On Deep Quasi-geostrophic Theory

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ABSTRACT

Deep quasi-geostrophic theory applies to large-scale flow whose vertical depth scale is comparable to the potential temperature scale height. The appropriate expression for the potential vorticity equation is derived from the general formulation due to Ertel. It is further shown that the *potential* temperature field on a lower boundary acts as a surface charge of potential vorticity.

Deep equivalent barotropic Rossby waves in the presence of a mean zonal wind exhibit an enhanced beta effect but a reduced phase speed. This behavior, analogous to that displayed in shallow water theory, arises due to the inclusion of compressibility effects in the deep theory. These results help clarify the applicability of shallow water theory to barotropic atmospheric flows.

A conceptual model of the role of a surface charge of potential vorticity gradient in generating a change in the relative vorticity of a fluid parcel is described.

1. Introduction

Quasi-geostrophic theory provides the conceptual cornerstone for much of our understanding of large-scale atmospheric motions. The standard quasi-geostrophic scale analysis (e.g., Pedlosky 1987) assumes that the depth scale, D, of the flow is small compared to H_θ, the potential temperature scale height:

$$D \ll H_\theta \equiv \left(\frac{1}{\theta_s}\frac{d\theta_s}{dz}\right)^{-1}. \qquad (1.1)$$

Here $H_\theta = g/N_s^2$ is the inverse of the static stability associated with the static potential temperature field, $\theta_s(z)$.

White (1977) introduced an alternative quasi-geostrophic theory that does not make the restrictive assumption (1.1). This theory for relatively deep flow for which $D \sim H_\theta$ has been referred to as the modified theory (White 1977) or the non-Doppler theory (White 1982). Here White's formulation is referred to as the deep quasi-geostrophic theory in contrast to the standard (i.e., shallow) formulation.

In the deep theory the governing equation for the geostrophic streamfunction, ψ, for adiabatic flow,

$$\frac{dq}{dt_g} = \frac{\partial q}{\partial t} + J(\psi, q) = 0, \qquad (1.2)$$

is unchanged from the standard theory (e.g., Pedlosky 1987). Here q is the quasi-geostrophic potential vorticity,

$$q = \nabla^2\psi + \beta y + \frac{f_0^2}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N_s^2}\frac{\partial\psi}{\partial z}\right), \qquad (1.3)$$

for a midlatitude tangent plane with Coriolis parameter $f = f_0 + \beta y$, where $\rho_s(z)$ is the static density field and J is the Jacobian. For a flat lower boundary at $z = 0$ the vertical motion field vanishes and the heat equation becomes

$$\frac{d}{dt_g}\left(\frac{g\delta\theta}{\theta_s}\right) = 0 \quad \text{at} \quad z = 0, \qquad (1.4)$$

where $\delta\theta$ is the dynamic contribution to the potential temperature field and g is the acceleration due to gravity. White (1977) showed that the hydrostatic relation for deep flow is

$$g\frac{\delta\theta}{\theta_s} = f_0\left(\frac{\partial\psi}{\partial z} - \frac{\psi}{H_\theta}\right), \qquad (1.5)$$

which reduces to the standard result, $g\delta\theta/\theta_s = f_0\partial\psi/\partial z$, for $D \ll H_\theta$. But as a consequence of (1.5), the lower boundary condition (1.4) contains an extra term,

$$-\frac{f_0}{H_\theta}\frac{\partial\psi}{\partial t},$$

which Lindzen (1968) in a linear barotropic study described as a non-Doppler term. The inclusion of this term for baroclinic flow has been examined for large-scale Rossby (i.e., stable) and baroclinic (i.e., unstable) waves by Geisler and Dickinson (1975), Geisler and

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Garcia (1977), White (1977, 1978, 1982), Blumen (1978), and Shutts (1978). Blumen, for example, has shown that this term significantly stabilizes a flow for stratospheric values of D/H_θ ($\sim$0.4) while ultralong waves in the tropospheric case are also stabilized.

The purpose of this paper is to reexamine White's system describing the dynamics of deep quasi-geostrophic flow. Three considerations motivate this reexamination. First, White's vorticity equation includes a solenoidal term and his continuity equation a dynamic compressibility term. Both terms are traditionally dropped. Section 2 presents an alternative derivation of (1.2) starting from Ertel's potential vorticity theorem. It is shown that (1.2) is indeed the correct quasi-geostrophic limit of the theorem for deep flow provided one uses (1.5). Equation (1.2) also holds for standard theory if all $O(D/H_\theta)$ terms are ignored.

