Linear Baroclinic Instability with the Geostrophic Momentum Approximation

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ABSTRACT

The linear Eady model of baroclinic instability with the geostrophic momentum (GM) approximation is solved analytically in physical space and shown to be identical to linear three-dimensional semigeostrophic theory. Both the growth rates and the wavenumber of the short-wave cutoff are larger than those predicted by quasi-geostrophic (QG) theory. This behavior arises because the effective static stability is reduced in the GM case. These results are opposite to those using standard nongeostrophic (NG) theory, and the discrepancy increases with decreasing Richardson number. Energetically, the unstable GM normal modes enhance the conversion of available potential energy compared to the QG modes and also convert available kinetic energy to eddy kinetic energy. With regards to the structure of the unstable modes, the northward tilt with height in the GM case is more consistent with NG theory than is the QG solution which displays no meridional tilt.

Additional analysis addresses the effect of assuming that either the meridional or zonal component of the perturbation wind field is geostrophic.

1. Introduction

Baroclinic instability theory plays a central role in our understanding of midlatitude synoptic disturbances. Introduced by Charney (1947), this theory provides a physical description of storm growth through the energy conversions proposed by Margules (1903). Independently, Eady (1949) gave a particularly convenient mathematical formulation of the problem. Both Charney and Eady restricted their analyses to quasi-geostrophic disturbances. Later Árnason (1963) and Stone (1966) showed that nongeostrophic effects tended to reduce the instability captured by quasi-geostrophic theory.

In contrast to quasi-geostrophic theory, Eliassen (1949) introduced the geostrophic momentum approximation. Hoskins (1975) combined this approximation with a coordinate transformation to develop a semigeostrophic theory of baroclinic instability which incorporates some ageostrophic effects. Since then, there have been numerous applications of this approach to instability problems as well as to frontogenesis and airflow over mountains.

The validity of the geostrophic momentum approximation warrants some discussion since it is not possible to justify it with a traditional scale analysis. Eliassen (1962) presented a Lagrangian argument suggesting that the advection of the ageostrophic wind, $\mathbf{u}_a$, is small compared to that of the geostrophic flow, $\mathbf{u}_g$. Hoskins (1975) made a geometric argument to support the approximation for relatively straight flow with small curvature such that the angle the wind makes with the isobars is small. Salmon (1985) employed a Hamiltonian formulation to derive the semigeostrophic shallow water equations in the transformed coordinates. All these approaches are based on the smallness of the ageostrophic flow:

$$|\mathbf{u}_a| \ll |\mathbf{u}_g|. \tag{1.1}$$

It is important to note that while the approximation has generally been developed in a nonlinear context, (1.1) does not preclude its application to a linear problem. Indeed, use of the approximation in baroclinic instability theory and frontogenesis implicitly assumes that the approximation holds in the early stages of the flow's development when the disturbance has a small amplitude.

The purpose of the present study is to assess the impact of the geostrophic momentum (GM) approximation on the linear Eady baroclinic instability problem through comparison with quasi-geostrophic (QG) and nongeostrophic (NG) theory. Section 2 describes the general formulation of the three-dimensional Eady model and summarizes the QG and NG results. Section 3 introduces the GM approximation and thus, the simplified problem is solved analytically in physical space. The solution, shown to be identical to that of linear semigeostrophic (SG) theory, indicates that the GM approximation enhances the instability (by increasing both the number of unstable modes and their growth rates) compared with either QG or NG theory. Comparisons of the wave structure and energetics are also

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presented. Section 4 discusses "half-geostrophic" approximations in which either the meridional or zonal component of the perturbation wind is geostrophic, denoted VG and UG respectively. The VG case corresponds to two-dimensional QG and SG theory of baroclinic instability, while the UG case is an approximation of symmetric instability. Section 5 provides some concluding remarks and a discussion of the implications of the present results to the nonlinear problem.

2. The Model

a. Formulation

The Eady model of baroclinic instability considers the adiabatic, hydrostatic flow of an inviscid Boussinesq fluid. The motion is described relative to a rotating Cartesian coordinate system (x, y, z) with constant angular rotation vector $\frac{1}{2} f \hat{z}$. Rigid horizontal boundaries at $z = 0$ and $z = H$ confine the fluid vertically but the flow is horizontally unbounded. The governing equations are

$$\frac{Du}{Dt} - fv = -\frac{\partial \phi}{\partial x}, \qquad (2.1a)$$

$$\frac{Dv}{Dt} + fu = -\frac{\partial \phi}{\partial y}, \qquad (2.1b)$$

$$\frac{\partial \phi}{\partial z} = g\theta/\theta_0, \qquad (2.1c)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \qquad (2.1d)$$

$$\frac{D\theta}{Dt} = 0, \qquad (2.1e)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u\partial}{\partial x} + \frac{v\partial}{\partial y} + \frac{w\partial}{\partial z},$$

and (u, v, w) is the velocity vector, ϕ the geopotential, θ the (potential) temperature anomaly, θ_0 a constant reference temperature and $-g\hat{z}$ is the acceleration due to gravity.

