NOTES AND CORRESPONDENCE

Period and Decay Rate of Amplitude Vacillations in a Low-Order General Circulation Model

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1. Introduction

In a recent paper, Thompson (1987) has studied the statistical behavior of a very low-order "general circulation model," consisting of a single finite-amplitude baroclinic wave interacting with a zonal mean shear flow, maintained against dissipation by differential heating. The physical basis of that study is the two-level quasi-geostrophic (Q-G) model for flow in a β-plane channel, with differential heating and eddy diffusion of heat and momentum.

The fluctuating velocity field was represented as the superposition of four sinusoidal modes, whose common wavenumber corresponds to marginal linear instability and whose amplitudes determine the phases and amplitudes of a baroclinic wave at the two levels. Four pairs of those modes interact nonlinearly with a fifth mode, representing the zonal mean vertical shear. The Q-G thermodynamic energy and vorticity equations, when applied to this simple representation, then lead to a closed system of five evolution equations for the zonal averages of five variables at 45° latitude—including poleward heat transport, meridional kinetic energy and vertical shear. With prescribed differential heating, those evolution equations enable us to calculate all five statistical quantities as functions of time only.

Owing to the existence of four independent invariants of the system of five nonlinear evolution equations (in the absence of dissipation and heating), its solutions exhibit periodic amplitude vacillation, and one might expect to find similar behavior under more general conditions. Indeed it was found by numerical experiment that, if the system were initially in equilibrium for a fixed rate of differential heating, then a small impulsive change in the rate of differential heating excited damped amplitude vacillations around a new equilibrium state. This qualitative result, which held universally over a considerable range of values of the rates of dissipation and differential heating and for fairly large departures from equilibrium, at least indicates that the equilibrium state is stable.

More specifically, Fig. 3 of Thompson (1987, hereafter referred to as T87) shows that the predicted period of vacillation is about 23 days for average summertime conditions at 45°S—a value that agrees fairly well with the observed periods reported by Randel and Stanford (1985) and Webster and Keller (1975). Figure 3 also shows a very slow decay of vacillation, with an *e*-folding time about an order of magnitude greater than the period of vacillation and more than an order of magnitude greater than the dissipation time associated with eddy diffusion over "synoptic" scales. The latter feature was especially puzzling.

Figure 4 of T87 summarizes the results of a series of numerical experiments, from which it appeared that the period of vacillation varies as the square root of the dissipation rate and inversely as the square root of the rate of differential heating. In a general way, this result suggested that the square of the frequency of vacillation depends linearly on some quantity whose equilibrium value is determined by the ratio of the rates of differential heating and dissipation, but the specific nature of such a relationship was not immediately clear.

Briefly stated, the purpose of this note is to give a clearer theoretical explanation of two ill-understood aspects of our earlier numerical experiments—namely, the unexpected persistence of vacillation cycles, and the peculiar dependence of the period of vacillation on the rates of dissipation and differential heating. To do so, we carry out an analysis of the equations for perturbations around an equilibrium state and derive a frequency equation for departures from equilibrium. One root corresponds to rapid damping without vacillation. The other two roots are complex and conjugate; they correspond to very slowly damped vacillations, with frequency proportional to the square root of meridional kinetic energy. In the limit of slow dissipation, the latter depends primarily on the ratio of the rates of differential heating and dissipation.

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2. The perturbation equations

Our point of departure is the system of evolution equations (21)–(25) of T87. They are

$$\frac{dT}{d\tau} - 2NSK - 2SV + RX + DT = 0 \quad (1)$$

$$\frac{dK}{d\tau} - ST + DK = 0 \quad (2)$$

$$\frac{dV}{d\tau} - NST + DV = 0 \quad (3)$$

$$\frac{dX}{d\tau} - RT + DX = 0 \quad (4)$$

$$\frac{dS}{d\tau} + 2LT + LDS - H = 0 \quad (5)$$

where the dimensionless variables T, K, V, X, S, τ and η are given by

$$\overline{v^2} = \frac{2\beta^2}{k^4} K \quad \overline{v'^2} = \frac{2\beta^2}{k^4} V \quad \overline{vv'} = \frac{\beta^2}{k^4} X$$

$$\overline{v\phi'} = \frac{\beta^2}{k^5} T \quad U' = \frac{\beta}{k^2} S \quad t = \frac{k\tau}{\beta} \quad y = \frac{\eta}{k}$$

and the dimensionless constants N, R, D and L are defined as

$$N = \frac{k_r^2 - k^2}{k_r^2 + k^2}, \quad R = \frac{k_r^2}{k_r^2 + k^2}$$

$$D = \frac{2\nu k^3}{\beta}, \quad L = \frac{\pi^2}{k^2 W^2}.$$

In this notation, v is half the sum of the y-(northward) components of velocity at 250 and 750 mb; v' is half the difference between those variables. The dimensional shear U' is half the difference between the mean zonal winds at 250 and 750 mb.

