Barotropic Instability of Basic States with a Realistic Jet and a Wave

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ABSTRACT

The stability of basic states consisting of a jet similar to the stratospheric polar night jet and a traveling wave with a single zonal wavenumber is examined in a linearized nondiabatic barotropic model on a sphere. Basic state waves are chosen to resemble observed traveling and stationary features in the winter stratosphere. Results are presented for disturbance growth rates, propagation characteristics, and energy conversion as a function of the basic state wave amplitude. The effects of small amplitude basic state waves on unstable disturbances arising from a zonally symmetric jet are discussed; results are shown where a small amplitude basic state wave dramatically affects the stability characteristics. Evidence is shown that the presence of a traveling wave may favor the appearance of disturbances that include other zonal wavenumbers which move with the basic state wave; this result is discussed in relation to the origin of observed quasi-nondispersive features in the polar winter stratosphere. Results for a stationary wavenumber 1 basic state wave suggest that a distorted polar vortex may be unstable to disturbances that would lead to further distortion. An unstable disturbance for a basic state with an eastward moving wavenumber 2 has components which resemble, in period and location, traveling waves that are observed in the winter stratosphere.

1. Introduction

Traveling planetary-scale waves are ubiquitous in the winter stratosphere. Hartmann (1976) studied the structure of the Southern Hemisphere stratosphere during late winter, and observed a circulation dominated by zonal wavenumbers 1 and 2 with propagation periods in excess of two weeks. He also reported an eastward traveling zonal wavenumber 3 component with a period near 6 days. The quasi-stationary wave-number 1 and eastward-traveling wavenumber 2 present in the winter and spring have been extensively reported in both Southern and Northern Hemispheres (Mechoso and Hartmann 1982; Yu et al. 1984; Randel 1987), and have been shown to be prominent during the time of the Southern Hemisphere final warming (Yamazaki and Mechoso 1985). Other planetary-scale disturbances observed in the Southern Hemisphere polar winter stratosphere include quasi-nondispersive features at high latitudes (Lait and Stanford 1988), which consist of zonal wavenumbers 1 through at least 4 moving with the same phase speed, such that a coherent feature is observed that moves around the globe in about 4 days. Due to the prominence of traveling planetary waves in the winter stratosphere, and their presence during such events as the Southern Hemisphere final warming, the stability of a state that includes not only a jet resembling the stratospheric polar night jet, but also a traveling wave is an important consideration in understanding the dynamics of the polar winter stratosphere.

Barotropic instability of the polar night jet has been examined in a zonally symmetric model on a sphere by Hartmann (1983), and by Manney et al. (1988). Hartmann (1983) used a hyperbolic secant jet profile with strengths and widths typical of the polar night jet, and examined the dependence of stability characteristics on these parameters. Similar jet profiles were used by Michelangeli et al. (1987) to examine barotropic stability on Mars and Venus. Manney et al. (1988) extended the study of Hartmann to include jet profiles that are asymmetric about the jet axis, and jets with a realistic summer and winter hemisphere structure, and concentrated on analyzing jet profiles taken from individual months of observational data. The above studies show unstable modes at wavenumbers 1 through 6 at high latitudes, which are approximately nondispersive and have periods such that they encircle the globe in about 4 days. Some jet profiles also give rise to lower frequency unstable modes on the equatorward side of the jet. While barotropically unstable features that might occur if the polar winter stratosphere were zonally symmetric have been discussed in considerable detail, a more realistic basic state must also include one or more of the wave features that are prominent in this region. Hartmann (1983) speculated...
that the presence of a zonal wavenumber 1 traveling with a period near 4 \text{d} might favor the appearance, through a barotropic wave instability mechanism, of a disturbance that includes higher zonal wavenumbers moving with the basic state wave. This point may also be addressed in the context of a wave stability study.

The basic states used in barotropic wave stability studies have usually been of two types, either highly idealized flows with a single wave, or climatological mean states. A detailed review of wave stability studies may be found in Grotjahn (1984). Briefly, Lorenz (1972) examined the stability of a pure, zonally propagating Rossby wave on an infinite beta plane, and showed that a wave of sufficient amplitude is barotropically unstable. Several studies of wave instability on an infinite beta-plane for idealized flows suggest the possibility of parametric instability with regions where the stability is increased or decreased depending on the wavenumber, frequency, and amplitude of the basic state wave (Gill 1974; Coaker 1977; Plumb 1977). Plumb (1977) also discussed some effects of a bounded domain on stability of a nonparallel flow. Other idealized studies of wave stability, carried out in a beta-plane channel, such as Przybylowicz and Loesch (1987), show cases where waves may become unstable only within a range of amplitudes that has both upper and lower bounds.

Hoskins (1973) and Baines (1976) examined the linear stability of a Rossby-Haurwitz wave on a sphere. Baines (1976) presents critical amplitudes for instability and curves of growth rate versus basic state wave amplitude of the most unstable modes for a number of planetary waves. Frederiksen (1978) compared the results of a two-layer model to those of Baines (1976) and found that barotropic waves produce large changes in disturbance streamfunctions and in momentum and heat fluxes.

At the other extreme, there have been several studies in spherical models dealing with the stability of climatological mean flows, such as Simmons et al. (1983), Branstator (1983), and, using layer models, Frederiksen (1985, and references therein). Although some studies have examined the stability of stratospheric flows (Frederiksen 1982), a time-averaged flow cannot be used to examine the stability of a basic state which includes a traveling wave; in a spherical model, the stability characteristics depend on the phase speed of the wave. Frederiksen (1980) examined instability in a two-layer model on a sphere with a basic state defined by a general zonal flow profile, and a single planetary wave; his basic states were typical of the troposphere. Frederiksen's (1980) results show complicated structures for barotropic basic waves, reflecting strong zonal wavenumber coupling in the disturbance streamfunction.

The modeling approach in this paper is similar to Frederiksen (1980) but focuses instead on barotropic instability of states typical of the winter stratosphere.

We utilize a barotropic nondivergent model on a sphere to examine the stability characteristics of zonally asymmetric basic states which include a jet with meridional structure and strength similar to the polar night jet, and a traveling wave typical of one observed in the polar winter stratosphere. A barotropic nondivergent model is chosen because 1) the simplicity of the model allows examination of a number of relatively realistic basic states, and 2) several observed waves in the stratosphere, most notably the quasi-nondispersive features reported by Lait and Stanford (1988), are equivalent barotropic in structure. To focus on the effect of a traveling wave on the stability of a state characterized by a strong zonal jet, we define a basic state that includes a realistic jet profile, as used by Manney et al. (1988), and a single spherical harmonic wave traveling with a specified frequency. The basic state waves used are typical of eastward-moving features that are observed in the stratosphere only during winter. While the use of a single wave implies that the meridional structure is not realistic over the entire globe, the local nature of barotropic instability suggests that results for regions where the wave structure is realistic should not be significantly affected by the structure at distant latitudes. To check this, we compare results for a case where the basic state wave is a sum of two spherical harmonics of the same zonal wavenumber with one of the cases with a single wave.

In view of the importance of the zonal jet in the winter stratosphere and the prominence of propagating planetary waves, we address several fundamental questions concerning the stability of the polar night jet in the presence of a traveling wave: 1) How does a small amplitude traveling wave in the basic state affect the properties of unstable modes that arise from a zonally symmetric basic state; 2) How do the structure and growth rates of these unstable disturbances change as the amplitude of the basic state wave is increased; 3) How sensitive are these results to changes in the frequency of the basic state wave; 4) Under what circumstances (amplitudes, structure, and frequency of basic state wave) do additional unstable disturbances occur, and what are their characteristics; and 5) What is the relative magnitude of the contribution from each part of the basic state to the energy of an unstable disturbance?