The flow consists of the sum of a basic state and a small perturbation. The basic state is assumed to be a stably stratified fluid with uniform buoyancy frequency N and to possess a zonal wind that varies linearly with height (i.e., constant shear U_s). The basic state is in thermal wind balance and is an exact solution of (2.1). The total flow is

$$\mathbf{u} = U_s z\hat{x} + (u', v', w'), \qquad (2.2a)$$

$$\phi = -fU_s yz + \phi', \qquad (2.2b)$$

$$g\theta/\theta_0 = N^2 z - fU_s y + g\theta'/\theta_0, \qquad (2.2c)$$

where primes denote the perturbation quantities. Substituting (2.2) into (2.1) and linearizing yields

$$\frac{du'}{dt} + U_s w' - fv' = -\frac{\partial \phi'}{\partial x}, \qquad (2.3a)$$

$$\frac{dv'}{dt} + fu' = -\frac{\partial \phi'}{\partial y}, \qquad (2.3b)$$

$$\frac{\partial \phi'}{\partial z} = g\theta'/\theta_0, \qquad (2.3c)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \qquad (2.3d)$$

$$\frac{d}{dt}(g\theta'/\theta_0) + N^2 w' - fU_s w' = 0, \qquad (2.3e)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + U_s z\frac{\partial}{\partial x},$$

and the boundary conditions are

$$w' = 0 \quad \text{at} \quad z = 0 \text{ and } H. \qquad (2.4)$$

b. Scaling

Nondimensionalization of (2.3) is based on the fact that the only external length scale to the problem is H, the depth of the fluid. A characteristic horizontal scale is then the deformation radius L:

$$L = NH/f. \qquad (2.5)$$

This choice of L is consistent *a posteriori* with the results of Eady (1949) and Emanuel (1979) that the characteristic scale of the unstable baroclinic or symmetric disturbance is the deformation radius. An advective time scale for the perturbation would then be

$$L/(U_s H) = N/(fU_s). \qquad (2.6)$$

These considerations lead to the following choice of the scaling:

$$(x, y) = L(x'', y''), \quad z = Hz'', \quad t = \frac{\sqrt{\mathrm{Ri}}}{f} t'', \qquad (2.7a)$$

$$(u', v') = U(u'', v''), \quad w' = U\frac{(f/N)}{\sqrt{\mathrm{Ri}}} w'', \qquad (2.7b)$$

$$\phi' = UfL\phi'', \quad g\theta'/\theta_0 = UN\theta'', \qquad (2.7c)$$

where U is the arbitrary (but small) characteristic horizontal wind speed of the perturbation, and $U \ll U_s H$. Here Ri denotes the Richardson number of the basic state:

$$\mathrm{Ri} = N^2/U_s^2. \qquad (2.8)$$

Substitution of (2.7) into (2.3) yields, after dropping the double primes,

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$$\frac{1}{\sqrt{\mathrm{Ri}}}\frac{du}{dt}+\frac{w}{\mathrm{Ri}}-v=-\frac{\partial\phi}{\partial x}, \tag{2.9a}$$

$$\frac{1}{\sqrt{\mathrm{Ri}}}\frac{dv}{dt}+u=-\frac{\partial\phi}{\partial y}, \tag{2.9b}$$

$$\frac{\partial\phi}{\partial z}=\theta, \tag{2.9c}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{\sqrt{\mathrm{Ri}}}\frac{\partial w}{\partial z}=0, \tag{2.9d}$$

$$\frac{d\theta}{dt}+w-v=0, \tag{2.9e}$$

where

$$\frac{d}{dt}=\frac{\partial}{dt}+z\frac{\partial}{\partial x}.$$

This scaling differs from the nondimensionalization of Stone (1966) who chose f^{-1} as the characteristic time scale and U_sH/f as the characteristic horizontal length scale.

c. Equation for W(z)

Following Eady (1949), we seek plane wave solutions to the nondimensional set (2.9) of the form; for example,

$$w(x, y, z, t)=W(z)\exp[i(k(x-ct)+ly)], \tag{2.10}$$

where (k, l) is the horizontal wavenumber vector, c the complex phase speed and $W(z)$ describes the vertical structure. Since the boundary conditions are in terms of the vertical motion field, it is natural to solve (2.9) for $W(z)$. The result (Eady 1949; Stone 1966) is

$$\left[1-\frac{k^2(z-c)^2}{\mathrm{Ri}}\right]\frac{d^2W}{dz^2}-2\left[\frac{1}{(z-c)}-\frac{il}{\sqrt{\mathrm{Ri}}}\right]\frac{dW}{dz}-\left[k^2+l^2+\frac{2il}{\sqrt{\mathrm{Ri}}\,(z-c)}\right]W=0, \tag{2.11}$$