The overbar denotes the zonal average, so that $\overline{v\phi'}$ is essentially the net poleward heat transport. The constants β and k are the Rossby parameter and two-dimensional wavenumber, respectively, and k_r is the reciprocal of the radius of deformation; ν is the coefficient of eddy viscosity and eddy conductivity, and W is the width of the channel. The dimensionless rate of differential heating H is

$$H = -\frac{p_0 k^3}{4 f \beta \rho} \cdot \frac{1}{\theta} \frac{\partial}{\partial \eta} \left(\frac{d\theta}{d\tau} \right)$$

where $p_0 = 1000$ mb; f is the Coriolis parameter at 45°; ρ is density at 500 mb; θ is potential temperature at 500 mb.

It will simplify matters if we write (1) as

$$\frac{dT}{d\tau} - 2SZ + RX + DT = 0 \quad (6)$$

where $Z = NK + V$. The evolution equation for Z is formed by multiplying (2) by N and adding it to (3), with the result that

$$\frac{dZ}{d\tau} - 2NST + DZ = 0. \quad (7)$$

The four equations (4)–(7) then comprise a closed system in which the variables are T, Z, X and S.

Let us now consider small departures from an equilibrium state for a fixed value of the differential heating H. The perturbation equations corresponding to (6), (7), (4) and (5) are

$$\frac{dT'}{d\tau} - 2S_0 Z' - 2Z_0 S' + RX' + DT' = 0 \quad (8)$$

$$\frac{dZ'}{d\tau} - 2NS_0 T' - 2NT_0 S' + DZ' = 0 \quad (9)$$

$$\frac{dX'}{d\tau} - RT' + DX' = 0 \quad (10)$$

$$\frac{dS'}{d\tau} + 2LT' + LDS' = 0 \quad (11)$$

where the prime denotes the departure of a variable from its equilibrium value. The subscript zero denotes the equilibrium values of variables, which are solutions of the steady-state forms of (6), (7), (4) and (5), namely:

$$-2S_0 Z_0 + RX_0 + DT_0 = 0 \quad (12)$$

$$-2NS_0 T_0 + DZ_0 = 0 \quad (13)$$

$$-RT_0 + DX_0 = 0 \quad (14)$$

$$2LT_0 + LDS_0 = H. \quad (15)$$

The nonlinearity of the system is reflected in the fact that the coefficients in the perturbation equations depend on equilibrium values of Z, T and S.

3. The frequency equation

Since (8)–(11) are linear with constant coefficients, we seek solutions of the form:

$$\left. \begin{matrix} T' \\ Z' \\ X' \\ S' \end{matrix} \right\} = \left. \begin{matrix} T'(0) \\ Z'(0) \\ X'(0) \\ S'(0) \end{matrix} \right\} e^{\sigma\tau}$$

where T', Z', X', S' and σ are generally complex. Equations (8)–(11) then become

$$(\sigma + D)T' - 2S_0 Z' - 2Z_0 S' + RX' = 0 \quad (16)$$

$$(\sigma + D)Z' - 2NS_0 T' - 2NT_0 S' = 0 \quad (17)$$

$$(\sigma + D)X' - RT' = 0 \quad (18)$$

$$(\sigma + LD)S' + 2LT' = 0. \quad (19)$$

We next solve (18) and (19) for X' and S' in terms of T', and substitute for X' and S' in (16) and (17), with the result that

$$\left(\sigma + D + \frac{R^2}{\sigma + D} + \frac{4LZ_0}{\sigma + LD}\right)T' - 2S_0 Z' = 0$$

$$\left(\frac{4LNT_0}{\sigma + LD} - 2NS_0\right)T' + (\sigma + D)Z' = 0.$$

Since these equations are linear and homogeneous, they have nonzero solutions only if the determinant of their coefficients vanishes. That is,

$$(\sigma + D)^2 + R^2 + 4LZ_0\frac{\sigma + D}{\sigma + LD} + \frac{8LNS_0T_0}{\sigma + LD} - 4NS_0{}^2 = 0. \quad (20)$$

This is the frequency equation, which determines the admissible values of σ.

