical basis functions for representing solutions to (2.4)–(2.7) may be expressed using a Fourier series. This series is used to represent each of the velocity fields, in which we use one horizontal harmonic and two vertical harmonics. The use of one horizontal harmonic is consistent with the observations of one characteristic horizontal wavelength (e.g., Brümmer, 1985). The use of two vertical harmonics is necessary because Shierer (1986) has shown that one vertical harmonic is not sufficient to capture the modes arising from the inflection point instability.

In the spectral expansion, all four possible combinations of trigonometric functions in the horizontal must be used if we are to represent all of the possible phase relationships created by the linear terms of the three-dimensional system (2.4)–(2.7) (Shirer, 1986). Thus, we do not limit the solutions to be two-dimensional, although they are easily seen as special cases in the expansion below. Also, the boundary conditions in the vertical restrict the functions that can be used in the Fourier series. An appropriate form for the expansion of the variable \( h = u^* + v^* + \varphi^* \) is
\[ \begin{align*}
\begin{align*}
& h = [h_1 \cos(y^*) + h_2 \sin(y^*) + h_3 \cos(x^*) \\
& + h_5 \sin(x^*)] \cos(qz^*) + [h_3 \cos(y^*) + h_4 \sin(y^*) \\
& + h_7 \cos(x^*) + h_8 \sin(x^*)] \cos(nz^*), \\
(2.10)
& \text{and for the expansion of } w^* \text{ is}
\quad w^* = [w_1 \cos(y^*) + w_2 \sin(y^*) + w_3 \cos(x^*) \\
& + w_6 \sin(x^*)] \sin(qz^*) + [w_3 \cos(y^*) + w_4 \sin(y^*) \\
& + w_7 \cos(x^*) + w_8 \sin(x^*)] \sin(nz^*), \\
(2.11)
& \text{in which } n \text{ and } q \text{ are vertical integral wavenumbers such that } n \neq q.
\end{align*}
\end{align*}
\]
c. The roll form

A 32-coefficient linear spectral model is obtained by substituting (2.10)–(2.11) in (2.4)–(2.7), multiplying the resulting equations by the appropriate basis functions, and integrating the result over the domain \( 0 \leq x^* \leq 2\pi, 0 \leq y^* \leq 2\pi, \) and \( 0 \leq z^* \leq \pi \) to form a system of ordinary differential equations. We can reduce the 32-coefficient linear spectral model to a 16-coefficient one by using the spectral form of \( \nabla \cdot \mathbf{v}^* = 0. \) This approach is standard for three-dimensional incompressible systems (Dutton, 1976) and is equivalent to forming a vorticity equation in a two-dimensional system.

An examination of the resulting 16-component model shows that the system is composed of two independent eight-component subsystems:
\[ R_1 = (u_1, \ldots, u_4, v_1, \ldots, v_4), \]
\[ R_2 = (u_5, \ldots, u_8, v_5, \ldots, v_8). \]

Physically, we can see from the spectral expansions (2.10)–(2.11) that \( R_1 \) is associated with rolls having axes parallel to the \( x^* \)-axis and \( R_2 \) is associated with rolls having axes parallel to the \( y^* \)-axis. Also, since the two roll systems have identical forms, we may choose either system to study the modes arising from the inflection point and parallel instability mechanisms. Here we choose the \( R_1 \) subsystem, so that the rolls are defined to be parallel to the \( x^* \)-axis. The resulting linearized form of the two vertical wavenumber model \((M2f)\) is
\[ \begin{align*}
\hat{u}_1 = \text{Re}[\Lambda_2 - \Lambda_4(q)]u_2 - 2\Lambda_3(q)v_2 - H_1(n, q)u_4 \\
- H_2(n, q)u_4 + f^* A^{-1} v_1 - D(q)u_1, \\
(2.12)
\hat{u}_2 = \text{Re}[\Lambda_2 - \Lambda_4(q)]u_1 + 2\Lambda_3(q)v_1 + H_1(n, q)v_3 \\
+ H_2(n, q)u_3 + f^* A^{-1} v_2 - D(q)u_2, \\
(2.13)
\hat{u}_3 = \text{Re}[\Lambda_2 - \Lambda_4(q)]u_4 - 2\Lambda_3(n)v_4 - H_1(q, n)v_2 \\
- H_2(q, n)u_2 + f^* A^{-1} v_3 - D(n)u_3, \\
(2.14)
\hat{u}_4 = \text{Re}[\Lambda_2 - \Lambda_4(n)]u_3 + 2\Lambda_3(n)v_3 + H_1(q, n)v_1 \\
+ H_2(q, n)u_1 + f^* A^{-1} v_4 - D(n)u_4, \\
(2.15)
\hat{v}_1 = \text{Re}[H_3(q)v_2 - H_4(n, q)v_4] \\
- f^* q^2 A^{-1}(A^2 + q^2) u_1 - D(q)v_1, \\
(2.16)
\hat{v}_2 = \text{Re}[H_3(q)v_1 + H_4(n, q)v_3] \\
- f^* q^2 A^{-1}(A^2 + q^2) u_2 - D(q)v_2, \\
(2.17)
\[ \dot{v}_3 = \text{Re}\{H_3(n)v_4 - H_4(q, n)v_2\} - f^*n^2A^{-1}(A^2 + n^2)^{-1}u_3 - D(n)v_3, \quad (2.18) \]