2. The model and calculations

Following Baines (1976), the linearized, nondivergent, barotropic vorticity equation on a sphere may be written in a coordinate system rotating with the angular velocity \( \omega_0 \) as

\[
\frac{\partial}{\partial t} \nabla^2 \psi' + J(\psi_b, \nabla^2 \psi') \\
+ J(\psi', \nabla^2 \psi_b + f) + 2\omega_0 \frac{\partial \psi'}{\partial \lambda} = 0
\]  
(1)

Where \( \psi \) is the streamfunction, \( \psi_b \) is the basic state streamfunction, and \( f \) is the Coriolis parameter.
where

\[
J(a, b) = \frac{1}{r_e^2} \left( \frac{\partial a}{\partial \mu} \frac{\partial b}{\partial \lambda} - \frac{\partial a}{\partial \lambda} \frac{\partial b}{\partial \mu} \right),
\]

(2)

\(\psi'\) is the perturbation streamfunction, \(\psi_b\) the zonally varying basic state streamfunction, \(f = 2\Omega\mu = 2\Omega \sin \phi\), \(\lambda\) is longitude, \(\phi\) is latitude, \(r_e\) the mean radius of the earth, and \(\Omega\) the angular velocity of the earth's rotation. In this coordinate system, the basic state streamfunction is written as

\[
\psi_b = \psi_{b_\mu} + \psi_{b_\lambda} = \psi_{b_\mu}(\mu) + 2\lambda P_0^\mu \cos \lambda \mu.
\]

(3)

Thus, the basic wave, denoted by \(P_0^\mu \cos \lambda\mu\), is traveling with an angular phase speed of \(\omega_b\), or a frequency of \(p_{0b}\), with respect to the earth. Since we work in a rotating coordinate system, only traveling basic stream waves with a single zonal wavenumber, or a set of waves moving with the same phase speed may be included; here we concentrate on basic stream waves with a single zonal wavenumber. The \(\psi_{b_\mu}\) is a realistic zonally symmetric jet profile and is expanded in Legendre polynomials as described in Manney et al. (1988). The perturbation streamfunction is expanded in spherical harmonics as

\[
\psi' = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \psi_{m}^n P_{m}^n e^{i(m\lambda - \sigma t)}.
\]

(4)

Substituting (3) and (4) into (1) results in an eigenvalue problem for the disturbance frequency, \(\sigma\). For computational efficiency, the basic state streamfunctions are chosen to be antisymmetric with respect to the equator (odd), so that modes with \(n - m\) even and \(n - m\) odd are decoupled (Baines 1976). We also show results only for the disturbances with streamfunctions that are antisymmetric with respect to the equator. This is commonly done since the antisymmetric streamfunctions, which have a zero at the equator, are considered to be more typical of the real atmosphere (Baines and Frederiksen 1978; Frederiksen 1980). The formulation of the eigenvalue problem is given in appendix A.

The growth of kinetic energy of an unstable disturbance can be written as

\[
\frac{\partial KE}{\partial t} = \text{CKX} + \text{CKYW} + \text{CKYJ},
\]

(5)

where KE is the globally averaged kinetic energy in the disturbance field, CKX represents energy conversion due to zonal variations in the basic state, CKYW conversion due to meridional variations of the wave part of the basic state, and CKYJ conversion due to the jet. The forms of CKX, CKYW, and CKYJ, and a discussion of some numerical difficulties in calculating these quantities are given in appendix B. The energy conversion terms described above are globally averaged values, thus giving a measure of the net contribution of each part of the basic state to the growth of an unstable disturbance. The results we present show CKX/|CKYJ| and CKYW/|CKYJ|; these give the energy conversion from the wave part of the basic state relative to that of the jet part, with positive values indicating that the disturbance is receiving energy from the basic state wave. The sum CKX + CKYW gives the total contribution to the disturbance energy from the basic state wave. Partitioning the basic state wave energy conversion into CKX and CKYW allows us also to show the relative importance of energy conversion from zonal and meridional variations of the basic state wave.

Manney et al. (1988) noted that very high meridional truncations were necessary to capture the behavior of unstable modes arising from high latitude jets. This difficulty was also noted by Grotjahn (1987) in sensitivity studies for a three-dimensional model using the same expansion in the horizontal as is used here. He showed that growth rates decrease as the truncation increases, and that rhomboidal truncation at wave-number 30 \((M = N = 30)\) may be inadequate for some flows. In tests of our model using a zonally symmetric basic state, the growth rates of unstable modes, particularly those at the higher zonal wavenumbers \((m \geq 4)\), were found to decrease with increasing meridional truncation up to values near \(N = 60\); however, the periods and spatial structure of the disturbance streamfunctions are well represented at lower truncation values, near \(N = 30\). Because of these truncation characteristics and considerations of computational efficiency, we do not use a standard rhomboidal truncation, but rather one with a larger number of meridional than zonal components; the truncation used is \(M = 10, N = 35\). For a zonally symmetric basic state at this meridional truncation, unstable modes with zonal wavenumbers greater than 3 have faster growth rates (by several days) than those obtained at truncations with \(N \geq 60\), but the spatial structure and periods are well represented. Some tests of the meridional truncation were also performed using the full model with a wave included in the basic state. These tests show that the disturbances that are altered by increasing \(N\) above 35 are those whose growth rates are slower at \(N = 35\). For instance, in a case where the most unstable disturbances at \(N = 35\) have e-folding times around 6 to 7 d, the disturbances that are altered significantly when \(N\) is increased are those with e-folding times greater than about 15 d. Since the presence of a wave in the basic state introduces coupling between zonal wavenumbers, we must also examine the zonal truncation. Tests with different zonal \((M)\) truncations show that only unstable disturbances with dominant amplitudes at zonal wavenumbers greater than 7 are significantly affected by the truncation at \(M = 10\). Because observed stratospheric wave spectra in the winter are dominated by zonal wavenumbers less than 6, the truncation at \(M = 10, N = 35\) is adequate for this study.
3. Basic states

We choose zonally symmetric jet profiles representative of the stratospheric polar night jet, and run the model with different amplitudes, phase speeds, zonal wavenumbers, and meridional structures for the basic state wave. The jet profile used is given by

\[
\bar{\omega} = U_0 r_0^{-1} \left\{ \text{sech}[2(\phi - \phi_0)B^{-1}] \right. \\
+ \text{sech}[2(\phi + \phi_0)B^{-1}] \right\},
\]

(6)

This form is used by Manney et al. (1988) and Hartmann (1983), and the resulting unstable modes for several values of the parameters are discussed in detail in those papers. With one exception, we use \( U_0 = 150 \text{ m s}^{-1}, B = 20^\circ, \) and \( \phi_0 = 60^\circ \) (Jet 1). This profile gives a maximum wind speed of about 80 m s\(^{-1}\) and is typical in strength, width, and latitudinal position of the stratospheric polar night jet in the Southern Hemisphere. As discussed in Manney et al., this profile gives rise to unstable modes at wavenumbers 1 through 6 whose periods and meridional structure closely resemble observed quasi-nondispersive features in the polar winter stratosphere (Lait and Stanford 1988). For Jet 1, there is a change in sign of the absolute vorticity gradient on the poleward side of the jet, fulfilling the necessary condition for barotropic instability. Unstable modes appear that are associated with the region of negative absolute basic state vorticity gradient, are confined to the poleward side of the jet, and are approximately nondispersive. These modes have periods such that it takes about 3.6 d to move around the globe, and the most unstable mode, in this case the zonal wavenumber 1, has an e-folding time near 6 d.

The meridional structures of \( \psi_{bw} \) (the wave part of the basic state) used with Jet 1 are those of the associated Legendre polynomials \( P_4^1, P_6^3, P_3^2, P_4^4, \) and the sum \( \frac{1}{2}(P_4^1 + P_6^3) \). Figure 1 shows the meridional structures of a) \( P_4^1 \) and \( \frac{1}{2}(P_4^1 + P_6^3) \), b) \( P_3^2 \), and c) \( P_4^3 \) and \( P_6^3 \). These structures, with the exception of \( P_4^1 \), are chosen to represent observed features. The eastward moving zonal wavenumber 2 and 3 features reported by Hartmann (1976) and by Mechoso and Hartmann (1982) have maximum amplitudes near 50 to 60° latitude, and the \( P_3^2 \) and \( P_6^3 \) structures are chosen to resemble these. While the reported periods vary considerably, the observed zonal wavenumber 3 generally has a period of less than a week, and the zonal wavenumber 2 a period in excess of 2 weeks. The periods of the basic state waves are 4 d for the \( P_3^2 \), and 16 d for the \( P_3^2 \).

To examine some effects of different meridional structures, we include results for a wavenumber 3 with the \( P_3^2 \) structure (Fig. 1c) and the same period as the \( P_6^3 \) basic state wave.

The meridional structure of the \( P_4^1 \) is chosen to represent both the observed quasi-stationary wavenumber 1 (Randall 1987), and the zonal wavenumber 1 component of the fast-moving, quasi-nondispersive feature.
A decomposition into associated Legendre polynomials with \( m = 1 \) for the wavenumber 1, 4 day wave shows the two largest components to be \( P_4^1 \) and \( P_6^1 \), which have similar magnitudes and the same sign. Thus, the profile \( \frac{1}{2}(P_4^1 + P_6^1) \) corresponds more closely to the observed fast-moving wavenumber 1 feature. Both the \( P_4^1 \) and the \( \frac{1}{2}(P_4^1 + P_6^1) \) profiles will be used as meridional profiles with a basic state wave period of 3.64 d. The \( P_4^1 \) profile will also be studied for a stationary wave (\( \omega_b = 0 \)).

Although generalized necessary conditions for barotropic instability of a zonally varying flow have yet to be derived, the structure and growth of disturbances to such a flow are expected to be related to the changes in the basic state absolute vorticity gradient. Figure 2 shows the absolute vorticity gradients at longitudes where \( \cos \lambda = -1, 0, \) and 1 for Jet I, with a) \( P_4^1 \), b) \( P_4^2 \), c) \( P_4^3 \), and d) \( P_6^3 \), with \( A = 1.0 \times 10^7 \text{ m}^2 \text{ s}^{-1} \) [see Eq. (3)]. The profile with \( \cos \lambda = 0 \) is the absolute vorticity gradient for the zonally symmetric jet profile used. For comparison, the streamfunction amplitude of the zonally symmetric jet at 85° latitude is 2.5 \( \times 10^8 \text{ m}^2 \text{ s}^{-1} \).