The second consideration lies in the observation that the lower boundary condition, (1.4), when written in terms of ψ using (1.5), is altered in the deep theory while the potential vorticity equation, (1.2), is unchanged. This situation raises the possibility that the deep set [(1.2)–(1.5)] may not exhibit the property that the lower boundary condition be consistent with a definition of a surface charge of potential vorticity. This important internal consistency was first noted by Bretherton (1966) for standard quasi-geostrophic theory, while James and Hoskins (1985) have shown it holds for the primitive equations. Section 3 demonstrates that the deep quasi-geostrophic set is also consistent.

The third consideration addresses the applicability of shallow water theory to barotropic atmospheric dynamics. For example, the zonal phase speed c_h for a Rossby wave in shallow water of mean depth, h, in the presence of a uniform zonal current U_0 is (Corby 1966; Pedlosky 1987),

$$c_h = U_0 - \frac{(\beta + f_0{}^2U_0/gh)}{k^2 + l^2 + f_0{}^2/gh}, \qquad (1.6)$$

where k and l are the zonal and meridional wavenumbers, respectively. In addition to a Doppler shift in the phase speed, the presence of the mean flow has introduced a non-Doppler modification to the beta effect. The analysis of section 4 suggests that (1.6) better approximates deep atmospheric Rossby waves if γH replaces h. Here H is the density scale height, $H = (\partial \ln\rho_s/\partial z)^{-1}$, and $\gamma = C_p/C_v$ is the ratio of the specific heats. Section 4 also describes the vorticity dynamics associated with the non-Doppler effect on deep Rossby waves, and presents a conceptual model of vorticity generation in association with a surface gradient in potential temperature.

2. Derivation using Ertel's theorem

The general expressions for the conservation of potential vorticity and potential temperature for adiabatic, inviscid flow are

$$\frac{d\Pi_*}{dt_*} = 0, \qquad (2.1)$$

$$\frac{d\theta_*}{dt_*} = 0, \qquad (2.2)$$

respectively, where

$$\Pi_* = (\omega_{a*} \cdot \nabla\theta_*)/\rho_*, \qquad (2.3)$$

is Ertel's potential vorticity. The notation in this section follows Pedlosky (1987, see pp. 358–359) with asterisks denoting dimensional quantities.

Scale analysis indicates that

$$(u_*, v_*, w_*) = U(u, v, \delta\epsilon w), \qquad (2.4a)$$

$$\theta_* = \theta_s(z)(1 + \epsilon F\theta), \qquad (2.4b)$$

$$\rho_* = \rho_s(z)(1 + \epsilon F\rho), \qquad (2.4c)$$

where $\epsilon = U/(f_0L)$ is the Rossby number, $\delta = D/L$ is the aspect ratio, and $F = f_0{}^2L^2/gD$ is a rotating Froude number. Here U, L, and D are characteristic windspeed, length, and depth scales, respectively. Then the material derivative may be written,

$$\frac{d}{dt_*} = \frac{U}{L}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \epsilon w\frac{\partial}{\partial z}\right), \qquad (2.5)$$

and the absolute vorticity on the β-plane is

$$\omega_{a*} = f_0\left[(1 + \beta\epsilon y + \epsilon\zeta)\hat{z} + \left(\epsilon^2\delta\frac{\partial w}{\partial y} - \frac{\epsilon}{\delta}\frac{\partial v}{\partial z}\right)\hat{x} + \left(\frac{\epsilon}{\delta}\frac{\partial u}{\partial z} - \epsilon^2\delta\frac{\partial w}{\partial x}\right)\hat{y}\right], \qquad (2.6)$$

where $\beta = \beta_*L^2/U$ is the nondimensional measure of the β-effect and $\zeta = \partial v/\partial x - \partial u/\partial y$ is the vertical component of the nondimensional relative vorticity. The scaled potential vorticity is

$$\Pi_* = \left[\frac{f_0/D}{\rho_s(1 + \epsilon F\rho)}\right]\left\{(1 + \beta\epsilon y + \epsilon\zeta) \times \frac{\partial}{\partial z}[\theta_s(1 + \epsilon F\theta)] + \epsilon^2 F\theta_s\left[\left(-\frac{\partial v}{\partial z} + \epsilon\delta^2\frac{\partial w}{\partial y}\right)\frac{\partial\theta}{\partial x} + \left(\frac{\partial u}{\partial z} - \epsilon\delta^2\frac{\partial w}{\partial x}\right)\frac{\partial\theta}{\partial y}\right]\right\}. \qquad (2.7)$$