\[ \dot{v}_4 = \text{Re}\{-H_3(n)v_3 + H_4(q, n)v_1\} - f^*n^2A^{-1}(A^2 + n^2)^{-1}u_4 - D(n)v_4, \quad (2.19) \]

in which we have used the definitions

\[ H_1(m, r) = (m - r)\Gamma_1(m, r)/m + (m + r)\Gamma_1(m, r)/m, \quad (2.20) \]

\[ H_2(m, r) = \Gamma_2(m, r) + \Gamma_4(m, r), \quad (2.21) \]

\[ H_3(r) = \Lambda_2 + [(A^2 - 3r^2)/(A^2 + r^2)]\Lambda_4(r), \quad (2.22) \]

\[ H_4(m, r) = [(-A^2r + 2mr^2 + r^3)/m(A^2 + r^2)]\Gamma_3(m, r) + [(A^2r + 2mr^2 - r^3)/m(A^2 + r^2)]\Gamma_4(m, r), \quad (2.23) \]

\[ D(r) = A^{-1}(A^2 + r^2), \quad (2.24) \]

and where the Fourier coefficients \( \Lambda_i \) and \( \Gamma_i \) (i = 1, 2, 3, 4) of the wind profile are defined below in (2.25)–(2.32). The model M2(q, n) is obtained by setting \( f^* = 0 \) in (2.12)–(2.19).

d. The controlling parameters

An examination of the spectral system reveals the existence of parameters that affect the critical value of \( \text{Re} \) for roll development. One such parameter is the aspect ratio \( A \), which is equivalent to twice the domain height \( z_T \) divided by the horizontal wavelength \( L \) of the roll. The other controlling parameters are the Fourier coefficients \( \Lambda_1 \) and \( \Lambda_2 \) of the mean wind, and the Fourier coefficients \( \Lambda_3(r), \Lambda_4(r) \) and \( \Gamma_1(m, r) \) of the mean wind shear, in which \( m \) and \( r \) represent the vertical wavenumbers \( q \) and \( n \) of the mean wind. Of primary concern in this article is a comparison of the model results with those of observations. Since observational data are given in terms of the mean wind, and not the mean wind shear, it is wise to integrate by parts the Fourier coefficients of the mean wind shear to obtain only Fourier coefficients of the mean wind. Here, the appropriate forms for the Fourier coefficients are

\[ \Lambda_1 = \pi^{-1} \int_0^\pi U^*dz^*, \quad (2.25) \]

\[ \Lambda_2 = \pi^{-1} \int_0^\pi V^*dz^*, \quad (2.26) \]

\[ \Lambda_3(r) = -\pi^{-1} \int_0^\pi U^* \cos(2rz^*)dz^*, \quad (2.27) \]

\[ \Lambda_4(r) = -\pi^{-1} \int_0^\pi V^* \cos(2rz^*)dz^*, \quad (2.28) \]

\[ \Gamma_1(m, r) = -\pi^{-1} \int_0^\pi U^* \cos[(m + r)z^*]dz^*, \quad (2.29) \]

\[ \Gamma_2(m, r) = -\pi^{-1} \int_0^\pi V^* \cos[(m + r)z^*]dz^*, \quad (2.30) \]

\[ \Gamma_3(m, r) = -\pi^{-1} \int_0^\pi U^* \cos[(m - r)z^*]dz^*, \quad (2.31) \]

\[ \Gamma_4(m, r) = -\pi^{-1} \int_0^\pi V^* \cos[(m - r)z^*]dz^*, \quad (2.32) \]

in which \( U^* \) and \( V^* \) denote components of the mean wind that are parallel to and perpendicular to the roll axis, respectively (Fig. 1).

Effects of the background wind are represented in the models by certain of the Fourier coefficients of a trigonometric series. To see which ones, we consider an idealized mean wind profile whose \( U^* \)-component obeys

\[ U^*(z^*) = U_0 + U_1 \cos(2z^*) + U_2 \cos(4z^*) + U_3 \sin(2z^*) + U_4 \sin(4z^*). \quad (2.33) \]

We set \( q = 1 \) and \( n = 2 \) and use the definitions of the Fourier coefficients (2.25)–(2.32) to obtain the identical expression

\[ U^*(z^*) = \Lambda_1 - 2\Lambda_3(1) \cos(2z^*) - 2\Lambda_3(2) \cos(4z^*) + \{[35\pi/128]\Gamma_1(1, 2) - [75\pi/128]\Gamma_3(1, 2)\}

\times \sin(2z^*) + \{-[175\pi/256]\Gamma_1(1, 2)

- [105\pi/256]\Gamma_3(1, 2)\} \sin(4z^*). \quad (2.34) \]

Thus, choices of 1 and 2 for \( q \) and \( n \) allow inclusion in M2f of the first three cosine terms and the first two sine terms of the Fourier series of the background wind profile.

More generally, we find that different choices for \( q \) and \( n \) isolate other trigonometric terms of the Fourier series. For example, the choices of 1 and 3 for \( q \) and \( n \)

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**Fig. 1.** The relationship between the original \((x_0, y_0)\) system and the usual \((x, y)\) roll coordinate system. The orientation angle between the roll axis and east is denoted by \( \theta \) (after Shirer, 1986).
yield only the first four cosine terms, while choices of
1 and 4 yield the first two cosine terms and both sine
terms, along with a higher order cosine term. For any
given profile, certain terms of the Fourier series are
larger in magnitude than the other terms. The wave-
number choice that captures these larger terms would
be expected to yield the best model results. Because
the mean wind varies smoothly in the boundary layer,
it is likely that only the first few terms of a Fourier
series are needed to accurately represent the mean wind
profile. From the above example, we therefore expect
that M2f(1, 2) would generally produce the best results,
since this choice of wavenumbers maximizes the infor-
mation from the lower order trigonometric terms
obtained from the mean wind profile.