with the boundary conditions

$$W=0 \quad \text{at} \quad z=0, 1. \tag{2.12}$$

Analytic solutions to the eigenvalue problem (2.11) with (2.12) have not been found. Numerical results for the complex phase speed c have been presented by Árnason (1963), Stone (1970), and Sela and Jacobs (1971). In the present notation, a convenient approximate expression for c in the case of baroclinically unstable modes is (Stone 1966)

$$c=\frac{1}{2}\pm\frac{i\sqrt{3}}{3}\left\{\frac{1}{2}-\frac{1}{15\,\mathrm{Ri}}\left[k^2(1+\mathrm{Ri})-l^2(1-\mathrm{Ri})-k^2l^2\frac{(12\,\mathrm{Ri}^2+25\,\mathrm{Ri}-62)}{210\,\mathrm{Ri}}\right]\right\}. \tag{2.13}$$

d. The quasi-geostrophic limit

The scaling (2.7) is consistent with that of quasi-geostrophic theory (Pedlosky 1979) with the stratification parameter, S, set to unity,

$$S=N^2H^2/f^2L^2=1,$$

and the Rossby number, ϵ, equal to the square root of the inverse of the Richardson number,

$$\epsilon=\frac{U_0}{fL}=\frac{U_sH}{fL}=\frac{1}{\sqrt{\mathrm{Ri}}}. \tag{2.14}$$

Then a perturbation series expansion in $\mathrm{Ri}^{-1/2}$, for example,

$$w=w_0+\mathrm{Ri}^{-1/2}w_1+\mathrm{Ri}^{-1}w_2+\cdots, \tag{2.15}$$

will lead to the quasi-geostrophic approximation to (2.9):

$$\frac{du_0}{dt}-v_1=-\frac{\partial\phi_1}{\partial x}, \tag{2.16a}$$

$$\frac{dv_0}{dt}+u_1=-\frac{\partial\phi_1}{\partial y}, \tag{2.16b}$$

$$\frac{\partial\phi_1}{\partial z}=\theta_1, \tag{2.16c}$$

$$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial w_0}{\partial z}=0, \tag{2.16d}$$

$$\frac{d\theta_0}{dt}+w_0-v_0=0, \tag{2.16e}$$

where

$$u_0=-\frac{\partial\phi_0}{\partial y},\quad -v_0=-\frac{\partial\phi_0}{\partial x},\quad \theta_0=\frac{\partial\phi_0}{\partial z}. \tag{2.17}$$

The set (2.16) may be solved for the vertical motion field. The result (Eady 1949) for plane wave solutions of w_0 of the form (2.10) is

$$\frac{d^2}{dz^2}W-\frac{2}{(z-c)}\frac{dW}{dz}-(k^2+l^2)W=0. \tag{2.18}$$

Comparison of (2.18) with (2.11) indicates that the quasi-geostrophic result is obtainable from the nongeostrophic one in the limit as $\sqrt{\mathrm{Ri}}\to\infty$. Eady (1949) solved the eigenvalue problem (2.18) with (2.12) analytically and found

$$c=\frac{1}{2}+\frac{i}{2\alpha}[(\alpha-\tanh\alpha)(\coth\alpha-\alpha)]^{1/2}, \tag{2.19}$$

where

$$\alpha^2=(k^2+l^2)/4. \tag{2.20}$$

Pedlosky (1979) and Gill (1982) review the features of this solution. It is convenient for the later comparison

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with the GM case to let $W = (z - c)\psi(z)$. Then (2.18) becomes

$$\frac{d^2\psi}{dz^2} - \left[K^2 + \frac{2}{(z-c)^2}\right]\psi = 0, \qquad (2.21)$$

where $K^2 = k^2 + l^2$.

3. Solution with the geostrophic momentum approximation

a. The W equation and its solution

The geostrophic momentum approximation consists in replacing the advected wind field by the geostrophic wind. Then the scaled equations (2.9) may be written as

$$\frac{1}{\sqrt{\mathrm{Ri}}}\frac{d}{dt}u_g + \frac{w}{\mathrm{Ri}} - v = -\frac{\partial\phi}{\partial x}, \qquad (3.1\text{a})$$

$$\frac{1}{\sqrt{\mathrm{Ri}}}\frac{d}{dt}v_g + u = -\frac{\partial\phi}{\partial y}, \qquad (3.1\text{b})$$

$$\frac{\partial\phi}{\partial z} = \theta, \qquad (3.1\text{c})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\sqrt{\mathrm{Ri}}}\frac{\partial w}{\partial z} = 0, \qquad (3.1\text{d})$$

$$\frac{d}{dt}\theta + w - v = 0, \qquad (3.1\text{e})$$

where

$$u_g = -\frac{\partial\phi}{\partial y}, \quad v_g = +\frac{\partial\phi}{\partial x}. \qquad (3.2)$$