Hartmann (1983) also presented results for stronger and narrower jets of the form (6). For the jet given by (6) with \( U_0 = 180 \text{ m s}^{-1}, B = 10^0, \) and \( \phi_0 = 60^0 \) (Jet II), the quasi-nondispersive poleward modes take approximately 3 d to encircle the globe. There are also unstable modes at zonal wavenumbers \( \geq 2 \) that have maximum amplitudes on the equatorward side of the jet; these equatorward modes have longer periods than the poleward modes and considerably broader merid-

**Fig. 2.** Basic state absolute vorticity gradients for cases with Jet I at longitudes where \( \cos \lambda = 0 \) (dotted line), +1 (solid line), and −1 (dashed line) (\( p \) is the basic state zonal wavenumber), at a basic state wave amplitude of \( A = 1.0 \times 10^7 \text{ m}^2 \text{ s}^{-1} \): (a) \( P_4^1 \), (b) \( P_4^2 \), (c) \( P_4^3 \), and (d) \( P_6^3 \) basic state waves.
ional structures (Hartmann 1983). The equatorward zonal wavenumber 3 mode for the parameters listed above has a period of about 3 d. Hartmann (1983) suggested that these unstable modes may interact strongly with waves propagating up from the troposphere that have similar periods, and occur in the same latitudinal regions. Manney et al. (1988) noted that during a month when the observed jet gave rise to unstable modes at zonal wavenumbers 3 and 4 on the equatorward side of the jet, the quasi-nondispersive feature was not observed (Lait and Stanford 1988). One possible explanation for this result is that the presence of a feature resembling one of the equatorward modes inhibits the growth of barotropically unstable disturbances on the poleward side of the jet. Although the case of Jet I plus a $P_3^3$ wave addresses this point to some extent, the profile of Jet II is closer to the jets that are present when unstable equatorward modes occur. To further address this point, and the suggestion of interaction with waves from another source, we examine the effect of a traveling zonal wavenumber 3 on unstable modes that arise from a strong, narrow jet by looking at a basic state that includes Jet II, and a wave with the meridional structure $P_3^3$ (Fig. 1c) with a period of 3.2 d. The absolute vorticity gradients for this basic state for $\cos^2 \lambda = -1$, 0, and 1 and $A = 1.0 \times 10^7$ m$^2$ s$^{-1}$ are shown in Fig. 3 (the streamfunction amplitude at 85° for this zonally symmetric profile is 5.6 $\times 10^8$ m$^2$ s$^{-1}$).

To provide a reference for the magnitude of the wave in the basic state relative to the jet profile, we give values for amplitudes as a percent of the contribution from the zonally symmetric jet, defined by $(A_i/A_j)$ × 100%, where $A_i$ is the amplitude of the streamfunction from the zonally symmetric jet at 85° latitude. The stream-

**4. Results**

**a. Procedures and general disturbance properties**

A fundamental difference between the barotropic instability problem for zonally symmetric and zonally varying basic states is the coupling between zonal wavenumbers in the latter problem. Figure 4 shows an example of the spectral distribution of amplitudes for several types of basic states. The distribution for the unstable wavenumber 1 mode resulting from the zonally symmetric basic state given by Jet I is shown in Fig. 4a; since each zonal wavenumber is linearly independent in the zonally symmetric problem, the non-zero amplitudes are confined to zonal wavenumber 1. As suggested by the need for large meridional truncations, the meridional spectrum is quite complicated, with significant amplitudes at large values of $n - |m|$. The waves that are used as a part of the basic state in this study can, in the absence of a jet, themselves become barotropically unstable if the amplitude, $A_i$ is sufficiently large. The orthogonality conditions for spherical harmonics are such that a wave with zonal wavenumber $p$ will interact only with other components that have zonal wavenumbers $m$ and $r$ given by $p = m \pm r$. Figure 4b shows an unstable disturbance for a basic state consisting entirely of the wave $P_3^3$, traveling eastward with a period near 4 d. In contrast to the zonally symmetric problem, several zonal wavenumbers are involved; the separation of the components at intervals of three in zonal wavenumber is due to the selection rule. The meridional structure is considerably simpler than that for unstable modes from the zonally symmetric jet; this is consistent with Baines' (1976) result that many unstable disturbances growing on a basic state consisting of a Rossby-Haurwitz wave were well represented by a triad interaction. To determine if the critical amplitude for instability of the wave alone is related to changes in the stability characteristics of disturbances arising from a basic state consisting of both a jet and a wave, we calculate approximate critical amplitudes for instability of the wave alone. These amplitudes will be marked on plots of $e$-folding time versus basic state wave amplitude by an open triangle on the amplitude axis.

Figure 4c shows a spectrum for the most unstable disturbance growing on a basic state consisting of Jet I plus the $P_3^3$ wave used above, at an amplitude of about 2% of the jet contribution (as defined above in section 3). The meridional distribution at zonal wavenumber $m = 1$ is nearly identical to that shown in Fig. 4a for the jet alone, and additional spectral amplitudes.
appear at zonal wavenumbers $1 \pm 3k$, where $k$ is an integer. The spatial structure of the disturbance streamfunction and the period of the unstable disturbance are nearly identical to those of the unstable mode shown in Fig. 4a. In later sections we will refer to this type of disturbance as one that "corresponds to the unstable wavenumber 1 mode from the zonally symmetric problem." This identification is made on the basis of the distribution of spectral amplitudes, the similarity in the appearance of the disturbance streamfunctions, and the similarity of propagation characteristics.

The real part of an eigenvalue obtained from the solution of this problem represents the propagation frequency of the unstable disturbance in the coordinate system rotating with respect to the earth with angular velocity $\omega_p$. Because an unstable disturbance may include several zonal wavenumbers, the disturbances are generally dispersive, and conversion to a frequency in a coordinate system at rest with respect to the earth is not straightforward. At each time, the structure must be calculated separately for each zonal component of the disturbance streamfunction field because components of different zonal wavenumbers with the same frequency have different phase speeds. To examine the propagation characteristics with respect to the earth of an unstable disturbance, a time–longitude plot of the disturbance streamfunction field may be constructed at a representative latitude. From such a plot, approximate periods with respect to the earth can be estimated. The plots of the "period", shown in the following sections are for the period in the rotating coordinate system, given by $2\pi/\sigma_r$, where $\sigma_r$ is the real part of the eigenvalue for the unstable disturbance.

b. Basic states with zonal wavenumber 1

We examine basic states with zonal wavenumber 1 which represent two observed features in the polar winter stratosphere, the quasi-stationary wavenumber 1, and the eastward-moving 4 day wave. Due to the wavenumber selection rule discussed above, the zonal structure of spectra for unstable disturbances growing on a basic state with zonal wavenumber 1 is considerably more complicated than for higher basic state wavenumbers, since nonzero amplitudes may appear at all zonal wavenumbers. Characteristics of the unstable disturbances are significantly altered from the zonally symmetric problem at basic state wave amplitudes much lower than for higher basic state wavenumbers.

Fig. 4. Spectral distribution of amplitude components of an eigenvector for an unstable disturbance growing on three types of basic states: (a) a zonally symmetric jet (Jet 1), (b) an eastward traveling zonal wavenumber 3 with no jet, and (c) Jet 1 plus the eastward traveling zonal wavenumber 3 shown in (b). The amplitude scale is arbitrary.
The first basic state is Jet I plus a stationary zonal wavenumber one \( (\omega_b = 0) \) with the meridional structure \( P_1^1 \) (Fig. 1a). Figure 5 shows (a) period in days, (b) e-folding time in days, and (c) energy conversion terms (terms for conversion due to the wave relative to that due to the jet, as discussed in section 2) as a function of basic state wave amplitude for the most unstable disturbance whose streamfunction is antisymmetric with respect to the equator. The dashed lines represent the characteristics of a disturbance which corresponds (in period and spatial structure, as discussed in section 4a) to the unstable zonal wavenumber 1 mode from Jet I at amplitudes where this disturbance is not the most unstable. The correspondence can be recognized for basic state wave amplitudes up to about 9% of the contribution from the jet. For basic-state wave amplitudes between the arrows, the most unstable disturbance is stationary (the real part of the eigenvalue is zero). At basic-state wave streamfunction amplitudes \( \leq 2.75 \times 10^7 \text{ m}^2 \text{ s}^{-1} \) (10% of the Jet I contribution, as defined in section 3), the e-folding time of the most unstable disturbance is increased over that for the zonally symmetric problem; the beginning of the sharp decrease in the e-folding time of the most unstable disturbance corresponds to the basic-state wave amplitude where the stationary disturbance first becomes the most unstable. Figure 6 shows (a) spectral amplitudes and (b) a contour plot of the disturbance streamfunction field for the most unstable disturbance \( A = 3.0 \times 10^7 \text{ m}^2 \text{ s}^{-1} \) (12% of the jet contribution). The stationary disturbance consists largely of wavenumber 2 components, with significant values also at wavenumber 1. At amplitudes between 1.5 and 2.5 \((\times 10^7 \text{ m}^2 \text{ s}^{-1})\) the most unstable disturbance has a period greater than 20 d, and is dominated by components with zonal wave numbers 1 and 2. The spatial structure is similar to the stationary disturbance that is most unstable at higher amplitudes. This similarity, and the absence of any other disturbance resembling the long period (>20 d) disturbance at higher amplitudes, suggests that the disturbance with a period near 20 d is of the same type as the stationary one, but that only in a limited range of amplitudes are the basic states favorable to phase locking of the basic state wave and an unstable disturbance.