In the previous discussion, we have included wave-
number 1 consistently as one of the vertical wavenum-
bers, because wavenumber 1 represents the actual roll
circulation that we assume fills the domain. In addition,
we find that the spectral models yield better results
when wavenumber 1 is included.

e. The energy cycle

In order to elucidate the available energy sources for the
roll circulation, we review the energetics analysis
of Kaylor and Faller (1972), but for a neutrally stable
atmosphere, and simultaneously identify the corre-
sponding terms in the model M2f. Following the spec-
tral model formulation, we choose the roll to be parallel
to the \( x^* \)-axis and the unit vector. Thus, \( U^*(x^*) \)
is the wind component parallel to the roll and \( V^*(x^*) \)
is the wind component perpendicular to the roll, so that

\[
V^*(x) = U^*(x^*)i + V^*(x^*)j. \tag{2.35}
\]

Thus, since the roll is defined to be parallel to the
\( x^* \)-axis, we may neglect the \( x^* \)-derivatives (e.g., \( \partial / \partial x^* \)
= 0) in the governing equations (2.4)–(2.7). For two-
dimensional flow, it is convenient to divide the kinetic
energy into two parts: one part is the roll kinetic energy

\[
K = \frac{1}{2} \int_0^{2\pi} \int_0^\pi (v^{*2} + A^2 w^{*2})dz^*dy^*, \tag{2.36}
\]

and the other part is the longitudinal kinetic energy,

\[
E = \frac{1}{2} \int_0^{2\pi} \int_0^\pi u^{*2}dz^*dy^*. \tag{2.37}
\]

The corresponding energy definitions for the spectral
model M2f are

\[
2K = v_1^2 + v_2^2 + v_3^2 + v_4^2
+ A^2 q^{-2}(v_1^2 + v_2^2) + A^2 n^{-2}(v_3^2 + v_4^2), \tag{2.38}
\]

\[
2E = u_1^2 + u_2^2 + u_3^2 + u_4^2, \tag{2.39}
\]

where we have used the spectral form of the continuity
equation (2.7) to relate \( w_i \) to \( v_i \) (\( i = 1, \ldots, 4 \)). Equa-
tions (2.4)–(2.7) may be transformed into energy rate
equations by multiplying (2.4)–(2.7) by \( u^*, v^* \) and
\( A^2 w^* \), respectively, and then by integrating over
the domain. If we denote temporal derivatives by an over-
dot, then we now have

\[
\dot{K} = \int_0^{2\pi} \int_0^\pi [-v^* w^* \text{ Re} \frac{\partial V^*}{\partial z^*}]dz^*dy^* - \int_0^{2\pi} \int_0^\pi [f^* A^{-1} u^* v^*]dz^*dy^*
+ \int_0^{2\pi} \int_0^\pi [-A^{-1} \nabla v^*]^2 - A [\nabla w^*]^2]dz^*dy^*, \tag{2.40}
\]

\[
\dot{E} = \int_0^{2\pi} \int_0^\pi [-u^* w^* \text{ Re} \frac{\partial U^*}{\partial z^*}]dz^*dy^* + \int_0^{2\pi} \int_0^\pi [f^* A^{-1} u^* v^*]dz^*dy^* + \int_0^{2\pi} \int_0^\pi [-A^{-1} \nabla u^*]^2]dz^*dy^*. \tag{2.41}
\]

The corresponding energy rate equations for M2f are

\[
\dot{K} = \text{Re}(n^2 - q^2)n^{-1} q^{-1} [\Gamma_4(n, q) - \Gamma_2(n, q)](v_3v_2 - v_4v_1) - f^* A^{-1}[u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4]
+ q^{-2}A^{-1}(A^2 + q^2)[v_1^2 + v_2^2] - n^{-2}A^{-1}(A^2 + n^2)[v_3^2 + v_4^2], \tag{2.42}
\]
\[ \dot{E} = 2 \text{Re}\Lambda_3(q)(v_1 u_2 - v_2 u_1) + 2 \text{Re}\Lambda_3(n)(v_3 u_4 - v_4 u_3) \]

II

\[ + \text{Re}[(n + q)\Gamma_1(n, q)/n + (n - q)\Gamma_3(n, q)/n][v_3 u_2 - v_4 u_1] \]

II

\[ + \text{Re}[(q + n)\Gamma_1(q, n)/q + (q - n)\Gamma_3(q, n)/q][v_4 u_4 - v_2 u_3] \]

II

\[ + f^* A^{-1}[u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4] - A^{-1}(A^2 + q^2)[u_1^2 + u_2^2] - A^{-1}(A^2 + n^2)[u_3^2 + u_4^2]. \]

(2.43)