It should be noted that the set (3.1) is also obtained if one linearizes the equation *after* making the geostrophic momentum approximation in (2.1). It is straightforward to solve (3.1) for the vertical motion field. The result for the plane wave solution (2.10) is

$$\frac{d^2W}{dz^2} - 2\left[\frac{1}{z-c} - \frac{il}{\sqrt{\mathrm{Ri}}}\right]\frac{dW}{dz} - \left[k^2 + l^2 - \frac{k^2}{\mathrm{Ri}} + \frac{2il}{\sqrt{\mathrm{Ri}}(z-c)}\right]W = 0. \qquad (3.3)$$

This equation reduces to the quasi-geostrophic case (2.18) in the limit as $\sqrt{\mathrm{Ri}} \rightarrow \infty$. However, (3.3) is not obtainable from the nongeostrophic result (2.11) in a straightforward manner (e.g., in the limit as $\mathrm{Ri} \rightarrow \infty$ with $\sqrt{\mathrm{Ri}}$ held finite). Equation (3.3) agrees with (2.11) for all O(1) and $O(\mathrm{Ri}^{-1/2})$ terms, but the two equations differ in the $O(\mathrm{Ri}^{-1})$ terms. In particular, (3.3) has none of the $O(\mathrm{Ri}^{-1})$ terms present in (2.11) and introduces a new term, k^2W/Ri, to the equation. Lastly it should be noted that (3.3) differs from various filtered models (Mak 1977).

To obtain an analytic solution for (3.3), we introduce an integration factor, and write

$$W(z) = (z-c)\exp\left(-\frac{ilz}{\mathrm{Ri}^{1/2}}\right)\psi(z). \qquad (3.4)$$

Then (3.3) becomes

$$\frac{d^2\psi}{dz^2} - \left[K^2\left(1 - \frac{1}{\mathrm{Ri}}\right) + \frac{2}{(z-c)^2}\right]\psi = 0, \qquad (3.5)$$

where $K^2 = k^2 + l^2$. Solution of (3.5) may be given in terms of Riccati–Bessel functions (e.g., Abramowitz and Stegun 1972).

b. Growth rates

Comparison of (3.5) with (2.21) indicates that the GM and QG eigenvalue problem differs by the factor $(1 - 1/\mathrm{Ri})$ multiplying the K^2 term. Thus,

$$c = \frac{1}{2} \pm \frac{i}{2\mu}[(\mu - \tanh\mu)(\coth\mu - \mu)]^{1/2}, \qquad (3.6)$$

where

$$\mu^2 = \frac{(k^2 + l^2)}{4}\left(1 - \frac{1}{\mathrm{Ri}}\right). \qquad (3.7)$$

Apart from the factor $(1 - 1/\mathrm{Ri})$ in the definition of μ, this result is identical to that for quasi-geostrophic theory (2.19) and (2.20). Thus, as $\mathrm{Ri} \rightarrow \infty$, the two results merge.

Figure 1 plots the growth rate $\sigma = kc_i$ where $c = c_r + ic_i$, for the nongeostrophic (NG), (2.13), the quasi-geostrophic (QG), (2.19), and the geostrophic momentum (GM), (3.6) cases. Comparison of the curves indicate that the GM result does not accurately portray the nongeostrophic effects on baroclinic instability. In particular, the GM case predicts larger growth rates and a larger wavenumber of the shortwave cutoff than the QG theory. This behavior is opposite to NG theory (Árnason 1963; Stone 1966, 1972b; Derome and Dolph 1970). [The prediction of an NG shortwave cutoff comparable to the GM case in Fig. 1 reflects the approximate nature of (2.13). Stone (1970, Fig. 10) demonstrates that (2.13) overpredicts the wavenumber of the shortwave cutoff and slightly underpredicts the maximum growth rate.]

It should be noted that each theory predicts that the steering level of the unstable disturbance is at middepth in the fluid, $c_r = ½$.

c. Relation to semigeostrophic theory

In order to ascertain the relationship of the present analysis with semigeostrophic (SG) theory (Hoskins 1975), it is advantageous to solve (3.1) for the geopotential ϕ. The result is a potential vorticity equation

$$\frac{d}{dt}q = 0, \qquad (3.8)$$

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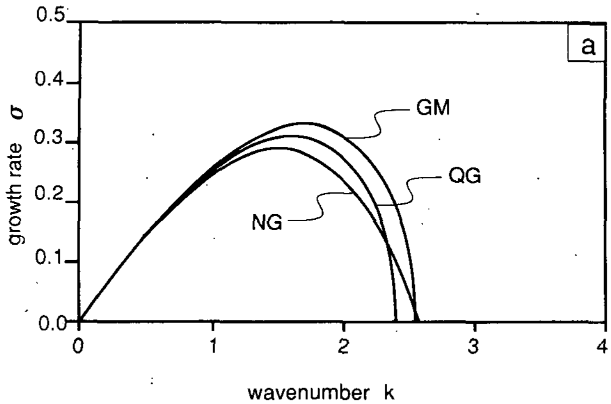