Figure 5c shows that at basic state wave amplitudes where the stability is either greatly increased [between \( A = 1.3 \) and 2.3 \((\times 10^7 \text{ m}^2 \text{ s}^{-1})\)] or decreased \((A > 2.6 \times 10^7 \text{ m}^2 \text{ s}^{-1})\), the magnitude of the energy conversion due to the basic state wave is increased; both terms (CKX and CKYW) are positive, so the basic state wave is a destabilizing influence to the most unstable disturbance at these amplitudes. In both of these regions, corresponding to the unstable wavenumber 1 mode in the zonally symmetric problem. The arrows indicate an amplitude range where the most unstable disturbance is stationary. The open triangle indicates the critical amplitude for instability of a stationary \( P_1^1 \) without a jet.
the term CKX, representing energy conversion due to zonal variations in the basic state, becomes the dominant source of energy for the most unstable disturbance; the term CKYW, representing energy conversion from meridional variations due to the basic state wave remains smaller than the conversion from the zonally symmetric jet. For amplitudes up to about 1.8 \times 10^7 \text{ m}^2 \text{ s}^{-1}, where disturbances corresponding to unstable modes of the zonally symmetric problem can be recognized, the energy conversion due to the basic state wave remains small, and in the higher amplitude part of that region, becomes negative, indicating that the basic state wave is a stabilizing influence to the disturbance corresponding to the unstable zonal wavenumber 1 in the zonally symmetric problem.

The other basic-state zonal wavenumber 1 cases include an eastward-moving wave representative of the zonal wavenumber 1 component of the observed quasi-dispersive feature discussed above. We use Jet I plus a basic state wave given by \( P_4 \) with \( \omega_b = 2.0 \times 10^{-5} \text{ rad s}^{-1} \), which gives a basic state wave period of 3.64 d. We also examine results for the same jet profile (Jet I) and a basic state wave with the meridional structure \( \frac{1}{2}(P_4 + P_5) \) and a 3.64 day period, and compare results for the two meridional structures.

Figure 7 shows (a) “period,” (b) e-folding time, and (c) energy conversion terms for the most unstable disturbance whose streamfunction is antisymmetric with respect to the equator in the eastward moving case with the meridional wave profile \( P_4 \). The “period” shown is \( 2\pi/\sigma \), corresponding to the period in the coordinate system rotating with a 3.64 day period. The period of the basic state wave is very near that of the unstable wavenumber 1 mode from the zonally symmetric case, and thus the phase speed of the basic state wave is near that of the unstable modes from the zonally symmetric problem. Hence a small change in the amplitudes of the zonal components in the disturbance field will correspond to a large change in the real part of the eigenvalue. The “period” plotted in Fig. 7a changes dramatically as the structure of the disturbance changes, and reveals little about the propagation characteristics. Figure 8 shows a series of spectra and contour plots of the most unstable disturbance at 3 amplitudes, \( A = 0.4, 1.0 \) and \( 2.0 \times 10^6 \text{ m}^2 \text{ s}^{-1} \) (0.2, 0.4 and 0.8% of the jet contribution, respectively). At the lower two amplitudes, the contour plot shows a structure dominated by zonal wavenumber 2, and the spectral distribution of amplitudes at wavenumber 2 closely resembles that for the unstable wavenumber 2 in the zonally symmetric problem. Figure 8c shows a highly distorted pattern, with the largest spectral amplitudes at zonal wavenumbers 1 and 2. The most unstable disturbance at higher amplitudes continues to resemble Fig. 8c. Construction of time–longitude plots of the disturbance streamfunction at 70° latitude at the three amplitudes shows that at each amplitude, although the disturbance includes increasing spectral amplitude at several zonal wavenumbers, and therefore is dispersive, the disturbance pattern shows recognizable features.
with eastward-moving periods in the range of 4 to 5 d; Fig. 9 shows the time–longitude plot at $A = 2 \times 10^6$ m² s⁻¹.

The general trend shown in Fig. 7b is for the basic state to become more stable as the amplitude of the basic state wave is increased. However, at very low amplitudes, up to about $2 \times 10^6$ m² s⁻¹, the most unstable disturbance grows faster than any unstable mode in the zonally symmetric problem. Also, the increase in stability with increasing basic state wave amplitude is nonmonotonic; near 4.0 and 8.0 ($\times 10^6$ m² s⁻¹) (1.6 and 3.2% of the contribution from the jet, respectively) there are also regions where the e-folding time reaches a local minimum. The plot of energy conversion terms (Fig. 7c) shows that the presence of the wave in the basic state destabilizes (the sum of the two terms shown is positive, indicating energy conversion from the basic state wave to the disturbance) at amplitudes below about $3.5 \times 10^6$ m² s⁻¹ and in a small region near $6.0 \times 10^6$ m² s⁻¹; over most of the amplitude range considered, the effect is stabilizing. The energy contribution from the basic state wave remains smaller than that from the jet at all amplitudes shown; for most amplitudes shown, the contribution from zonal variations in the basic state (CKX) is larger in magnitude than the contribution from meridional variations of the wave.

Another unstable disturbance of interest appears between basic state wave amplitudes of 1.1 and 1.7 ($\times 10^6$ m² s⁻¹) (0.4 and 0.7% of the contribution from the jet). The real part of the eigenvalue for this disturbance is zero, indicating that the disturbance is moving with the basic state wave (it is “stationary” in the rotating coordinate system). Figure 10 shows (a) e-folding time and (b) energy conversion terms for this disturbance. The rapid transition to instability and again to stability at a higher amplitude is typical of all “stationary” disturbances obtained. Figure 11 shows the structure of this disturbance at $A = 1.2 \times 10^6$ m² s⁻¹, where it is most unstable. Figure 11c shows the total streamfunction field where the maximum disturbance streamfunction amplitude is taken to be 15% of the maximum basic state streamfunction amplitude. The disturbance is dominated by zonal wavenumber 1 components, as well as having some amplitude at other zonal wavenumbers, which, when combined with the basic state results in an isolated disturbance (which is traveling with a 3.64 d period) that is more localized than a simple wavenumber 1 mode. The energy conversion from the basic state wave shown in Fig. 10b is very small, and the sum CKX + CKYW is always negative, indicating that the wave in the basic state is stabilizing.

To check the sensitivity of the results to the merid-
Fig. 8. Spectral amplitudes and contour plots for the disturbance streamfunction field for the most unstable disturbance shown in Fig. 7, for (a) $A = 4.0 \times 10^5$ m$^2$ s$^{-1}$, (b) $A = 1.0 \times 10^6$ m$^2$ s$^{-1}$, and (c) $A = 2.0 \times 10^6$ m$^2$ s$^{-1}$. Contours are uniformly spaced.
ional profile of the basic state wave, we show an additional case with a basic state consisting of Jet I plus a wave with the meridional structure $\frac{1}{2}(P_4^1 + P_6^1)$ (Fig. 1a) moving with the same period, 3.64 d, as the previous case. As mentioned above (Fig. 1a) this meridional profile is narrower than the $P_4^1$ alone, and thus closer to the observed structure of the 4-day wave (Lait and Stanford 1988). Figure 12 shows (a) “period,” (b) $e$-folding time, and (c) energy conversion terms for this case as a function of basic state wave amplitude. Comparing Fig. 12 to Fig. 7, we observe behavior that is qualitatively the same in both cases; the “period” decreases in a smooth curve, the $e$-folding times are characterized by stabilization with increasing basic state wave amplitude, punctuated by several local minima, and the energy conversion terms are stabilizing, except at very low amplitudes, with CKX being larger in magnitude than CKYW. The results are also similar in that “stationary” disturbances appear at low amplitudes and are dominated by zonal wavenumber 1 components. This indicates that the basic-state wave period, and the general latitudinal location of the peak are more important than the exact meridional structure away from the region of interest in modeling the general barotropic stability characteristics of a jet and a traveling wave.