3. Preferred roll geometry

The stability of the basic solution to each of the non-linear forms of the models is determined by an analysis of the linearized equations (e.g., Gelaro, 1987a). A nontrivial solution to the linear model, which signals a transition from one solution to another in the non-linear model (Gelaro, 1987b), exists if the determinant of the linearized system, written in matrix form, vanishes; here this calculation gives a value for the critical eddy Reynolds number $Re_c$. This point of neutral stability, or bifurcation point, is the value of $Re$ at which a nonlinear solution emanates from the basic solution. Finding the minimum value of this bifurcation point, which here is the smallest value at which stability is exchanged from the motionless solution to the dynamic solution, is equivalent to finding the fastest growing wave, because this value is a function of the aspect ratio $A$ and the orientation angle $\theta$ of the roll (Laufsweiler, 1987; Stensrud, 1987). Therefore, minimum values of $Re_c$ correspond to maximum values of the growth rate, and so the preferred roll alignments and aspect ratios are assumed to be those that yield the smallest values of $Re_c$. These preferred alignments and aspect ratios can be easily compared with observations such as those from the KonTur experiment (Shirer and Brümmer, 1986). We discuss in detail only the analysis of $M2$ for the pure inflection point instability case, since the methodology is identical for the analysis of $M2_f$.

The linearized equations for studying the stability of the basic, or trivial, solution to $M2$ are given by (2.12)-(2.19) with $f^* = 0$. Solutions to these equations have the form $\exp(\alpha t)$, and so the values of the characteristic exponent $\alpha$ measure the growth rates of the perturbations. The bifurcation point $Re_c$ is given by the vanishing of the real part of $\alpha$. When the imaginary part $\omega$ of $\alpha$ does not vanish, then by definition a Hopf bifurcation from a steady to a temporally periodic solution occurs. The period of this branching solution approaches $2\pi/\omega$ as the value of $Re$ approaches that of $Re_c$ (Pyle, 1987). Here the transition from a motionless solution to a temporally periodic one represents the development of propagating rolls from the roll-free background state.
Values of the characteristic exponent $\alpha$ are governed by an eighth-degree polynomial in $\alpha$, which can be factored into two quadratic equations and their complex conjugates. The value of $\text{Re}_c$ is determined by the vanishing of the real part of any of the roots of the four quadratic equations. We find that $\text{Re}(\alpha) = 0$ occurs when

$$|\omega| = \text{Re}_c[(H_3(n)D(q) + H_3(q)D(n))/[D(n) + D(q)],$$

(3.1)

and so the Hopf bifurcation equation for $M2$ is

$$\text{Re}_c^2([H_3(n) - H_3(q)]^2D(n)D(q) + [H_4(n, q)H_4(q, n)][D(n) + D(q)]^2) + [D(n) + D(q)]^2D(n)D(q) = 0,$$

(3.2)

where $H_3(r)$, $H_4(m, r)$ and $D(r)$ are defined in (2.22)–(2.24) and $\text{Re}_c$ is the critical Reynolds number for the pure inflection point instability case. Upon inspection of (3.2), we see that real roots for $\text{Re}_c$ exist, provided that $H_4(n, q)H_4(q, n) < 0$. We find that (3.2) depends only on Fourier coefficients of the mean wind perpendicular to the roll, the aspect ratio $A$, and the vertical wavenumbers $q$ and $n$.

The Hopf bifurcation equation for the two vertical wavenumber model $M2f$ representing the mixed inflection point and parallel instability mechanisms depends on the Fourier coefficients of both the roll-parallel and roll-perpendicular components of the mean wind.

4. Idealized wind profiles

Using simple wind profiles to approximate the mean wind in the atmospheric boundary layer, we are able to study the abilities of the various models to represent the onset of modes originating from the inflection point and parallel instabilities. For each model, we calculate the Fourier coefficients and then determine the preferred values of the orientation angle $\theta$ and the aspect ratio $A$ for the various modes, as well as the corresponding critical eddy Reynolds number $\text{Re}_c$. We use these preferred values to compare the model results for the cases of pure inflection point, pure parallel, and the mixed inflection point/parallel instability modes. These comparisons give crucial evidence that the simple models are capturing the modes properly and provide a basic understanding of the differences between the two dynamic instability mechanisms.

a. Sinusoidal profiles

A simple profile that supports both inflection point and parallel modes is given by (Stensrud, 1987)

$$U^*(z^*) = \sin(z^*) + \epsilon \sin(2z^*) + 1,$$

(4.1)