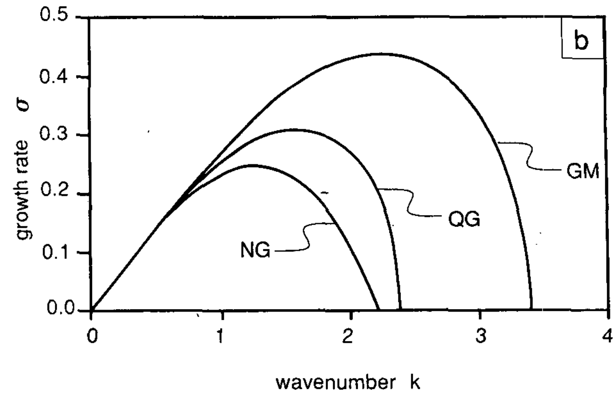

FIG. 1. Nondimensional growth rate σ as a function of zonal wavenumber k for (a) Ri = 9, and (b) Ri = 2. Here the meridional wavenumber $l = 0$ and QG, NG, and GM denote the quasi-geostrophic, nongeostrophic, and geostrophic momentum results respectively.

where

$$q = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{1}{(1-\mathrm{Ri}^{-1})}\left(\frac{\partial^2\phi}{\partial z^2} + \frac{2}{\sqrt{\mathrm{Ri}}}\frac{\partial^2\phi}{\partial y\partial z} + \frac{1}{\mathrm{Ri}}\frac{\partial^2\phi}{\partial y^2}\right). \quad (3.9)$$

As Ri → ∞, q reduces to the quasi-geostrophic result. In order to apply the boundary conditions (2.4), it is convenient to relate the vertical motion field to the geopotential by eliminating v from the heat equation (3.1e) using (3.1a) to obtain

$$\frac{d}{dt}\left(\frac{\partial\phi}{\partial z} + \frac{1}{\sqrt{\mathrm{Ri}}}\frac{\partial\phi}{\partial y}\right) + \left(1 - \frac{1}{\mathrm{Ri}}\right)w - \frac{\partial\phi}{\partial x} = 0. \quad (3.10)$$

Following SG theory, we make a coordinate transformation from physical (x, y, z, t) to geostrophic (X, Y, Z, T) space where

$$X = x, \quad Y = y - z/\sqrt{\mathrm{Ri}}, \quad Z = z, \quad T = t, \quad (3.11)$$

are the linearized geostrophic coordinates. Then by the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial Y},$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial Z} - \frac{1}{\sqrt{\mathrm{Ri}}}\frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial T},$$

and the potential vorticity equation becomes

$$\frac{d}{dT}q = 0, \quad (3.12)$$

where

$$q = \frac{\partial^2\Phi'}{\partial X^2} + \frac{\partial^2\Phi'}{\partial Y^2} + \left(\frac{1}{1-\mathrm{Ri}^{-1}}\right)\frac{\partial^2\Phi'}{\partial Z^2}, \quad (3.13)$$

and

$$\frac{d}{dT} = \frac{\partial}{\partial T} + Z\frac{\partial}{\partial X}.$$

The heat equation becomes

$$\frac{d}{dT}\left(\frac{\partial\Phi'}{\partial Z}\right) - \frac{\partial\Phi'}{\partial X} = 0 \quad \text{at} \quad z = 0, 1, \quad (3.14)$$

where Φ' is the linearized SG geopotential. Here $\Phi' = \phi$ and

$$u_g = -\frac{\partial\Phi'}{\partial Y}, \quad v_g = +\frac{\partial\Phi'}{\partial X},$$

$$\theta = \frac{\partial\Phi'}{\partial Z} - \frac{1}{\sqrt{\mathrm{Ri}}}\frac{\partial\Phi'}{\partial Y}. \quad (3.15)$$

The problem posed by (3.12)–(3.14) is similar to that treated by Hoskins (1975, 1976) in geostrophic coordinates and is identical in form to a modern (e.g., Pedlosky 1979) formulation of the Eady problem. Here, however, the factor $(1 - \mathrm{Ri}^{-1})^{-1}$ arises in (3.13) and it is this factor which accounts for the differences in the results (2.20) and (3.7) for the QG and GM phase speeds. Careful examination of Hoskins (1975, p. 240) indicates that N^2 in his equations for Φ' should be replaced by $N^2 - U^2/H^2 = N^2(1 - \mathrm{Ri}^{-1})$. Then his results are consistent with the present analysis. Hoskins (1976, pp. 107–108) alludes to this factor but does not assess its consequences.