c. Basic states with zonal wavenumber 2

The wavenumber 2 case is chosen to represent the observed traveling wavenumber 2 in the winter and early spring stratosphere (Hartmann 1976; Mechoso and Hartmann 1982). The basic state consists of Jet I plus a wave with meridional structure $P_2^3$, and $\omega_b = 2.3 \times 10^{-6}$ rad s$^{-1}$, giving a period of 16 d. Figure 13 shows (a) “period,” (b) $e$-folding time, and (c) energy conversion terms for the most unstable disturbance (solid line) as a function of basic state wave amplitude. In the range $1.0 \times 10^7 \leq A \leq 3.5 \times 10^7$ m$^2$ s$^{-1}$ (4 to 14% of the jet contribution), the most unstable disturbance is “stationary,” i.e., it is moving with the basic state wave. Fig. 13b shows that the growth rate of the most unstable disturbance increases with the amplitude of the basic state wave. The energy conversion terms shown in Fig. 13c indicate that the basic state wave is contributing energy to the most unstable disturbance for all basic state wave amplitudes, except at $A = 1.5$.
Fig. 11. (a) Spectral amplitudes, (b) disturbance streamfunction field, and (c) total streamfunction field for the disturbance shown in Fig. 10, at $A = 1.2 \times 10^6$ m$^2$ s$^{-1}$. The disturbance streamfunction field in (c) is taken to be 15% of the largest basic state streamfunction value. Latitudes are from 30° to 80°. Contours are uniformly spaced.

$x = 10^7$ m$^2$ s$^{-1}$ where both CKX and CKYW go to zero. As in the previous cases, the energy conversion from zonal variations in the basic state (CKX) is generally larger than that due to the meridional variations introduced by the basic state wave (CKYW). Over much of the range of amplitudes studied, both CKX and CKYW are larger than the conversion due to the zonally symmetric jet.
The spectral structure and a contour plot of the "stationary" and most unstable disturbance at $A = 2.5 \times 10^7 \text{ m}^2 \text{s}^{-1}$ (10% of contribution from jet) are shown in Fig. 14. This disturbance is dominated by spectral amplitudes at zonal wavenumbers 1 and 3. Since the phase speed of the disturbance is $2.3 \times 10^6 \text{ rad s}^{-1}$, the wavenumber 1 components have periods near 32 d, and the wavenumber 3 components near 10 d.

The dashed lines in Fig. 13a and 13b represent the unstable disturbance which most closely corresponds to the unstable zonal wavenumber 1 mode in the zonally symmetric jet problem. The dashed lines with triangles and open circles in Fig. 13c represent, respectively, the values for CKX and CKYW for this disturbance. This disturbance is much more stable at amplitudes where the most unstable disturbance is "stationary," and becomes increasingly more unstable again when the most unstable disturbance is no longer the stationary one. The energy conversion terms from the wave remain large and positive throughout the range of amplitudes studied, and CKX is considerably larger than CKYW; both are larger at most amplitudes than the contribution from the jet.

d. Basic states with zonal wavenumber 3

The first zonal wavenumber 3 case we examine is chosen to represent the observed eastward-moving wavenumber 3 feature in the winter stratosphere, reported by Hartmann (1976). We use a basic state with Jet 1 plus a wave of the form $P_6^3$, with $\omega_b = 6.0 \times 10^{-6}$ rad s$^{-1}$, giving a basic state wave period of 4 d. The $P_6^3$ structure peaks between 50° and 60° latitude (Fig. 1c). To examine some effects of the latitudinal position of the peak, we include results for a meridional structure $P_4^3$ (Fig. 1c), which peaks near 30°, moving with the same period. Figure 15 shows (a) "period," (b) e-folding time, (c) energy conversion terms for $P_4^3$, and (d) energy conversion terms for $P_6^3$, as a function of basic state wave amplitude for the most unstable disturbance. The disturbance that is most unstable for the $P_4^3$ case, and at low amplitudes for the $P_6^3$ case, is one which corresponds to the wavenumber 1 mode seen in the zonally symmetric problem (Fig. 4c) showed this disturbance for the $P_6^3$ case at $A = 5.0 \times 10^6$ m$^2$ s$^{-1}$). For $P_6^3$, at the higher amplitudes shown, the "period" of the most unstable disturbance exhibits abrupt changes as the amplitude of the basic state wave is changed. The dashed lines in Fig. 15 show the characteristics of this disturbance where it is not the most unstable one for $P_6^3$. The abrupt changes in the period at adjacent amplitudes suggest that of several disturbances which are unstable, conditions at a particular amplitude may favor a different one of these disturbances to be most unstable. This is supported by the fact that we can identify the disturbance corresponding to the zonal wavenumber 1 mode of the symmetric problem at amplitudes where it is not the most unsta-
The general trend shown in Fig. 15b is for the basic state to become more unstable as the amplitude of the basic state wave is increased; in the \( P_6^3 \) case, however, the stability of the flow is increased between \( A = 7.0 \) and \( 17.0 \times 10^6 \text{ m}^2 \text{ s}^{-1} \) (3% and 7% of the jet contribution), with increased instability for \( A > 17 \times 10^6 \text{ m}^2 \text{ s}^{-1} \).

In the \( P_4^3 \) case, the energy conversion terms (Fig. 15c) are always positive, indicating that the wave in the basic state is always a destabilizing influence. The magnitudes of the conversion terms remain small compared to the conversion from the jet; and CKYX is larger than CKX at amplitudes above about 2% of the jet contribution. In the \( P_6^3 \) case (Fig. 14d) both energy conversion terms are positive at low basic-state wave amplitudes, with a region in the amplitude range 1.2 to \( 1.5 \times 10^7 \text{ m}^2 \text{ s}^{-1} \) where the sum CKX + CKYX is negative. Above amplitudes of about \( 1.2 \times 10^7 \text{ m}^2 \text{ s}^{-1} \), CKX is generally larger in magnitude than CKYX. At amplitudes higher than the sharp increase in both terms near \( 1.5 \times 10^7 \text{ m}^2 \text{ s}^{-1} \), the most unstable disturbance is one for which the jet is a stabilizing influence (i.e., in the relative units used here, CKYJ = -1, rather than +1; see discussion in section 2). As mentioned above, and illustrated by comparing Figs. 2c and 2d, the structure of the \( P_4^3 \) wave at a given amplitude, \( A \), has less effect on the shape of the basic state vorticity gradient than the other waves studied; thus it is not surprising that the results in the \( P_6^3 \) case are reminiscent of those at lower amplitudes in the \( P_4^3 \) case.

The other case examined using the \( P_6^3 \) wave includes the zonally symmetric profile of Jet II, the stronger and narrower jet discussed in section 3 (Fig. 3 shows the absolute vorticity gradient). The basic state wave has phase speed \( \omega_b = 7.5 \times 10^{-6} \text{ rad} \text{ s}^{-1} \), corresponding to a period of 3.2 d. As noted in section 3, the phase speed of the basic state wave is near that of the unstable wavenumber 3 in the zonally symmetric problem. Figure 16 shows (a) “period,” (b) \( e \)-folding time, and (c) energy conversion terms for the most unstable disturbance whose streamfunction is antisymmetric with respect to the equator (\( n - m \) odd) as a function of basic state wave amplitude. Again, the arrows indicate a range of amplitudes (between 0.2% and 1.1% of the jet contribution) where the most unstable disturbance is one which is moving with the basic state wave. The dashed lines in Fig. 16 represent a disturbance corresponding to the unstable zonal wavenumber 2 on the poleward side of the jet in the zonally symmetric problem. This disturbance can be identified for all basic state wave amplitudes considered. The most unstable mode in the zonally symmetric problem with Jet II is the equatorward wavenumber 4 mode. At basic state

![Fig. 13.](image-url) (a) "Period" (in rotating frame), (b) \( e \)-folding time, and (c) energy conversion from the basic state wave as a function of \( A \), for the most unstable disturbance (solid line) for a basic state with Jet I, and an eastward-moving wave \( P_2^2 \text{cos}2\Delta \), with a period of 16 d. Dashed lines show the disturbance corresponding to the unstable wavenumber 1 mode in the zonally symmetric problem. The arrows indicate an amplitude range where the most unstable disturbance is moving with the basic state wave. The open triangle indicates the critical amplitude for instability of the eastward-moving \( P_2^2 \) without a jet.
wave amplitudes below $1.0 \times 10^6$ m$^2$ s$^{-1}$, the most unstable disturbance is dominated by zonal wavenumber 4 components, and corresponds to the equatorward wavenumber 4 in the zonally symmetric problem. The e-folding times shown in Fig. 16b for the most unstable disturbance show a slight general trend towards greater stability as the basic state wave amplitude is increased. There are two regions, near $A = 2.5 \times 10^6$ and $1.3 \times 10^7$ m$^2$ s$^{-1}$, where the e-folding time shows a local minimum.