$$V^*(z^*) = 0,$$

(4.2)
where we vary the value of \( \varepsilon \) between 0.0 and 1.0 and we choose \( f^* = 0.5 \) in M2f. We note that because of the form of the dimensionless terms, we must specify the profiles so that \( U_\infty^2(\pi) + V_\infty^2(\pi) = 1 \). For \( \varepsilon = 0.0 \) in (4.1)–(4.2), we expect that the parallel instability dominates, since the wind profile does not contain an inflection point (Fig. 2a). For all of these profiles, the value of \( \mbox{Re}_p \) is the same since the analysis of Shirer (1986) includes only one Fourier coefficient \( \Delta_3(1) \), which is represented here by \( \sin(z^*) \). An inflection point is present in the wind profile when \( \varepsilon > 0 \), and the value of \( \varepsilon \) controls the height \( z_{ip} \) of the inflection point within the domain (Fig. 2). Once the value of \( \varepsilon \) has been increased sufficiently, we expect the inflection point instability to have an effect on the roll development. For \( \varepsilon = 0.0 \) (Fig. 2a), Mf yields \( \theta_p = 0^\circ \), \( A_p = 0.5 \) and \( \mbox{Re}_p = 39 \), while M2(1, 2) yields only negative values of \( \mbox{Re}_p \), so that instability only occurs via the parallel mechanism. However, when \( \varepsilon = 0.3 \) (Fig. 2b), we find that \( \theta_i = 90^\circ \), \( A_i = 1.4 \), and \( \mbox{Re}_i = 38 \), while the preferred values of \( \theta_p, A_p \), and \( \mbox{Re}_p \) remain the same. For this case, the inflection point modes dominate since \( \mbox{Re}_i < \mbox{Re}_p \), and the preferred values of \( \theta \) and \( A \) are \( 90^\circ \) and 1.4, respectively. The roll orientation has switched from being parallel to the mean wind shear vector to being perpendicular to it. As we further increase the value of \( \varepsilon \), we find that the value of \( \mbox{Re}_i \) continues to decrease. At \( \varepsilon = 0.6 \) (Fig. 2c), we have \( \theta_i = 90^\circ \), \( A_i = 1.2 \) and \( \mbox{Re}_i = 18 \), while at \( \varepsilon = 1.0 \) (Fig. 2d) we find \( \mbox{Re}_i = 11 \).

The mixed case of inflection point and parallel instabilities given by M2f agrees with the combined results from Mf and M2 because the preferred orientation angle shifts abruptly from 0° to 90° once the value of \( \varepsilon \) is increased past 0.3. The fact that the shift in preferred values occurs near \( \varepsilon = 0.3 \) is illustrated in Fig. 3, where the values of \( \mbox{Re}_m \) are contoured as functions of both \( \theta \) and \( A \). The preferred values of \( \theta \) and \( A \) are those that yield the lowest real values of \( \mbox{Re}_m \), so the preferred values for this case are \( \theta_m = 90^\circ \) and \( A_m = 1.2 \). Notably, the behavior of the contoured values of \( \mbox{Re}_m \) is the same as the behavior of the contoured values of the growth rate seen in other studies (e.g., Lilly, 1966); but minimum values in \( \mbox{Re}_m \) are equivalent to maximum values in the growth rate. Moreover, a local minimum value of \( \mbox{Re}_m \) occurs at \( \theta = 2^\circ \) and \( A = 0.5 \), and these values correspond to the parallel modes.

b. The Ekman profile

One commonly used wind profile for approximating the wind in the boundary layer is the Ekman profile, and it has been used extensively to study the development of boundary layer rolls (Lilly, 1966; Asai and Nakasuji, 1973). We recall from the Introduction that one way to test the spectral models is to determine if the preferred values of \( \theta \) and \( A \) given by M2, Mf and M2f are similar to those given by these other studies. Here a 0° orientation denotes rolls that are aligned parallel to the westly geostrophic wind vector, and positive orientations are to the left of this wind vector (Fig. 1). For an Ekman profile having Ekman depth \( D \), the inflection point instability is associated with orientation angles of 10° to 20° and a dimensionless wavelength \( L/D \) of 12, and the parallel instability is associated with orientation angles of -10° to -20° and a dimensionless wavelength \( L/D \) of 20.

The Ekman profile can be written as

\[
\begin{align*}
U_\infty^2(z^*) &= |V_\infty^2| \left[ 1 - \exp(-z^*/D^*) \cos(z^*/D^*) \right], \\
V_\infty^2(z^*) &= |V_\infty^2| \exp(-z^*/D^*) \sin(z^*/D^*),
\end{align*}
\]

where \( D^* = D_T/z_T \) is a dimensionless Ekman depth. Owing to the choice of dimensionless forms, we find that \( U_\infty^2(\pi) + V_\infty^2(\pi) = 1 \). Thus, we must choose a value for \( |V_\infty^2| \) to ensure that this relationship holds. For example, we find that \( 0.99 \leq |V_\infty^2| \leq 1.55 \) for 0.25 \( \leq D^* \leq 3.0 \).

In order to compare the results of M2f with those from larger models, we must determine the average
circulation depth in these studies. The results of Faller and Kaylor (1966) suggest that when $D$ is used as a vertical scaling parameter, the average circulation depth $z_D$ is $4D$ to $6D$ for most studies using an Ekman profile; here this corresponds approximately to a value of $D^*$ between 0.5 and 0.8. In the following example, we choose $D^* = 0.7$.

Results of M2f(1, 2) using an Ekman profile (4.3)–(4.4) are summarized in Fig. 4, where we have contoured the values of $Re_m$ as functions of $\theta$ and $A$. There are two minimum values of $Re_m$: the global minimum is at $Re_m = 55$ for which $\theta = -8^\circ$ and $A = 0.5$, and the local minimum is at $Re_m = 91$ for which $\theta = 22^\circ$ and $A = 0.5$. The results of Lilly (1966) for $Re_c = 93$ yield $\theta = -20^\circ$ and $A = 0.43$, and for $Re_c = 157$ yield $\theta = 8^\circ$ and $A = 0.76$. These results qualitatively agree well with those given by M2f(1, 2). The values of the Reynolds numbers are slightly different, as are the values of $A$, but the change of $\theta$ from negative values to positive values as the magnitude of $Re_c$ is increased is seen. This behavior indicates that the spectral model M2f can successfully capture the mixed dynamic modes produced by a much larger model. Moreover, since we have seen that Mf and M2 have also produced results for the sinusoidal wind profiles that agree with those of M2f, we suspect that the spectral models can produce reasonably good results using observed wind data. A comparison using data from the 1981 West German KonTur experiment is the topic of the next section.