Neglecting terms of $O(\epsilon^2)$ or higher yields

$$\Pi_* = \left(\frac{f_0}{\rho_s D}\right)\left\{\frac{d\theta_s}{dz} + \epsilon\left[(\zeta + \beta y)\frac{d\theta_s}{dz} + F\frac{\partial}{\partial z}(\theta_s\theta) - F\rho\frac{d\theta_s}{dz}\right]\right\}. \qquad (2.8)$$

The scaled form of Ertel's theorem, (2.1), becomes

$$\frac{d\Pi_*}{dt_*} = \frac{U}{L}\left[\frac{d\Pi_*}{dt_g} + \epsilon w\frac{\partial\Pi_*}{\partial z}\right] = 0. \qquad (2.9)$$

Again we neglect $O(\epsilon^2)$ terms to obtain

$$\frac{d}{dt_g}\left[(\zeta+\beta y)\frac{d\theta_s}{dz}+F\frac{\partial}{\partial z}(\theta_s\theta)-F\rho\frac{d\theta_s}{dz}\right]+\rho_s w\frac{d}{dz}\left(\frac{1}{\rho_s}\frac{d\theta_s}{dz}\right)=0. \quad (2.10)$$

A similar scaling of the thermodynamic equation (2.2) yields

$$\frac{d\theta}{dt_g}+Sw=0, \quad (2.11)$$

where $S = 1/(FH_\theta)$ is the stratification parameter or Burger number. Solving (2.11) for w and substituting into (2.10) yields the conservation statement,

$$\frac{dq}{dt_g}=0, \quad (2.12)$$

where

$$q=\zeta+\beta y+\frac{F}{(d\theta_s/dz)}\frac{\partial}{\partial z}(\theta_s\theta)-\frac{\rho_s\theta}{S(d\theta_s/dz)}\frac{d}{dz}\left(\frac{1}{\rho_s}\frac{d\theta_s}{dz}\right)-F\rho, \quad (2.13)$$

is the quasi-geostrophic potential vorticity. The third and fourth terms may be combined to yield

$$q=\zeta+\beta y+\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s\theta}{S}\right)-F\rho. \quad (2.14)$$

Introduction of the diagnostic relations

$$\zeta=\nabla^2\psi,\ \theta=\frac{\partial\psi}{\partial z}-\frac{D\psi}{H_\theta},\ \rho=-\frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s\psi), \quad (2.15)$$

which can be derived using the geostrophic and hydrostatic approximations and the definition of ψ, enables (2.14) to be written as

$$q=\nabla^2\psi+\beta y+\frac{1}{\rho_s}\frac{\partial}{\partial z}\left[\frac{\rho_s}{S}\left(\frac{\partial\psi}{\partial z}-\underline{\frac{D\psi}{H_\theta}}\right)\right]+\underbrace{\frac{F}{\rho_s}\frac{\partial}{\partial z}(\rho_s\psi)}. \quad (2.16)$$

For the deep quasi-geostrophic theory, the underlined terms cancel to produce the final result:

$$q=\nabla^2\psi+\beta y+\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\partial\psi}{\partial z}\right), \quad (2.17)$$

which is the nondimensional version of (1.3).

In the standard theory the first underlined term in (2.16) does not appear since $\theta = \partial\psi/\partial z$ replaces (2.15b). However the second underlined term can also be ignored for the same reason:

$$O\left[\frac{F}{\rho_s}\frac{\partial(\rho_s\psi)}{\partial z}\Big/\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\partial\psi}{\partial z}\right)\right]=D/H_\theta\ll 1. \quad (2.18)$$

Thus (2.17) holds for both the standard ($D/H_\theta \ll 1$) and deep ($D/H_\theta \lesssim 1$) theories.

3. Surface charge of potential vorticity

We perform a density-weighted integration of (1.2) across a thin vertical thermal boundary layer adjoining a rigid horizontal boundary at $z = 0$:

$$\int_{bl}\rho_s\frac{dq}{dt_g}dz=0. \quad (3.1)$$

This hypothetical boundary layer (Bretherton 1966; James and Hoskins 1985) is the narrow transition region where the oblique isentropes in the fluid interior bend horizontally to form an isentropic surface at $z = 0$.