The contribution of the present analysis is the demonstration that the linear SG model of the Eady problem is identical to making the geostrophic momentum approximation in physical space. The demonstration indicates that (i) the linearized geostrophic coordinate transformation should be used, and (ii) the potential temperature field $\theta(X, Y, Z, T)$ is *not* identical to the quasi-geostrophic result expressed in geostrophic coordinates [see (3.15)].

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d. Structure of the unstable waves

Expressions for the total velocity field in terms of Φ' are

$$u = -\frac{\partial\Phi'}{\partial Y} - \frac{1}{\sqrt{\mathrm{Ri}}}\frac{d}{dT}\left(\frac{\partial\Phi'}{\partial X}\right), \quad (3.16a)$$

$$v = \left(\frac{\mathrm{Ri}}{\mathrm{Ri}-1}\right)\left\{\frac{\partial\Phi'}{\partial X} - \frac{1}{\mathrm{Ri}}\frac{d}{dT}\left(\frac{\partial\Phi'}{\partial Z}\right)\right\} - \frac{1}{\sqrt{\mathrm{Ri}}}\frac{d}{dT}\left(\frac{\partial\Phi'}{\partial Y}\right), \quad (3.16b)$$

$$w = \frac{\mathrm{Ri}}{\mathrm{Ri}-1}\left[\frac{\partial\Phi'}{\partial X} - \frac{d}{dT}\left(\frac{\partial\Phi'}{\partial Z}\right)\right]. \quad (3.16c)$$

In the transformed space (3.11), the geopotential and geostrophic winds are identical to the Eady model. Equation (3.16c) indicates that the vertical motion field is increased in amplitude but identical in form. By (3.15) the potential temperature field is cooler (warmer) in regions where the geostrophic flow is westward (eastward). Transformation of these fields into physical space will introduce an "upwards–northwards" tilt in the meridional plane with an angle $\alpha = \tan^{-1}(1/\sqrt{\mathrm{Ri}})$. The total meridional wind may be written using (3.16b) as

$$v = ik\Phi'[1 - il\lambda(Z-c)] + O(\lambda^2), \quad (3.17)$$

where $\lambda = 1/\sqrt{\mathrm{Ri}}$. Since

$$\exp[-il\lambda(Z-c)] = 1 - il\lambda(Z-c) + O(\lambda^2), \quad (3.18)$$

(3.17) becomes

$$v = ik\Phi' e^{-il\lambda(Z-c)} + O(\lambda^2). \quad (3.19)$$

Ignoring terms of the order of the inverse Richardson number, the total meridional wind has a tilt in physical space $\alpha = \tan^{-1}(2/\sqrt{\mathrm{Ri}})$ which is approximately twice that of the other fields. This structure is consistent with the NG results of McIntyre (1965), Derome and Dolph (1970), and Mak (1977).

e. Energetics

Equations for the perturbation kinetic and potential energies are obtained from (3.1) by multiplying (3.1a) by u_g and (3.1b) by v_g and multiplying (3.1c) by θ. Thus

$$\frac{\partial}{\partial t}\bar{K}_g = \overline{w\theta} - \frac{\overline{u_g w}}{\sqrt{\mathrm{Ri}}}, \quad (3.20a)$$

$$\frac{\partial}{\partial t}\bar{P} = -\overline{w\theta} + \overline{v_g\theta}, \quad (3.20b)$$

where $K_g = (u_g^2 + v_g^2)/2$ and $P = \theta^2/2$ are the eddy kinetic and potential energies, respectively, and the overbar denotes an integration over a full horizontal wavelength of the disturbance and over the total fluid depth. The energy conversion terms on the right-hand side of (3.20) have their usual interpretation.

The corresponding energy equations for the QG case have a form similar to (3.20) but without the term $-\overline{u_g w}/\sqrt{\mathrm{Ri}}$. Thus, GM theory allows for the conversion of available kinetic energy (AKE) to eddy kinetic energy (EKE), while QG theory does not. Besides this additional energy source, GM energetics differ quantitatively from QG theory since the expression for the potential temperature and vertical motion field differ [see (3.15) and (3.16)]. Figure 2 provides an example for the most unstable square ($k = l = 1.131$) Eady mode. It is seen that the GM case enhances the conversion of available potential energy (APE) to eddy kinetic energy.