**5. Observed wind profile from the KonTur experiment**

KonTur was a convection and turbulence experiment that was conducted over the German Bight, a small area in the North Sea to the west of Denmark, during September and October 1981 (Brügger et al., 1985). During the KonTur experiment, data were collected on several days containing boundary layer rolls capped by cloud streets. From these days, we chose a case with a near-neutral stratification in the boundary layer to test the ability of the low-order models to adequately represent the observed roll circulations.

As measured by the buoyancy term in the roll kinetic energy analysis performed by Brünger (1985), the day that had a stratification closest to neutral was 26 September 1981. The observations on this day were taken ahead of an approaching warm front where boundary layer rolls under a layer of altostratus were observed. The data collection procedure consisted of flying two aircraft, equipped with gust probes, in L-shaped patterns at eight different levels in the boundary layer (Brünger, 1985). Each leg of an L-shaped pattern was approximately 40 km in length. Instantaneous vertical soundings were taken before and after the flight levels were flown. The data collection was completed in approximately two hours.

According to Brünger (1985), the top $H$ of the boundary layer was estimated to be the level at which the turbulence dropped markedly; although he estimated $H$ to be 500 m ± 50 m, probably the level was near 530 m, the highest cross-roll flight leg. In addition, at 1120 UTC, the inversion height $z_i$ was reported to be 440 m. Because of the absence of clouds during most of the measurements, the distance between the maxima in the relative humidity time series that were obtained during the cross-roll legs flown in the upper part of the roll layer was used to estimate the roll spacing $L$; an average value of 1675 m was found so that the average value of the aspect ratio $A = 2H/L$ was equal to approximately 0.63 for $H = 530$ m. However, the roll spacing varied along the cross-roll flight paths from as narrow as 800 m to as wide as 2700 m. These values may be used to calculate the limits of the range of observed aspect ratios for comparison with the model results. The orientation of the rolls was taken to be along the mean layer wind vector, with a conservative estimated error of ±10°. Because the winds were reported in roll coordinates, the observed orientation angle $\theta$ was $0^\circ$ ± 10°. Finally, the mean eddy Reynolds number $Re$ was calculated using (2.3) with $v = 17$ m$^2$ s$^{-1}$; this value was obtained from the usual definition by integrating over the high-frequency (sub-roll scale) portion of the observed momentum cospectra and then by multiplying the result by the vertical gradient of the mean wind (Brünger, 1985). However, this value of $v$ has a factor of two uncertainty, leading to a range of estimated values of $Re$ (see Table 1).

Vertical profiles of potential temperature $\theta$, mixing ratio $m$, along-roll wind component ($U$), and cross-roll wind component ($V$) are shown in Fig. 5. The solid lines denote instantaneous profiles, while the dots and horizontal bars denote means and standard deviations calculated at each of the eight flight levels. The dashed lines in Fig. 5b represent the estimated mean wind profile that we determined by eye to maintain the general shape of the instantaneous profiles as well as to remain within the standard deviations at the eight flight levels. In addition, we required that the mean profile agreed with the observed 15 m s$^{-1}$ minimum surface wind speed.
An examination of Fig. 5a shows that the value of the virtual potential temperature was approximately constant between 100 m and the inversion height $z_i$ and then rapidly increased above that height. Thus, the well-mixed boundary layer was capped by a stable layer, producing a net slightly stable domain containing the roll circulations.

A moist energetics analysis of the data from 26 September was conducted by Brümmer (1985) as well. The results presented in Fig. 6 of Brümmer (1985) suggest that a nearly steady state had been reached, with a balance being maintained between energy gain from the Reynolds stress [Term I in (2.40)] and energy loss from the buoyancy term, which was formulated using virtual potential temperature (e.g., see Stensrud, 1987). The transfer of energy to the roll circulations through the action of the Coriolis term was negligible. Because the Reynolds stresses were positive throughout the roll layer, Brümmer (1985) concluded that the inflection point mechanism was responsible for the roll circulations on that day.

In order to use an infinitesimal perturbation approach to test the above conclusions, we ideally would use an observed mean wind profile that was measured before the rolls existed. However, during the KonTur experiment, the aircraft were not sent on a mission unless cloud streets already were observed on satellite pictures. Hence, we are forced to use winds that were measured after the rolls were formed, but to do so we must argue that the mean wind was not altered significantly by the roll circulations. Unfortunately, we do not know how long the roll circulations existed before being observed from the aircraft. However, a very crude estimate of the maximum distance upwind that cloud streets might have existed was obtained from the 1247 UTC NOAA 7 satellite picture. This picture reveals that clouds were present for approximately 200 km upwind, or southeast, of the measuring site. As the boundary layer wind speed was approximately 18 m s$^{-1}$, the air would have interacted with the streets for no more than three hours.