By the transport theorem (see Dutton 1976, section 5.3), (3.1) becomes, using (1.3),

$$\frac{d}{dt_g}\int_{bl}\left[\rho_s(\nabla^2\psi+\beta y)+f_0{}^2\frac{\partial}{\partial z}\left(\frac{\rho_s}{N_s^2}\frac{\partial\psi}{\partial z}\right)\right]dz=0. \quad (3.2)$$

In the limit of a vanishingly thin boundary layer, the absolute vorticity contribution to the integral vanishes since ψ is continuous. But, as noted by Bretherton (1966), the third contribution does not since $\partial\psi/\partial z$ need not be continuous across the boundary layer. Thus (3.2) reduces to

$$\frac{d}{dt_g}\left[f_0{}^2\frac{\rho_s}{N_s^2}\frac{\partial\psi}{\partial z}\right]_0^+=0, \quad (3.3)$$

or, using (1.5),

$$\frac{d}{dt_g}\left[g\frac{\delta\theta}{\theta_s}+\frac{f_0\psi}{H_\theta}\right]_0^+=\frac{d}{dt_g}\left[g\frac{\delta\theta}{\theta_s}\right]_0^+=0, \quad (3.4)$$

since ψ, ρ_s, and N_s are continuous. Because the bottom boundary is isentropic, $\delta\theta|_0 = 0$, (3.3) becomes

$$\frac{d}{dt_g}g\frac{\delta\theta}{\theta_s}\bigg|_+=0, \quad (3.5)$$

which is the correct boundary condition, (1.4), for the problem posed without the boundary layer. Thus the deep theory is consistent in the sense of Bretherton (1966). Since the dynamic temperature and potential temperature fields are identical in the standard theory, the present derivation emphasizes the result that horizontal gradient of *potential* temperature on a boundary represents a surface-charge gradient in potential vorticity.

4. Dynamics of stable waves

a. Derivation of dispersion relation

We next examine the dynamics of small amplitude perturbations in a uniform zonal flow, U_0. The streamfunction for the basic state, denoted by a capital letter, is

$$\Psi = -U_0 y. \tag{4.1}$$

Taking U_0 to be positive, there is low pressure on the poleward side of the flow. From (1.5), a positive linear meridional potential temperature gradient exists with relatively warm air poleward. Since

$$\frac{\delta\rho}{\rho_s} = \frac{\delta p}{\gamma p_s} - \frac{\delta\theta}{\theta_s}, \tag{4.2a}$$

$$\frac{\delta T}{T_s} = \frac{f_0}{g}\frac{\partial\psi}{\partial z}, \tag{4.2b}$$

the density field decreases poleward while the dynamic temperature field vanishes. Consistent with the thermal wind relation,

$$\frac{\partial u}{\partial z} = \frac{g}{f_0}\frac{\partial}{\partial y}\left(\frac{\delta T}{T_s}\right), \tag{4.3}$$

there is no vertical wind shear despite the potential temperature gradient. Variations in $\delta\theta$ arise from compressibility effects associated with the dynamic pressure perturbation.

The dynamics of small amplitude perturbations to this basic state satisfy the linearized potential vorticity equation,

$$\left(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}\right)\left[\nabla^2\phi + \frac{f_0^2}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N_s^2}\frac{\partial\phi}{\partial z}\right)\right] + \beta\frac{\partial\phi}{\partial x} = 0, \tag{4.4}$$

subject to the lower boundary condition

$$\left(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}\right)g\frac{\delta\theta}{\theta_s} + v\frac{\partial}{\partial y}\left(g\frac{\delta\Theta}{\theta_s}\right) = 0, \quad \text{at} \quad z = 0, \tag{4.5}$$

where ϕ and $\delta\theta$ are the perturbation streamfunction and potential temperature, respectively. Here $\delta\Theta$ is the dynamic potential temperature associated with the basic state (4.1). The solution (e.g., White 1978) is

$$\phi = Ae^{ik(x-ct)}\cos ly \exp\left[\frac{cz}{(c-U_0)H_\theta}\right], \tag{4.6}$$

where A is an arbitrary amplitude. The zonal phase speed c satisfies

$$\left[k^2 + l^2 + \frac{f_0^2}{gH}(1-\kappa)\right](c-U_0)^2 + \left[\beta + \frac{U_0 f_0^2}{gH}(1-2\kappa)\right](c-U_0) - \frac{U_0^2 f_0^2}{gH}\kappa = 0, \tag{4.7}$$

where $\kappa = R/C_p = (\gamma - 1)/\gamma = 1 - C_v/C_p = H/H_\theta$. Equivalent forms of (4.7) were obtained by Lindzen (1967) and White (1978).