Figure 17 shows the spatial structure of the disturbance streamfunction for the most unstable disturbance at $A = 2.5 \times 10^6$ m$^2$ s$^{-1}$ (0.45% of the jet contribution); at this amplitude the most unstable disturbance is "stationary." The spatial structure is dominated by wavenumber 3 components, corresponding to the equatorward zonal wavenumber 3 in the zonally symmetric problem. Other nonzero spectral amplitudes occur at zonal wavenumbers 0 (a zonal flow) and 6. Figure 16c shows that the energy conversion from the basic state wave in the amplitude range where the most unstable disturbance is "stationary" is very small, with conversion due to zonal variations in basic state (CKX) negative (stabilizing), and conversion due to meridional variations in the wave (CKYW) positive (destabilizing).

The magnitudes of both CKX and CKYW remain very small until $A = 1.3 \times 10^7$ m$^2$ s$^{-1}$ and then increase rapidly (both positive) with CKX becoming larger than CKYW. The most unstable disturbance at these amplitudes is still dominated by zonal wavenumber 3 components, with a meridional structure similar to the equatorward zonal wavenumber 3 mode in the zonally symmetric problem, but no longer moving with the basic state wave.

e. Summary of general characteristics of results

In the following paragraphs, we elucidate some similarities and differences in the results presented above for cases with different basic state waves. Although the cases have been chosen as representative of observed stratospheric features, a comparison of the results for basic state waves with different spatial structures and propagation periods reveals some general characteristics of unstable disturbances growing on a basic state with a realistic jet and a traveling wave.

An obvious contrast is between cases with a basic state wave of zonal wavenumber 1 and higher basic state zonal wavenumbers. The zonal wavenumber selection rule for spherical harmonics allows disturbance amplitudes to appear at all zonal wavenumbers when the basic state wave is zonal wavenumber 1. The disturbances in the zonal wavenumber 1 cases cease to correspond to those in the zonally symmetric problem at lower basic state wave amplitudes.

Referring to the absolute vorticity gradients shown in Figs. 2 and 3, we see that the wavenumber 2 and 3 cases show larger differences in the shape of the vorticity gradient on the equatorward side of the jet, whereas the wavenumber 1 case shows much smaller changes on the equatorward side, but greater changes in shape on the poleward side of the jet. This difference may also contribute to more rapid alterations in the character of disturbances corresponding to unstable pole-
Fig. 15. (a) "Period" (in rotating frame), (b) e-folding time, and (c) and (d) energy conversion terms as a function of $A$ for the most unstable disturbances for a basic state given by Jet I and a wave, either $P_5^0 \cos 3\lambda$ (dots) or $P_5^3 \cos 3\lambda$ (x's), traveling eastward with a 4 d period. The dashed line is a disturbance in the $P_5^3$ case that corresponds to the unstable wavenumber 1 mode from the zonally symmetric problem with Jet I. The open triangle indicates the critical amplitude for instability of the $P_5^3$ without a jet.

ward modes in the zonally symmetric problem when the basic state wave has zonal wavenumber 1. Figures 2b and 2d (for the $P_5^2$ and $P_5^3$ cases, respectively) show that at some longitudes, there is a change in sign of the absolute vorticity gradient on the equatorward side of the jet, which, by analogy to the zonally symmetric problem, we might expect to allow the occurrence of unstable disturbances which have largest amplitudes on the equatorward side of the jet. In fact, there are unstable disturbances in both of these cases at amplitudes above approximately $7.5 \times 10^4$ m$^2$ s$^{-1}$ that are unrelated to any in the zonally symmetric problem and that have spatial structure with largest amplitudes on the equatorward side of the jet. In the $P_5^2$ case, the "stationary" disturbance shown in Fig. 14 also has large amplitudes on both the poleward and equatorward sides of the jet.

Plots of energy conversion terms for the most unstable disturbance show that the conversion due to zonal variations of the basic state wave (CKX) is generally larger in magnitude than that due to meridional variations of the basic state wave (CKYW) at amplitudes where the most unstable disturbance does not correspond to an unstable mode of the zonally symmetric problem. Although the basic state wave amplitudes shown are generally less than about 15% of the contribution from the jet, the energy conversion due to the wave (CKX + CKYW) exceeds that from the jet in several of the cases studied; in the $P_4^1$ case with a stationary wave (Fig. 5c) CKX alone exceeds the en-
energy conversion from the jet, and in the $P_3^2$ case both CKX and CKYW individually exceed the conversion from the jet (Fig. 13c). Thus, zonal variations in the basic state not only allow additional unstable disturbances to appear, but also contribute the largest portion of the energy for these disturbances.

In situations where unstable disturbances can be identified that correspond to unstable modes in the zonally symmetric problem, the $\epsilon$-folding times of these disturbances may change dramatically for a relatively small amplitude basic state wave. For example, in the $P_3^2$ case, when the basic state wave amplitude is $8\%$ of the jet contribution, the $\epsilon$-folding time for the disturbance corresponding to the zonal wavenumber 1 mode in the zonally symmetric problem is approximately $75\%$ longer than that in the zonally symmetric case (Fig. 13b). At higher amplitudes, around $15\%$ of the jet contribution, it is about $75\%$ shorter. The $P_3^3$ case with Jet I shows similar behavior (Fig. 15b). Thus, a small-amplitude basic state wave can give rise to large changes in the $\epsilon$-folding times of unstable disturbances corresponding to unstable modes of the zonally symmetric problem, while the spatial structure and propagation periods remain near the values in the zonally symmetric problem.

Plots of $\epsilon$-folding time generally show nonmonotonic behavior with increasing basic state wave amplitude. For example, Fig. 5b shows that in the stationary $P_3^1$ case, between 1.5 and $2.5 \times 10^7$ m$^2$ s$^{-1}$ the most unstable mode is much more stable, and at higher amplitudes it becomes much more unstable. Nonmonotonic behavior is also apparent in Figs. 7b, 12b, 15b, and 16b. This behavior is reminiscent of the results of several idealized wave instability studies mentioned in the introduction, where resonant behavior, regions of increased and decreased stability (Gill 1974; Coaker 1977; Plumb 1977), and upper and lower bounds on the amplitudes of basic state waves that may become unstable (Przybylowicz and Loesch 1987) are reported.

Three cases were examined where the basic state wave period was near that of an unstable mode with the same zonal wavenumber in the corresponding zonally symmetric problem. We note several differences in the general results between these three cases (which we denote as type I cases) and the other cases studied (denoted as type II cases) where the period of the basic state wave was sufficiently different from that of an unstable mode from the zonally symmetric problem:

- The $\epsilon$-folding times for type I cases (Figs. 7b, 12b and 16b) show a (nonmonotonic) trend towards greater stability as the basic-state wave streamfunction ampli-

![Fig. 16. (a) "Period" (in rotating frame), (b) $\epsilon$-folding time, and (c) energy conversion from the basic state wave as a function of $A_1$ for the most unstable disturbance (solid line) for a basic state with Jet II, and an eastward-moving wave $P_3^3 \cos 3\lambda$ with a period of 3.2 d. Dashed lines in (b) show the disturbance corresponding to the unstable poleward wavenumber 2 mode in the zonally symmetric problem. The arrows indicate an amplitude range where the most unstable disturbance is moving with the basic state wave. The open triangle indicates the critical amplitude for instability of the eastward-moving $P_3^1$ without a jet.](image)
Fig. 17. (a) Spectral amplitudes and (b) polar contour plot of disturbance streamfunction field for the case shown in Fig. 16, at $A = 2.5 \times 10^7$ m$^2$ s$^{-1}$. The disturbance is moving with the basic state wave. Latitudes in (b) are from 30° to 80°.

**tude is increased, with less stable regions where the e-folding time reaches a local minimum. In contrast, type II cases show a general trend towards greater instability as the basic state wave amplitude is increased; in cases where the meridional structure of the wave peaks near the jet peak, there is first a region of increased stability (Figs. 5b and 15b).

- In type I cases, a region where the most unstable disturbance is more unstable than the most unstable mode in the zonally symmetric problem occurs at very low amplitudes (where the contribution from the wave is less than 1% of that from the jet), and “stationary” features appear in this amplitude range, with very little energy conversion from the basic state wave. “Stationary” disturbances appear at much higher amplitudes in type II cases.

- The dominant component of “stationary” disturbances in type I cases is mainly that of the basic state wave (Figs. 11 and 17), while type II cases include large amplitudes at other zonal wavenumbers (Figs. 6 and 14).