To crudely estimate how much the mean wind might have changed while interacting with the roll circulations, we use Fig. 6 of Brümmer (1985). He shows the dimensional values of the roll-scale Reynolds stresses corresponding to Terms I and II in (2.40) and (2.41); these terms represent energy sources for the rolls but energy sinks for the mean wind. Because these values were obtained from the 40 km flux legs, these stresses, which are the integrands of the energy conversion integrals, provide good estimates of the rate of change of mean kinetic energy in the volume containing the streets. The maximum values of the integrands of I and II are between 1 and $1.5 \times 10^{-4} \, \text{m}^2 \, \text{s}^{-3}$, implying that the maximum rate of change of the mean wind speed was approximately $0.4 \, \text{m} \, \text{s}^{-1} \, \text{h}^{-1}$. Thus, after three hours, the mean wind would have been altered by no more than 1 m s$^{-1}$, leading us to conclude that the rolls did not exist long enough to alter the background winds appreciably. Rather, the large changes in the instantaneous winds given in Fig. 5 were more likely caused by the changing synoptic conditions associated with the approaching warm front.

Thus, we may assume that the rolls did not alter the mean wind profile. We calculate the values of the Fourier coefficients (2.25)–(2.32) by using the values of the mean wind given in Fig. 5b at 25 m intervals. With the procedure discussed in section 3, we determine the preferred values of orientation angle $\theta$ and aspect ratio.
Table 1. Summary of model results (labeled “Preferred”) using $z_T = 550$ m for the mean wind profile in Fig. 5b. For each model, only the vertical wavenumber combination in parentheses giving the smallest value of the critical eddy Reynolds number is shown. The aspect ratio is given by $2z_T/L$, where the roll spacing $L$ had a mean value of 1675 m and varied from 800 m to 2700 m. The cited range of observed eddy Reynolds number originates from the factor of two uncertainty in the value of the eddy viscosity that was estimated by Brümmer (1985) to be 17 m$^2$ s$^{-1}$.

<table>
<thead>
<tr>
<th>Orientation angle</th>
<th>Aspect ratio</th>
<th>Eddy Reynolds number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model version</td>
<td>Observed</td>
<td>Range</td>
</tr>
<tr>
<td>M2(1,3)</td>
<td>0°</td>
<td>-10° to 10°</td>
</tr>
<tr>
<td>M2f(1,3)</td>
<td>0°</td>
<td>-10° to 10°</td>
</tr>
<tr>
<td>M3(1,3,4)</td>
<td>0°</td>
<td>-10° to 10°</td>
</tr>
<tr>
<td>Mf(1)</td>
<td>0°</td>
<td>-10° to 10°</td>
</tr>
</tbody>
</table>

$A = 2z_T/L$ for a domain height $z_T = 550$ m. This domain height is the upper limit of the depth of the roll circulation reported in Table 1 of Brümmer (1985) and on Fig. 5b. However, inspection of the height dependence of the variances and roll-scale fluxes in his article reveals that the most likely top of the roll circulation is near 530 m, which is the level of the highest flight legs. Moreover, use of this domain height is consistent with the observation that the roll circulations, and even the clouds early in the observation period, extended well into the capping inversion. In the analysis that follows, we conclude that a dynamic instability mechanism is responsible for the roll circulations if the results from several of the models are nearly identical, with preferred values of $\theta$, $A$, and $Re_e$ that are close to their observed ranges.

The results from the models Mf, M2, M2f and M3, which is a three-wavenumber pure inflection point model M2, are listed in Table 1; only the wavenumber combination, given in parentheses, producing the smallest value of $Re_e$ is cited for each model. From the three inflection point models M2, M2f and M3 we obtain preferred values of orientation angle, aspect ratio and critical eddy Reynolds number that are close to their observed values. In Fig. 6 we show the value of $Re_e$ produced by M2f(1, 3) for a range of values of $A$ and $\theta$, and we see that two minima occur, implying that two modes are possible; significantly, the one having the smaller value of $Re_e$ and hence producing the fastest growing wave, is closer to the stippled region corresponding to the observed ranges of $A$ and $\theta$. We expect that this mode, because it is nearly identical to the mode described by the inflection point model M2(1, 3), is associated with the inflection point instability mechanism, which would be the dominant mode. The three wavenumber version M3 of the two wavenumber inflection point model M2 substantiates and improves upon these results. Notably, we find that the values of $A$, $\theta$ and $Re_e$ would fall well within the stippled region on Fig. 6. Finally, the parallel instability model Mf yields a poor orientation angle, implying that the parallel instability mechanism was not active on this day. These inflection point and parallel instability mode results are consistent with those given in section 4b for the Ekman profile.

We conclude that the inflection point instability mechanism was driving the rolls observed on 26 September 1981, in agreement with the results of Brümmer (1985). Furthermore, we have demonstrated the utility of the spectral model approach that allows simple, direct comparison of the theoretical results with the observed data.

6. Summary and conclusions

Using several three-dimensional spectral models of flow in a neutral atmosphere, we have examined the roles of the two dynamic instabilities known to describe the development of boundary layer rolls. We developed model M2, with two vertical wavenumbers, to study the pure inflection point instability case and we de-
veloped M2f, with two vertical wavenumbers and non-zero Coriolis terms, to study the mixed inflection point and parallel instability case. Finally, we applied the model Mf of Shirer (1986) to examine the pure parallel instability case.

Using the Ekman profile, we tested the ability of the various spectral models to reproduce the modes arising from the parallel and inflection point instability mechanisms by comparing the results from the spectral models with the results from larger models used by other investigators (e.g., Lilly, 1966; Asai and Nakasuji, 1973). We found that the spectral models yielded preferred values of orientation angle \( \theta \) and aspect ratio \( A \) that agreed with those produced by the larger models. This analysis together with one using a sinusoidal profile indicated that the truncated spectral models were able to capture the essential information contained in a wind profile.