b. Comparison of phase speeds

We now analyze the dispersion relation (4.7) in various limits and compare the results for the phase speed c with other waves. For an incompressible flow, $C_p = C_v$ and $\kappa = 0$. Then the two solutions of (4.7) are

$$c_+ = U_0, \quad c_- = c_H, \tag{4.8}$$

where

$$c_H \equiv U_0 - \frac{(\beta + f_0^2 U_0/gH)}{k^2 + l^2 + f_0^2/gH}.$$

Here the subscripts (+) and (−) denote the sign of the radicand in the quadratic solution of (4.7). The c_+ solution, representing an advective wave, is excluded by the boundedness condition at $z = \infty$ [see (4.6)]. The c_- solution describes Rossby waves in a zonal current for a stratified semi-infinite incompressible fluid. In addition to a Doppler shift, there is a non-Doppler enhancement (for $U_0 > 0$) of the retrograde β-effect. We note that (4.8) is identical to the phase speed, c_h, [see (1.6)] for a divergent Rossby wave in a homogeneous fluid with a free surface and mean depth $h = H$.

Figure 1 is a plot of the solution to (4.7) as a function of, K_H, the nondimensional horizontal wavenumber,

$$K_H^2 = gH(k^2 + l^2)/f_0^2, \tag{4.9}$$

for various values of K_H. Also plotted is c_H, the homogeneous fluid result, and c_S,

$$c_S = U_0 - \frac{\beta}{k^2 + l^2}, \tag{4.10}$$

the phase speed for standard quasi-geostrophic theory. In plotting the retrograde phase speed, we normalize the speeds by c_β:

$$c_\beta = \beta gH/f_0^2, \tag{4.11}$$

which is the fastest Rossby group speed for (4.8) in the absence of a mean flow. As noted by White (1978), the deep theory predicts weaker retrograde motion than standard theory. Moreover the homogeneous fluid analogue, c_H, underestimates the retrograde motion. These results are larger for an eastward zonal current.

An approximate form for (4.7) facilitates explanation of these results. For $|U_0| \ll c_\beta$, the last term on the right-hand side of (4.7) may be ignored and the solutions reduce to

$$c_+ = U_0, \quad c_- = U_0 - \frac{\beta + \frac{f_0^2 U_0}{gH}(1-2\kappa)}{k^2 + l^2 + \frac{f_0^2}{gH}(1-\kappa)}, \tag{4.12}$$