Perturbation energy equations for the nongeostrophic (NG) problem are

$$\frac{\partial\bar{K}}{\partial t} = \overline{w\theta} - \frac{\overline{uw}}{\sqrt{\mathrm{Ri}}}, \quad (3.21a)$$

$$\frac{\partial\bar{P}}{\partial t} = -\overline{w\theta} + \overline{v\theta}, \quad (3.21b)$$

where $K = (u^2 + v^2)/2$. Comparison of (3.20) and (3.21) indicates that, unlike GM theory, NG energetics allows the ageostrophic wind components to assist in the conversion of both APE and AKE to EKE. In the present notation, approximate expressions for the conversion terms in (3.21) for the most unstable baroclinic disturbances with $l = 0$ (Stone 1972a) are

$$\overline{v\theta} = \frac{9}{25}\sqrt{\frac{2}{15}}\left(1 + \frac{1}{\mathrm{Ri}}\right)^{5/2}, \quad (3.22a)$$

$$\overline{w\theta} = \frac{1}{10}\sqrt{\frac{3}{10}}\left(1 + \frac{1}{\mathrm{Ri}}\right)^{3/2}, \quad (3.22b)$$

$$-\frac{\overline{uw}}{\sqrt{\mathrm{Ri}}} = \frac{1}{60}\sqrt{\frac{3}{10}}\frac{1}{\sqrt{\mathrm{Ri}}}\left(1 + \frac{1}{\mathrm{Ri}}\right)^{1/2}, \quad (3.22c)$$

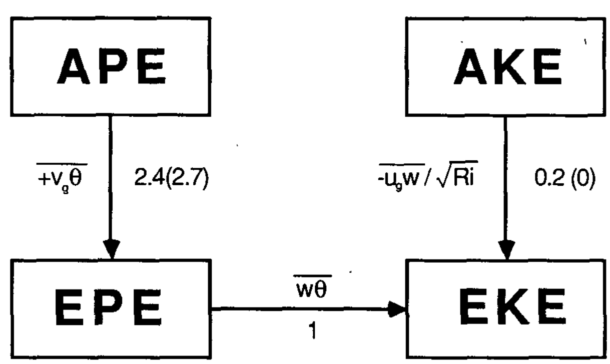

FIG. 2. Energy conversion diagram for the unstable square wave $k = l = 1.131$ for the geostrophic momentum approximation with Ri = 9. Values in parentheses denote the quasi-geostrophic case (Ri = ∞) for which this wave is the most unstable. In each case, $\overline{w\theta} = 1$.

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where the most unstable mode corresponds to

$$k = \left(\frac{5/2}{1 + 1/\mathrm{Ri}}\right)^{1/2}, \quad (3.23a)$$

$$c = \frac{1}{2} + i\frac{\sqrt{3}}{9}. \quad (3.23b)$$

Table 1 provides a comparison between the GM and NG cases of the features of the most unstable mode. Since the amplitudes of the modes in this linear problem are arbitrary, the energy conversion terms have been normalized such that $\overline{w\theta} = 1$ to facilitate comparison. As noted earlier, the GM case overpredicts the growth rate of the most unstable mode. Consistent with this greater growth rate, the conversion of APE to EPE is more efficient for the GM modes and this efficiency increases with decreasing Richardson number. For the $l = 0$ mode, $u_g = 0$, and the GM modes cannot tap the AKE. In contrast the efficiency of the APE conversion of the NG mode decreases with decreasing Richardson number and there is a small conversion of AKE.

4. Solutions with other approximations

a. VG case: v-field geostrophic

Here the meridional wind is assumed to be in geostrophic balance. This assumption is made in two-dimensional semigeostrophic theory (Hoskins and Bretherton 1972). To avoid confusion with the three-dimensional formulation of SG theory (Hoskins 1975), this case is denoted VG. Then (3.1) holds, except that (3.1a) is replaced with

$$-v = -v_g = -\frac{\partial \phi}{\partial x}. \quad (4.1)$$

Combination of this modified set yields an equation for the plane wave vertical motion field (2.10):

$$\frac{d^2W}{dz^2} - \frac{2}{(z-c)}\frac{dW}{dz} - k^2W = 0, \quad (4.2)$$

which holds for arbitrary l.

This result corresponds to the QG case (2.18) in the limit as $l \to 0$ and solutions of (4.2) are directly obtainable from QG theory. Thus the VG case yields growth rates identical to two-dimensional QG theory and the solution is independent of Ri. In contrast, the limit of (3.3) as $l \to 0$ does not yield (4.2). Thus, three-dimensional SG theory does not reduce to its two-dimensional counterpart.

In terms of their structure these modes have fields for u_g, v_g, ϕ, θ, and w that are identical to the Eady solution. In particular, the fields display no tilt in the meridional plane.

b. UG case: u-field geostrophic

If the zonal wind is assumed to be geostrophic (hereafter denoted the UG case), then (3.1) holds with (3.1b) replaced by

$$u = u_g = -\frac{\partial \phi}{\partial y}. \quad (4.3)$$

The associated $W(z)$ equation is

$$\frac{d^2W}{dz^2} - 2\left(\frac{1}{(z-c)} - \frac{il}{\sqrt{\mathrm{Ri}}}\right)\frac{dW}{dz} - \left[l^2 + \frac{2il}{\sqrt{\mathrm{Ri}}(z-c)}\right]W = 0, \quad (4.4)$$

which holds for arbitrary k.