We have also seen that the spectral models M2 and M2f yield preferred values of \( \theta \) and \( A \) that agreed with the observations that were taken on 26 September 1981 during the KonTur experiment. The controlling dynamic instability mechanism on this day was the inflection point instability mechanism in agreement with the results of Brümmer (1985). This comparison with observed data shows the utility of the low-order spectral approach.

These results suggest that for any observed or idealized wind profile we can use simple spectral models such as M2 or M2f to determine easily whether or not the dynamic instabilities may provide the dominant mechanisms for roll development. It is this easy comparison between the model results and observations that is the significant contribution of the low-order spectral approach. Indeed, since we reinforced the conclusion of Brümmer (1985) that, at least on 26 September 1981, the parallel instability had little influence on the roll development, we expect that a spectral model that includes both the thermal and inflection point instability mechanisms would be general enough to determine the dominant instability mechanism responsible for boundary layer rolls.

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APPENDIX A

Orientation Angle

We may separate the orientation angle \( \theta \) for the roll alignment from the Fourier coefficients \( \Lambda_i(r) \) and \( \Gamma_i(m, r) \) by expressing (2.25)-(2.32) as

\[
\Lambda_1 = -\gamma_u(0) \cos(\theta) - \gamma_v(0) \sin(\theta), \\
\Lambda_2 = -\gamma_u(0) \cos(\theta) + \gamma_v(0) \sin(\theta), \\
A_3(r) = \gamma_u(2r) \cos(\theta) + \gamma_v(2r) \sin(\theta), \\
A_4(r) = \gamma_u(2r) \cos(\theta) - \gamma_v(2r) \sin(\theta), \\
\Gamma_1(m, r) = \gamma_u(m + r) \cos(\theta) + \gamma_v(m + r) \sin(\theta), \\
\Gamma_2(m, r) = \gamma_u(m + r) \cos(\theta) - \gamma_v(m + r) \sin(\theta), \\
\Gamma_3(m, r) = \gamma_u(m - r) \cos(\theta) + \gamma_v(m - r) \sin(\theta), \\
\Gamma_4(m, r) = \gamma_u(m - r) \cos(\theta) - \gamma_v(m - r) \sin(\theta).
\]

Here we introduce the definitions

\[
\gamma_u(s) = -\pi^{-1} \int_0^\infty U^*_\xi(z^*) \cos(sz^*) dz^* \\
= -(|V(z_T)|z_T)^{-1} \int_0^{z_T} U(z) \cos(s\pi z/z_T) dz, \\
\gamma_v(s) = -\pi^{-1} \int_0^\infty V^*_\xi(z^*) \cos(sz^*) dz^* \\
= -(|V(z_T)|z_T)^{-1} \int_0^{z_T} V(z) \cos(s\pi z/z_T) dz,
\]

and \( U^* \) and \( V^* \) are the dimensional mean wind components in the standard eastward/northward coordinate system.

APPENDIX B

Terms of the Preferred Orientation Angle Formula (3.3) for the Pure Inflection Point Instability Case

A simple expression for the preferred orientation angle \( \theta_i \) can be determined for the pure inflection point instability case modeled by M2. We find that

\[
\tan(2\theta_i) = B_1/B_2,
\]

in which we have used the definitions

\[
B_1 = \{2\alpha^2 \gamma_u(2n)\gamma_y(2n) + 2\alpha^2 \gamma_u(2q)\gamma_y(2q) \\
- 2\alpha \beta \gamma_u(2n)\gamma_y(2q) \gamma_v(2n)\gamma_y(2q)\} \times (A^2 + n^2/q^2) + \{2a_3 \gamma_u(n + q)\gamma_v(n + q) + 2a_2 \gamma_u(n - q)\gamma_v(n - q) + (a_1 + a_2 + a_3) \gamma_u(n + q)\gamma_v(n - q) + \gamma_v(n + q)\gamma_v(n - q)\}(2A^2 + n^2/q^2),
\]

(4.1)
\[ B_2 = \{a_6^2[\gamma_u^2(2n) - \gamma_v^2(2n)] + a_5^2[\gamma_u^2(2q)] - \gamma_v^2(2q)\} + 2a_5 a_6 [\gamma_u(2n) \gamma_v(2q)] \\
- \gamma_v(2n) \gamma_u(2q)\} (A^2 + n^2) (A^2 + q^2) \\
+ \{a_1 a_3 [\gamma_u^2(n + q) - \gamma_v^2(n + q)] + a_2 a_4 [\gamma_u^2(n - q) - \gamma_v^2(n + q)] \gamma_u(n - q) \\
- \gamma_v(n + q) \gamma_v(n - q)\} (2 A^2 + n^2 + q^2)^2 \]  

and where

\[ a_1 = (-A^2 q + 2 n q^2 + q^3)/(n (A^2 + q^2)), \]  

\[ a_2 = (A^2 q + 2 n q^2 - q^3)/(n (A^2 + q^2)), \]  

\[ a_3 = (-A^2 n + 2 q n^2 + n^3)/(q (A^2 + n^2)), \]  

\[ a_4 = (A^2 n + 2 q n^2 - n^3)/(q (A^2 + n^2)), \]  

\[ a_5 = (A^2 - 3 q^2)/(A^2 + q^2), \]  

\[ a_6 = (A^2 - 3 n^2)/(A^2 + n^2). \]

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