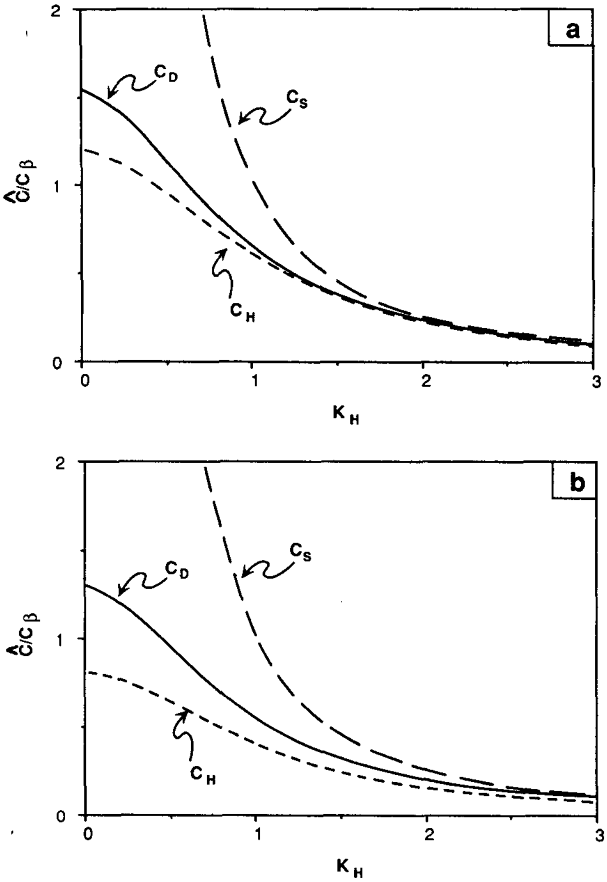

FIG. 1. The retrograde phase speed, $\hat{c} = -(c - U_0)$, (in units of $c_\beta = \beta gH/f_0^2$) as a function of the nondimensional horizontal wavenumber, $K_H = [gH(k^2 + l^2)/f_0^2]^{1/2}$, in the presence of (a) an eastward zonal wind, and (b) a westward zonal wind of magnitude $|U_0| = 0.2c_\beta$. The curves labeled c_s, c_D, and c_H correspond to the standard, deep, and homogeneous results, respectively.

which agrees with White (1978). The approximate ($|U_0| \ll c_\beta$) result (4.12) for the zonal phase speed indicates that the deep theory reduces both the non-Doppler and the stratification contributions compared with the homogeneous case c_H. Further, the weaker retrograde motion of the deep wave compared to that of the standard theory, (4.10), can be attributed to the stratification contribution as represented by the last term in the denominator of (4.12). This feature, entirely absent from the standard theory, is responsible for bounded phase speeds as $K_H \rightarrow 0$ in the deep theory. The phase speeds in the standard theory are unbounded in this limit.

In order to isolate the dynamical effects of the deep theory, we let $\beta = 0$. Then the quadratic (4.7) has the exact solutions

$$c_+ = U_0 + \frac{\kappa U_0}{\alpha H + 1 - \kappa}, \qquad (4.13)$$

$$c_- = U_0 - \frac{U_0}{1 + \alpha H_\theta}, \qquad (4.14)$$

where

$$\alpha = \left[\frac{N^2(k^2 + l^2)}{f_0^2} + \frac{1}{4H^2}\right]^{1/2} - \frac{1}{2H}.$$

Since $c_+ - U_0 > 0$, the (+) root grows exponentially with height and must be excluded as being unphysical in a semi-infinite atmosphere. The (−) root decays with height and is a bottom trapped edge wave of the deep system. Alternatively we can write (4.14) as

$$c_- = U_0 - \frac{\dfrac{f_0^2 U_0}{gH}(1 - 2\kappa)\left[\dfrac{1}{4} + \kappa \dfrac{gH}{f_0^2}(k^2 + l^2)\right]^{1/2}}{k^2 + l^2 + \dfrac{f_0^2}{gH}(1 - \kappa)}. \qquad (4.15)$$

This latter form enables direct comparison with (4.12), which suggests that (4.12) only approximates the form of the non-Doppler term.

Additional insight is obtained if we express (4.14) in terms of the basic state potential temperature gradient,

$$c_- = U_0 - \frac{gH_\theta}{f_0\theta_s(1 + \alpha H_\theta)} \frac{\partial \delta\Theta}{\partial y}, \qquad (4.16)$$

and compare it to the phase speed, c_E, of an Eady edge wave (see section 13.2 of Gill 1982):

$$c_E = U(z = 0) - \frac{gH_R}{f_0\theta_0} \frac{\partial \delta\Theta}{\partial y}, \qquad (4.17)$$

where

$$H_R = [f_0^2/N_0^2(k^2 + l^2)]^{1/2},$$

is the Rossby height. The Eady edge wave exists in a semi-infinite Boussinesq quasi-geostrophic fluid with a linear basic state potential temperature gradient. Since $\delta\theta = \delta T$ in the Boussinesq case, this gradient implies a linear wind shear by the thermal wind relation

$$f_0 \frac{dU(z)}{dz} = -\frac{g}{\theta_0} \frac{\partial \delta\Theta}{\partial y}. \qquad (4.18)$$

Comparison of c_- and c_E indicates that both waves are associated with the potential temperature gradient. Since the interior potential vorticity field is uniform in each case, a potential vorticity explanation of the wave propagation would involve Bretherton's interpretation of the surface gradient of $\delta\Theta$ as a gradient of potential vorticity (i.e., an effective β). Consider the northward displacement of a fluid column for, say, a positive meridional potential temperature gradient, $\partial\delta\Theta/\partial y > 0$. Then, to conserve its potential vorticity, a fluid column would generate negative relative vorticity. The retrograde phase propagation can then be explained by a number of fluid columns alternately displaced northward and southward undergoing a "vorticity dance"