To examine further the effect of the basic state wave phase speed on the behavior of unstable modes at low basic state wave amplitudes, we return to Jet I, and a basic state wave with the meridional structure $P_4^1$, with $A = 1.0 \times 10^7$ m$^2$ s$^{-1}$ (0.4% of the contribution from the jet), and examine the stability for a number of basic state wave phase speeds. Figure 18 shows the e-folding time of the two most unstable disturbances as a function of the period of the basic state wave. At the amplitude used, we can identify these disturbances as ones corresponding to the zonal wavenumber 1 (dots) and 2 (squares) modes of the zonally symmetric problem. The arrows indicate the period and e-folding time of the unstable zonal wavenumber 1 mode in the zonally symmetric problem. At $\omega_1 = 2.0 \times 10^{-3}$ rad s$^{-1}$ (period of 3.64 d), several unstable disturbances appear that are dominated by zonal wavenumber 1 or 2 components. Figure 18 shows the most unstable of these. At this low amplitude, the growth rates of unstable disturbances are near those for the corresponding unstable modes in the zonally symmetric case, until the basic state wave period becomes less than about 5 d (eastward moving). For eastward moving periods less than this the e-folding times change rapidly. The disturbance corresponding to the zonal wavenumber 1 mode is the most unstable, except near a basic state wave period of 3.64 d. There it becomes more stable, and the disturbance corresponding to zonal wavenumber 2 becomes the most unstable. It is only near this 3.64 d period that the most unstable disturbance is more unstable than the most unstable mode in the zonally symmetric problem.

The results for this test, and the similarities discussed above between results in cases where the basic state wave phase speed is near that of an unstable mode of the zonally symmetric problem, suggest that even a very small amplitude basic state wave can dramatically affect the stability characteristics and the structure of
a disturbance when there is matching between the structure and phase speed of the basic wave and an unstable mode in the zonally symmetric problem.

Referring again to the plots of e-folding time (Figs. 5b, 7b, 13b, 15b and 16b), we note that the position of the mark indicating the critical amplitude for instability of a basic state including only the wave does not consistently bear a clear relation to changes in energy conversion or e-folding time for the most unstable disturbances growing on the basic state including both the jet and the wave. Thus we find that the critical amplitude for instability of a wave alone is not fundamentally related to the amplitude at which that wave begins to have a substantial effect on the stability of a state including both the wave and a jet.

5. Applications to the stratosphere

In the Introduction, we discussed cases where a barotropic wave stability model can be used to examine the stability characteristics of observed stratospheric features. Basic state waves were chosen to resemble planetary-scale waves that are observed in the winter stratosphere.

Several cases studied are relevant to examination of the possible origin of fast-moving, quasi-nondispersive features that are observed in the Southern Hemisphere winter stratosphere (Lait and Stanford 1988). Barotropic instability studies of zonally symmetric jets have shown that there are unstable modes which are approximately nondispersive and resemble the observed quasi-nondispersive feature in meridional structure and propagation period (Manney et al. 1988). Hartmann (1983) speculated that the presence of a zonal wave-number 1 mode, with a period near 4 d, might favor the appearance of higher wavenumbers moving with the same phase speed. The cases studied here for an eastward moving wavenumber 1 provide an idealized test of this suggestion. In both cases studied, disturbances appear that are moving with the same speed as the basic state wave, which are largely composed of zonal wavenumber 1, with smaller amplitude components at higher wavenumbers (Fig. 11). The appearance of this type of disturbance, when combined with the basic state (Fig. 11c) is less localized than the "warm pools" reported by Lait and Stanford (1988), but does contain amplitude at higher wavenumbers. In the model, these disturbances appear only for a narrow range of basic-state wave amplitudes, which are less than 1% of the maximum streamfunction amplitude in the jet. Since observations of these features are in temperature data, the amplitude of the observed
features cannot be directly compared to the barotropic streamfunction amplitudes used here; the amplitude of the 4-day wavenumber 1 shown by Lait and Stanford (1988) is in the range \( \frac{1}{4}\% \) to 2\% of the mean temperature shown. The exact amplitudes for which stationary disturbances appear in the model also depend on the details of the basic state wave used. Modeling evidence for such disturbances supports Hartmann's (1983) suggestion that the presence of a zonal wavenumber 1 mode can lead to the appearance of an unstable disturbance which includes higher wavenumbers moving with the basic state wave.

The most unstable disturbance for the eastward-moving \( P_0^1 \) case discussed in section 4b is also of some interest in examining possible origins for observed quasi-nondispersive features. Although this disturbance does not have exactly the same period as the basic state wave, and therefore is not completely nondispersive, a longitude–time plot (Fig. 9) for this disturbance at 70\° latitude shows features moving with a period between 4 and 5 days, which can be identified for several periods.

Another problem relevant to the study of observed quasi-nondispersive features is the effect of other waves which may be present in the polar winter stratosphere on the stability characteristics of the polar night jet. In particular, the two cases (with Jet I and Jet II) for a \( P_0^3 \) wave are relevant since Manney et al. (1988) found that the polar night jet during a month where the quasi-nondispersive feature was not observed was unstable to modes of zonal wavenumber 3 and 4, which peak on the equatorward side of the jet. Our results show that unstable disturbances corresponding to the approximately nondispersive poleward modes of the zonally symmetric problem can be identified in these cases for all the amplitudes studied. In the case with Jet I, the disturbance corresponding to the zonal wavenumber 1 ceases to be the most unstable, and at some amplitudes has a much longer e-folding time than the most unstable disturbance. In the case with Jet II, which more closely resembles the observed jet at that time, the equatorward modes are the most unstable in the zonally symmetric case; disturbances corresponding to poleward modes remain more stable, but can still be identified. Thus, the presence of a feature resembling one of the equatorward modes does not preclude the appearance of disturbances corresponding to the quasi-nondispersive poleward modes in a linear model, although for specific situations the growth rates of these features may be sufficiently slow that their appearance in the real atmosphere would be unlikely.

We note further that the disturbances corresponding to the poleward modes in the zonally symmetric case can be identified at all amplitudes considered for the \( P_0^3 \) case, but the presence of a stationary zonal wavenumber 1 feature does lead to their disappearance at an amplitude of about 4\% of the basic state wave amplitude. Thus, if barotropic instability is the origin of the observed quasi-nondispersive feature, one might not expect to observe this feature during times when the observed quasi-stationary wavenumber 1 is particularly strong. The observational study by Lait and Stanford (1988) looked only at fast-moving features, and thus did not address this question. Further observational studies for the months examined by Lait and Stanford would be of interest to catalog what other stationary or traveling wave features are present during times when the quasi-nondispersive feature is and is not observed.

Another suggestion raised by Hartmann (1983) was that barotropically unstable equatorward modes which arise from a basic state that includes a narrow jet may interact strongly with waves propagating up from the troposphere which have similar periods. The case run in section 4d above with Jet II (a strong and narrow jet) and the basic state wave \( P_0^3 \) addresses the question of what effect the presence of a zonal wavenumber 3 with a period similar to that of a barotropically unstable wavenumber 3 mode has on the stability of that basic state. The results show that with a basic state wave of small amplitude (0.2\% to 1.1\% of the jet contribution), the instability of the state is increased (by about 20\% over the most unstable mode in the zonally symmetric problem, and about 30\% over the equatorward wavenumber 3 mode), and the most unstable disturbance becomes one which is dominated by wavenumber 3 components, with nonzero amplitudes also at wavenumbers 0 (zonal flow) and 6. For the amplitude range mentioned, the most unstable disturbance moves with the same period as the basic state wave; at higher amplitudes it does not, but still is dominated by zonal wavenumber 3 components. Thus the presence of a zonal wavenumber 3 wave moving with a phase speed near that of a barotropically unstable mode enhances the instability of that state to disturbances including components of wavenumber 3 and multiplies thereof, consistent with the suggestion made by Hartmann (1983).

As a further application to observed stratospheric phenomena, we note that previous studies, such as Frederiksen (1982), have shown that wave instability may be an important factor in stratospheric warmings; it is likely to be important in understanding the final warming in the Southern Hemisphere. The stratospheric circulation at the time of the final warming is highly distorted, with a quasi-stationary wavenumber 1 and an eastward traveling wavenumber 2 being prominent (Yamazaki and Mechoso 1985). Both the stationary basic state wavenumber 1, and the eastward traveling wavenumber 2 that we considered show a region, at comparatively large basic wave amplitudes (around 8\% to 12\% of the jet contribution), of greatly increased instability where the most unstable disturbance is one which moves with the same phase speed as the basic state wave. In the case of the stationary zonal wavenumber 1, the disturbance has large am-
amplitudes at zonal wavenumbers 1 and 2 (Fig. 6). These results suggest that barotropic instability of a stratospheric state, which is initially distorted, would contribute to the further distortion of the circulation. The unstable disturbance in the wavenumber 2 case has large amplitudes at zonal wavenumbers 1 and 3, with wavenumber 1 and 3 periods near 32 and 10 days, respectively. These periods are near those reported in observations of the late winter stratosphere by Hartmann (1976); he reported a wavenumber 1 with a period near 34 d and a wavenumber 3 with a period near 6 d. Thus, instability of a basic state with a jet and an eastward-moving zonal wavenumber 2 may lead to the appearance of additional traveling waves that resemble observed features.