Equation (20) may be simplified considerably by making use of the equilibrium conditions (12)–(14). For example, solving (13) and (14) for Z_0 and X_0, and substituting for Z_0 and X_0 in (12), we find that

$$R^2 - 4NS_0{}^2 = -D^2. \quad (21)$$

Moreover, from (13),

$$2NS_0T_0 = DZ_0. \quad (22)$$

Thus, substituting from (21) into the second and fifth terms of (20), substituting from (22) into the fourth term of (20), expanding and factoring, we arrive at

$$(\sigma + 2D)(\sigma^2 + LD\sigma + 4LZ_0) = 0.$$

One root, $\sigma = -2D$, corresponds to rapid damping without vacillation. The other two roots are given by

$$\sigma = -\frac{LD}{2} \pm \left[\left(\frac{LD}{2}\right)^2 - 4LZ_0\right]^{1/2}. \quad (23)$$

For typical atmospheric conditions, the first term in the square brackets of (23) is three orders of magnitude smaller than the second, so that

$$\sigma \approx -\frac{LD}{2} \pm 2i(LZ_0)^{1/2}. \quad (24)$$

This expression for the complex frequency is in general accord with the results of our earlier numerical experiments.

4. Period and decay rate of vacillation

The values of the dimensionless constants that entered into the calculation of the curves shown on Fig. 3 of T87 were

$$N = 0.1716, \quad R = 0.5858, \quad D = 0.202$$

$$L = 0.065, \quad H = 0.051, \quad k/k_r = 2^{-1/4}.$$

As noted in the Introduction, these values are typical of summertime conditions at 45°S. Under such conditions, the decay rate given by (24) is

$$\frac{LD}{2} = 0.006565.$$

The corresponding e-folding time is about 152 units of dimensionless time or, with our present scaling, about 198 days. This is at least in semi-quantitative agreement with the numerical calculations of T87. In any event, our present result verifies that the vacillations are extremely persistent. It should be noted that the frequency LD is inversely proportional to the time required to transport heat by eddy conduction from one boundary to the other: thus the slow decay of vacillation is due to the long time needed to transport heat over distances comparable with the earth's radius by motions on subsynoptic scales.

According to (24), the frequency of vacillation is given by

$$\sigma_i = 2(LZ_0)^{1/2}. \quad (25)$$

From (21) we infer that

$$S_0 = \frac{1}{2}\left(\frac{R^2 + D^2}{N}\right)^{1/2}.$$

For the constants listed above,

$$S_0 = 0.748.$$

Substituting this value into (15), we find that

$$T_0 = 0.322$$

whence, from (13),

$$Z_0 = 0.409.$$

Then (25) yields

$$\sigma_i = 0.326.$$

This frequency corresponds to a vacillation period of 19.27 dimensionless units of time, or about 25 days. It compares favorably with the period of 23 days for the finite-amplitude vacillations calculated in T87.

In calculating T_0 from S_0 and H, it became apparent that the second term on the left-hand side of (15) is an order of magnitude less than H, for a reasonable range of D. Approximately, then:

$$T_0 \approx \frac{H}{2L}$$

and, from (13),

$$LZ_0 \approx NS_0\left(\frac{H}{D}\right).$$

Thus, from (21) and (25),

$$\sigma_i \approx [4N(R^2 + D^2)]^{1/4}\left(\frac{H}{D}\right)^{1/2}. \quad (26)$$

For $D < R$, the fourth root of the bracketed quantity in (26) is not sensitive to changes in D, so that the frequency of vacillation varies approximately as $(H/D)^{1/2}$. In other words, the period of vacillation varies as the square root of the dissipation rate and inversely as the square root of the rate of differential heating. This appears to explain the numerical results summarized in Fig. 4 of T87.

5. Summary and conclusions

By carrying out an analysis of the equations for small departures from an equilibrium state of a low-order general circulation model, we have derived theoretical estimates of the period and rate of decay of amplitude vacillations around that equilibrium state. The results explain and confirm several conclusions drawn from a series of numerical experiments described in Thompson (1987). One is that the decay rate of vacillations is very small, the e-folding time being of the order of 200 days. Another is that the period of vacillation is about 25 days, for average summer conditions at 45°S: this value agrees well with our earlier numerical results and with the observed periods reported by Webster and Keller (1975) and Randel and Stanford (1985).

Finally we show that an empirical conclusion, drawn from our previous numerical experiments, is confirmed by our present analysis: the period of vacillation varies as the square root of the dissipation rate and inversely as the square root of the rate of differential heating, for a range of realistically small dissipation rates.

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