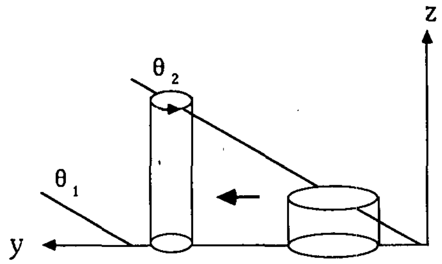

FIG. 2. Schematic illustration of the vorticity generation mechanism for a prograde Eady edge wave. The basic state for the semi-infinite Boussinesq fluid consists of a zonal wind increasing linearly with height. A fluid column displaced poleward is stretched and cyclonic vorticity is generated. The relative warmth of the column relative to its environment explains the required low pressure anomaly. Here the lid of the column lies at the steering level and its trajectory in the meridional plane is along the basic state isentropes (solid lines). Lids above (below) the steering level have steeper (shallower) trajectories.

(Pedlosky 1987). Such an explanation applies equally to both (4.16) and (4.17).

c. Vorticity dynamics

A more complete interpretation of the phase speed formulae, however, requires a discussion of the mechanism of the vorticity generation. For the Eady edge wave, the linearized vorticity equation is

$$\left(\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right)\zeta = f_0\frac{\partial w}{\partial z}, \qquad (4.19)$$

and one sees that vortex stretching and squashing produces the changes in vorticity. A schematic of this process is presented in Fig. 2 for the typical midlatitude case of an eastward mean zonal flow that increases linearly with height. The associated isentropes tilt poleward and upward ($\partial\delta\Theta/\partial y < 0$). Consider a fluid column with its base on the rigid lower boundary. As it is displaced poleward, its base is required to remain fixed to the boundary by the kinematic lower boundary condition. Its lid enters a region where it is positively buoyant and is free to rise. The net result is a stretching of the fluid column and a generation of positive relative vorticity. Since this generation is opposite to that of a barotropic fluid on the β-plane, the phase propagation is prograde.

The vorticity dynamics of the deep-edge wave is more subtle. The vorticity equation in the deep theory is

$$\frac{d}{dt_g}(\zeta + f) = \underbrace{-f_0\nabla\cdot\mathbf{u}_a}_{\mathrm{C}} - \underbrace{\frac{\hat{z}}{\rho_s}\cdot\left[\nabla\left(\frac{\delta\theta}{\theta_s}\right)\times\nabla\delta p\right]}_{\mathrm{B}}, \qquad (4.20)$$

which states that the absolute vorticity following the geostrophic flow will change due to ageostrophic horizontal convergence, C, and baroclinity, B. Here we show that there is large cancellation between these two terms, with the baroclinic term offsetting some of the effects of the convergence term. By the deep continuity equation (White 1977), we can write the convergence term as

$$\mathrm{C} = \frac{f_0}{\rho_s}\frac{\partial}{\partial z}(\rho_s w) + f_0\frac{d}{dt_g}\left(\frac{\delta\rho}{\rho_s}\right). \qquad (4.21)$$

Use of the streamfunction ψ and the definition of the Jacobian in the baroclinic term yields

$$\mathrm{B} = +f_0 J\left(\psi, \frac{\delta\theta}{\theta_s}\right) = -f_0 J\left(\psi, \frac{\delta\rho}{\rho_s}\right), \qquad (4.22)$$

using (4.2) and the fact that $J(\psi, \psi) = 0$. Thus B is proportional to the advection of density and cancels the convective derivative in C. Then the vorticity equation (4.20) simplifies to

$$\frac{d(\zeta + f)}{dt_g} = \frac{f_0}{\rho_s}\frac{\partial(\rho_s w)}{\partial z} + f_0\frac{\partial}{\partial t}\left(\frac{\delta\rho}{\rho_s}\right), \qquad (4.23)$$

and absolute vorticity changes are due to vortex stretching and the local rate of change of the density field, two processes embedded in the convergence term.