This result corresponds to either the NG (2.11) or the GM (3.3) case in the limit as $k \to 0$. Stone (1966) has discussed (4.4) as the approximation near the symmetric axis valid for $\mathrm{Ri} \geq 1$. His solution (2.9) for c may be shown to be identical to (3.6) with $k = 0$.

5. Conclusions

This paper focuses on the consequences of making the geostrophic momentum (GM) approximation in the Eady model of baroclinic instability. This GM model has been shown to be identical to the linear three-dimensional semi-geostrophic (SG) model which uses the geostrophic coordinate transformation. As Fig. 1 and Table 1 demonstrate, this theory incorrectly "opens up" the wedge of instability, leading to larger growth rates and greater values for the wavenumber of the most unstable mode and of the shortwave cutoff. This behavior occurs because the effective static stability, N_e^2, in linear SG is proportional to the basic state potential vorticity,

$$N_e^2 = N^2(1 - 1/\mathrm{Ri}),$$

and N_e^2 decreases with decreasing Richardson number. While this effect is small for large values of the Richardson number (Ri), significant differences with nongeostrophic theory arise as Ri tends to unity. This result suggests that three-dimensional models of moist semigeostrophic baroclinic instability (e.g., Thorpe and Emanuel 1985, appendix), in which the buoyancy frequency is greatly reduced, will overestimate the impact of the latent heat release. In contrast, as shown in sec-

TABLE 1. Wavenumber k, growth rate σ, and energy conversion terms of the most unstable $l = 0$ mode for the geostrophic momentum case as a function of the Richardson number. Values in parentheses are the approximate nongeostrophic results.

Ri	k	σ	$\overline{v\theta}$	$\overline{w\theta}$	$\overline{-uw}/\sqrt{\mathrm{Ri}}$
∞	1.60 (1.58)	0.31 (0.30)	2.75 (2.40)	1 (1)	0 (0)
9	1.70 (1.50)	0.33 (0.29)	2.44 (2.67)	1 (1)	0 (0.05)
2	2.26 (1.29)	0.44 (0.25)	1.37 (3.60)	1 (1)	0 (0.08)

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tion 4a, two-dimensional semigeostrophic theory, in which the cross-shear wind is in geostrophic balance, predicts unstable wave behavior identical to quasi-geostrophic theory. Thus, the differences in growth rate with nongeostrophic theory are not as large in the two-dimensional formulation of semigeostrophic theory. It should be noted, however, that the structure of the three-dimensional GM solution agrees with NG theory better than either its two-dimensional counterpart or QG theory.

These disparate results for the GM theory may be reconciled in the following manner. The nongeostrophic w-equation contains terms of order 1, $\mathrm{Ri}^{-1/2}$ and Ri^{-1}. The quasi-geostrophic approximation to this equation contains all terms of order 1 but no higher terms. The corresponding geostrophic momentum equation contains all terms of order 1 and $\mathrm{Ri}^{-1/2}$ and an order Ri^{-1} term that is not present in the nongeostrophic analysis. As the leading order correction to the quasi-geostrophic result for the growth rate is order Ri^{-1}, the geostrophic momentum approximation does not give this correction properly but overpredicts the instability. However, the leading correction to the structure of the unstable Eady wave is order $\mathrm{Ri}^{-1/2}$ and the geostrophic momentum approximation captures this feature accurately.

An important limitation of this study is its restriction to a linear analysis. Since most applications of GM theory have been nonlinear ones, it is natural to conclude with some discussion of the implications of the present results to finite-amplitude effects. It has been shown for small amplitude perturbations that GM theory overpredicts the growth rate of three-dimensional baroclinic waves. In order for the theory to describe accurately the wave's behavior as it grows to finite amplitude, it would be necessary for the action of the nonlinearity to compensate for the too rapid growth initially. Since nonlinearities (e.g., the stretching of relative vorticity) generally accelerate the cyclogenesis, it is speculated that the linear results will hold qualitatively at finite amplitude.

The present study draws a distinction between the two-dimensional (2-D) and three-dimensional (3-D) versions of semigeostrophic theory. This distinction may also exist in the nonlinear regime since the nonlinear definitions of the potential vorticity (q) differ for the Eady model. Using the definition for q given by Hoskins (1975) and Hoskins and Bretherton (1972), one finds that the 3-D value is a factor $(1 - 1/\mathrm{Ri})$ less than the 2-D value, suggesting a reduced stability in the 3-D case. It is emphasized that further research is required in order to confirm these ideas in the nonlinear regime.

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