6. Summary and conclusions

The linear stability of zonally asymmetric basic states has been examined in a nondivergent barotropic model on a sphere. The basic states consist of a jet, similar in strength and meridional structure to the stratospheric polar night jet, and a traveling wave with a spherical harmonic structure. The waves examined are chosen to represent observed stratospheric features, and have a spatial structure given by one or two spherical harmonics of a particular zonal wavenumber. We focus on the effect of a small-amplitude basic state wave on unstable modes which arise from a zonally symmetric basic state. To address these questions, we have identified unstable disturbances that correspond in period and spatial structure to unstable modes of the zonally symmetric problem. In examining these disturbances, we find:

- In cases where the basic state zonal wavenumber is greater than 1, disturbances corresponding to unstable modes of the zonally symmetric problem can be identified at all amplitudes studied.

- These disturbances show a spatial structure with spectral amplitudes at the zonal wavenumber of the unstable mode in the symmetric problem and smaller amplitudes at zonal wavenumbers multiples of $p$ from this value, where $p$ is the basic state zonal wavenumber.

- The e-folding times for disturbances corresponding to unstable modes of the zonally symmetric problem may be changed dramatically by the presence of a small amplitude basic state wave while their periods and spatial structure are relatively unaltered.

- In the stationary wavenumber 1 case, disturbances corresponding to unstable modes of the zonally symmetric problem appear only at low amplitudes.

Question 4 addresses the occurrence of other unstable disturbances that are not related to unstable modes in the zonally symmetric problem. We note that:

- Additional unstable disturbances do occur in all the cases considered here.

- In cases (denoted as type 1) where the period of the basic state wave is near that of an unstable mode in the zonally symmetric problem, unstable disturbances occur at low amplitudes, which move with the basic state wave, and have spatial structures dominated by the zonal wavenumber of the basic state wave.

- In the cases with a stationary zonal wavenumber 1 and with an eastward moving zonal wavenumber 2, disturbances occur at relatively high amplitudes which move with the basic state wave. These are the most unstable disturbances over a range of amplitudes where the most unstable disturbance is very quickly growing (e-folding times of about 3 and 2 d for the wavenumber 1 and 2 cases, respectively). The spatial structure of these disturbances is not dominated by the zonal wavenumber of the basic state wave.

- The e-folding time as a function of amplitude for the most unstable disturbance generally shows non-monotonic behavior with increasing amplitude.

Examination of energy conversion from the basic state wave (Question 5) shows the general results that:

- Where the most unstable disturbance is one that does not correspond to an unstable mode of the zonally symmetric problem, the zonal variations of the basic state wave contribute more to the disturbance energy than the meridional variations of the wave.

- In several of these cases, the magnitude of the total energy conversion from the wave exceeds that from the jet, even though the basic-state wave amplitude may be only a few percent of the contribution from the jet.

The results summarized above have been applied to several questions concerning observed stratospheric features. The occurrence of an unstable disturbance moving with the same phase speed as an eastward moving zonal wavenumber 1 basic state wave, and the structure of other unstable disturbances for this basic state, supports the speculation by Hartmann (1983) that the presence of an eastward moving wavenumber 1 may give rise to a disturbance including higher wave-numbers which moves with the basic state wave; this disturbance resembles the observed quasi-nondispersive feature. Another suggestion made by Hartmann was that unstable equatorward modes from a narrow jet may interact strongly with waves of similar period from other sources. For a basic state with a narrow jet and a zonal wavenumber 3 typical of one observed in the winter stratosphere, this is supported by the appearance of a very unstable disturbance with a structure dominated by zonal wavenumber 3 components. Results for the stationary wavenumber 1 case and the wavenumber 2 case suggest that a distorted state such as is observed in the late winter polar stratosphere is likely to be barotropically unstable to disturbances with
characteristics that would further distort the field, and to disturbances which consist of traveling waves that resemble observed features.

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APPENDIX A

The Eigenvalue Problem

The eigenvalue problem obtained from substituting (3) and (4) into (1) can be written as

\[ \sigma \psi_r = \sum_{m=-M}^{M} \sum_{n=m}^{N+N}\{ [A_{nm}^m \delta_{mn} + B_{nm}^m \delta_{m,n+p}] + \hat{B}_{nm}^m \delta_{m-p,n} + F_{nm}^m \delta_{m,n}\} \psi_m \} \]  

(A1)

with the following forms of the coefficients:

\[ A_{nm}^m = -\frac{1}{2c_s} \sum_{r} a_r \left( \frac{c_n}{r_e} - \frac{c_r}{r_e} \right) L_{nmr}^0 \]  

(A2)

\[ B_{nm}^m = -\frac{A}{2c_s} \left( \frac{c_n}{r_e} - \frac{c_r}{r_e} \right) L_{nmq}^m \]  

(A3)

\[ \hat{B}_{nm}^m = -\frac{A}{2c_s} \left( \frac{c_n}{r_e} - \frac{c_r}{r_e} \right) L_{nmq}^m \]  

(A4)

\[ F_{n}^m = -\left( 2\Omega + \omega_b \right) c_n \]  

(A5)

\[ c_n = n(n+1) \]

\[ L_{nmq}^m = \int_{-1}^{1} P_r^m \left( mP_s \frac{d}{d\mu} P_q - pP_r, \frac{d}{d\mu} P_m \right) d\mu. \]  

(A6)

For a zonally symmetric basic state, the problem reduces to the terms involving \( A_{nm}^m \) and \( F_{n}^m \), and the zonal wavenumbers are decoupled. The calculation of the \( a_r \) is described in Manney et al. (1988). The problem is formulated as two separate eigenvalue problems for even \( n - |m| \) and odd \( n - |m| \) modes, following Baines (1976); for a given truncation at \( M \) and \( N \), this results in two matrices of dimension \( M(N+1) + (N - 1)/2 \), after omitting coefficients corresponding to \( \phi_0^0 \) and \( \psi_0^0 \) (Simmons et al. 1983). The coefficients of the matrix may be calculated by two different methods: 1) by using Gaussian quadrature, as described by Branstator (1983), and 2) by calculating the integrals in (A2), (A3), and (A4) analytically as a sum of products of Clebsch–Gordan coefficients (Silverman 1954; Edmunds 1974). For the truncation used here, the computing time for the two methods is nearly the same. A series of EISPACK subroutines is used to solve the matrix eigenvalue problem, and there are several methods within EISPACK to achieve this solution (Smith et al. 1976). The values of the coefficients calculated by the two methods mentioned above typically differ in the fifth or sixth significant figure. Although calculations with two slightly different matrices result in no significant differences in the eigenvalues, the values for some of the eigenvectors calculated by EISPACK are considerably more sensitive to these small differences. The results are checked by running the model for both methods of calculating the coefficients, and, where necessary, using different series of EISPACK routines to achieve consistent results for the slightly different values of the coefficients.

APPENDIX B

Energy Calculations

The forms of the energy conversion terms given in (5) are

\[ \text{CKX} = -\frac{1}{r_e} \int_{-1}^{1} \int_{0}^{2\pi} (u^2 - v^2) \times \frac{1}{\cos \phi} \frac{\partial u_{bw}}{\partial \phi} - v_{b} \tan \phi d\lambda d\mu, \]  

(B1)

\[ \text{CKY} = \text{CKYW} + \text{CKYJ}, \]  

(B2)

\[ \text{CKYW} = -\frac{1}{r_e} \int_{-1}^{1} \int_{0}^{2\pi} u'v' \times \left[ \frac{\cos \phi}{\cos \phi} \frac{\partial u_{bw}}{\partial \phi} + \frac{1}{\cos \phi} \frac{\partial v_{b}}{\partial \phi} \right] d\lambda d\mu, \]  

(B3)

\[ \text{CKYJ} = \int_{-1}^{1} \int_{0}^{2\pi} u'v' \cos \phi \frac{\partial \tilde{\omega}}{\partial \phi} d\lambda d\mu \]  

(B4)

where \( u_{bw} \) is the basic state zonal wind calculated from the basic state wave, \( \tilde{\omega}(r_e, \cos \phi) \) the basic state zonal wind from the jet, and \( v_{b} \) the basic state meridional wind. The derivation of this energy equation is virtually identical to that given by Simmons et al. (1983); the only difference being that we are working in a coordinate system rotating at \( \Omega + \omega_b \), rather than \( \Omega \), as is usually done. Specific forms for CKYW and CKYJ come from substituting in a basic state of the form (3), and for the jet part, a jet of the form (6).

As mentioned in appendix A, the EISPACK routines used to calculate the eigenvectors are very sensitive to the values of the coefficients used. The energy calculations, in turn, may be very sensitive to the value of
the eigenvectors since the calculation involves a sum of products of the components of the eigenvectors. Again, to check that the values obtained are consistent, we compare results for runs where two different methods were used to calculate the coefficients of the matrix eigenvalue problem, and for runs using different EISPACK routines to calculate the eigenvectors. All of the results we present are cases where the two methods give energy values that agree to better than 15%; most are in much closer agreement.

REFERENCES