The vorticity dynamics of the deep edge wave simplifies further when it is noted that the vertical motion field vanishes. [This result follows from the fact that (4.5), with the solution (4.6), holds at all heights and not just at the surface.] The linearization of (4.23) with $w = 0$ and $\beta = 0$ yields

$$\left(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}\right)\zeta = f_0\frac{\partial}{\partial t}\left(\frac{\delta\rho}{\rho_s}\right), \qquad (4.24)$$

which indicates that local density changes produce vorticity anomalies.

Figure 3 provides a schematic of the vorticity generation for the deep edge wave with a uniform positive

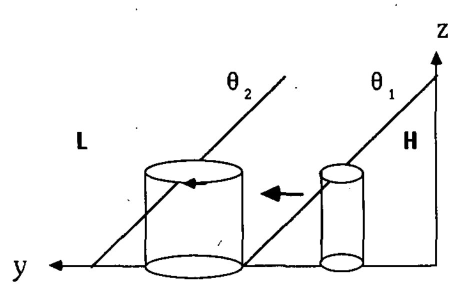

FIG. 3. Schematic illustration of the vorticity generation mechanism for a retrograde deep edge wave. The basic state consists of a uniform positive zonal wind with relatively low pressure (L) and density and potentially warm air towards the pole. A fluid column displaced poleward expands horizontally and anticyclonic vorticity is generated without a change in the column's height. To support the required high pressure anomaly, the column's density is larger than the local ambient density field.

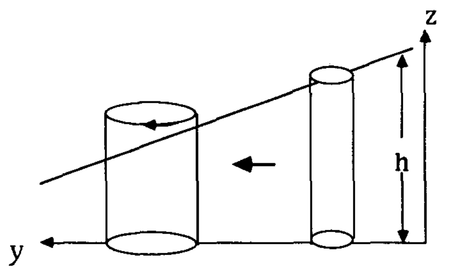

FIG. 4. Schematic illustration of the vorticity generation mechanism for a retrograde shallow water wave. The basic state, consisting of a uniform eastward flow, requires a sloping free surface to balance the Coriolis force. A fluid column displaced poleward enters shallower water and shrinks. Anticyclonic vorticity is generated by the decrease in the column's height. To support the required high pressure anomaly, the column's height is larger than the local fluid depth.

zonal flow. In the basic state there is relatively low pressure and hence potentially warm air toward the pole. Thus the isentropes tilt upward and equatorward. When displaced poleward, a bottom-trapped fluid column will have its lid enter a region where it is negatively buoyant. However, the implied sinking motion is negated by the tendency of the column to expand as it moves towards low pressure. The net result is that the fluid column expands horizontally with no change in its height. By Kelvin's circulation theorem, the increased horizontal area requires the generation of a negative vorticity anomaly. The sign of this generation is similar to that of a barotropic fluid on the β-plane and the phase propagation is retrograde. Shallow water theory (see Fig. 4) also predicts a retrograde motion but fluid columns change their height and vorticity anomalies are produced by vortex stretching.

5. Conclusions

The major contribution of deep quasi-geostrophic theory is the retention of compressibility effects. Examples include the hydrostatic relation (1.5), which includes the influence of the pressure field on the rate of pressure fall with height (e.g., higher dynamic pressures imply a denser fluid and a larger decrease with height) and the lack of equivalence between the temperature and potential temperature anomalies [cf. (1.5) and (4.2b)]. Furthermore, the quasi-geostrophic potential vorticity (1.3) may be written as

$$q = \zeta + \beta y + \frac{f_0}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N_s^2} g \frac{\delta\theta}{\theta_s}\right) - f_0 \frac{\delta\rho}{\rho_s}, \quad (5.1)$$

which was derived in nondimensional form (2.14). Standard quasi-geostrophic theory includes the effect of stratification on the potential vorticity as represented by the third term in (5.1) but lacks the effect of compressibility, the last term in (5.1). The deep theory captures both processes.

The inclusion of compressibility effects by the deep theory has several consequences. First, the relation of stratified theory to shallow water theory is clarified. To a good approximation, low-frequency shallow water wave theory can be applied to deep atmospheric flows if the mean fluid depth is taken to be $\gamma H \sim 11.2$ km. Applications include stable waves (e.g., Rossby et al. 1939) and divergent barotropic instability (e.g., Lipps 1963). Second, a uniform mean flow alters the potential vorticity gradient with a surface potential temperature gradient. This enhances the β-effect for eastward flow and explains the stabilization of baroclinic flows reported by Blumen (1978).

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