Reply

DALE R. DURRAN

Department of Atmospheric Sciences, University of Washington, Seattle, Washington

8 January 1990 and 26 February 1990

Fiedler (1990) has shown that when the energy flux through each boundary of some control volume is zero, the "modified-anelastic equations" will not support unstable linear waves. Thus, if one were to use the modified-anelastic equations to model a disturbance inside a closed rigid box, the volume averaged energy within the box would be conserved (at least for small-amplitude disturbances). Fiedler's result helps clarify the conditions under which the modified-anelastic equations fail to conserve energy and may be of significant interest to numerical modelers using these equations. Fiedler's result is not, however, in conflict with any of the results in Durran (1989). Furthermore, I believe that Fiedler oversells the energy "conservation" properties of the modified-anelastic system.

Let us begin by examining Fiedler's "energy" equation (15). This is an equation in flux form and it can be used to construct "energy" budgets, but its physical significance is unclear. There are two problems. The first difficulty is that the "energy" in (15) is not conserved by the full nonlinear modified-anelastic system. To be specific, Fiedler has not provided expressions for E^* and p^* that would render Durran's Eq. (61) equivalent to the energy equation for the (nonlinear) modified-anelastic system. The second problem is the nature of the conserved "energy." Fiedler's (15) shows that, whenever the "perturbation modified-anelastic energy" is conserved, the quantity ordinarily referred to as perturbation energy [see Fiedler's (14)] must vary in proportion to $\bar{\theta}$. There are two ways to describe the situation: following Fiedler one might say, " the conserved energy is slightly inaccurate," or following Durran one might say, "energy is not quite conserved." In the case of small amplitude motions, both statements are correct; in the case of nonlinear flow, only the latter statement is correct.

Let us now consider Fiedler's concern about my analysis of vertically propagating internal gravity waves in an unbounded isothermal atmosphere with no vertical wind shear. Equations (38) and (40) in Durran imply that for sinusoidal wave solutions to this problem,

$$\bar{w}(\bar{\pi})^* = |w_0|^2\left(\frac{\omega^2 - N^2}{\omega m - i\omega\Gamma}\right)^*, \qquad (1)$$

which is a nonzero constant. Thus, the waves in Durran's example do not satisfy Fiedler's condition (12),

$$w'(\pi')^* = 0, \qquad (2)$$

and Fiedler's results do not apply. The difference between my example and the cases considered by Fiedler centers around the distinction between zero energy flux and zero energy flux *divergence*.

The energy conservation properties for *linear* solutions to the modified-anelastic system depend crucially on the value of $w'(\pi')^*$. When $w'(\pi')^* = 0$ there is no vertical energy flux and the modified-anelastic equations are energy consistent. When the integral of $w'(\pi')^*$ is zero on two surfaces of constant z, there is no vertical energy flux into the volume bounded by those surfaces, and the modified-anelastic equations conserve volume-averaged energy. Fiedler has derived these results in his comment; they also follow from (52) of Durran. The energy conservation problems of the modified-anelastic system, and the spurious instability discussed in Durran, appear when there is a nonzero energy flux through a control volume within the fluid. Fiedler seems to feel that this last situation is not physically relevant because it only occurs when there is an energy source outside the control volume. I believe the case with nonzero energy flux is both physically relevant and fundamental. Many examples exist where there is a nonzero flux of energy through a limited region of the atmosphere, e.g., vertically propagating gravity waves. The governing equations should permit one to perform a sensible energy balance over that limited area. The modified-anelastic system will not properly approximate the energy balance in the compressible system unless the vertical energy flux is zero.

At the risk of belaboring the point, let us return to the example of vertically propagating waves in Durran and examine the relationship between the stability of the solution and the true vertical energy flux. Following

Corresponding author address: Dr. Dale R. Durran, Dept. of Atmospheric Sciences, AK-40, University of Washington, Seattle, WA 98195.

the notation in Durran, the true vertical energy flux is $p'w' = c_p\bar{\rho}\bar{\theta}\pi'w' = \tilde{\pi}\tilde{w}/\rho_0$, where ρ_0 is a constant reference density. Using Durran's (38) and (40), one finds that the energy flux in the linearized solution to the compressible equations is

$$F_e = \frac{1}{\rho_0}\left(\frac{\omega^2 - N^2}{\omega m - i\omega\Gamma}\right)[w_0 e^{i(kx+mz-\omega t)}]^2.$$

Assuming that k and m are real, the horizontal integral of F_e over one wavelength is a nonzero constant independent of height. Thus, the energy flux is nondivergent and the wave amplitude should not change. These results are consistent with the dispersion relation for the linearized compressible equations [(42) in Durran], which shows that ω will be real whenever k amd m are real (N^2 is positive since the basic state is isothermal).

If one tries to describe vertically propagating waves in an isothermal atmosphere using the modified-anelastic equations, one obtains the dispersion relation

$$\omega_{ma}^2 = \frac{N^2k^2}{k^2 + m^2 + \Gamma^2 + \Gamma N^2/g + imN^2/g}. \quad (3)$$

This dispersion relation clearly admits sinusoidal solutions with real wavenumbers k and m. The frequency for such solutions will be complex; let it be denoted by $\omega_r + i\omega_i$ where ω_r and ω_i are both real. When k and m are real, the flux is

$$F_{m1} = \frac{1}{\rho_0}\left(\frac{\omega_{ma}^2 - N^2}{\omega_{ma}m - i\omega_{ma}\Gamma}\right)[w_0 e^{i(kx+mz-\omega_r t)}]^2 e^{2\omega_i t}.$$

Although it will vary with time, the horizontal average of F_{m1} will not vary with height, i.e., the vertical energy flux, averaged over one horizontal wavelength, is nondivergent. Thus, the solution should not change amplitude, so ω_{ma} should be real, contradicting our earlier result. We conclude that the modified-anelastic equations do permit "spurious growth" (or perhaps spurious decay) in subdomains through which there is a nonzero energy flux.

The dispersion relation (3) also permits solutions with real ω_{ma} and complex m. However, the energy consistency of these solutions is no better than those with complex ω_{ma}. In order to demonstrate the problem, suppose that ω_{ma} is real and $m = m_r + im_i$. Then the energy flux becomes

$$F_{m2} = \frac{1}{\rho_0}\left(\frac{\omega_{ma}^2 - N^2}{\omega_{ma}m - i\omega_{ma}\Gamma}\right)[w_0 e^{i(kx+m_r z-\omega_{ma}t)}]^2 e^{-2m_i z}.$$

In this case, the vertical energy flux varies with height, but the solution does not change amplitude. Thus, the modified-anelastic solution exhibits a spurious "failure to amplify" (or decay). In the spirit of Fiedler's comment, one might attempt to make the preceding examples "energy" consistent by redefining the energy flux so that F_{m1} is divergent and F_{m2} is nondivergent. Such redefinition will not, however, change the fact that (1) the modified-anelastic equations do not replicate the energy conservation properties of the true solution, and (2) the energy consistency of the modified-anelastic system is not enhanced by simply choosing a real value for ω.

Before leaving the topic of energy conservation, it should be remarked that several other variants of Ogura and Phillips' (1962) original anelastic equations are also in existence. Their energy conservation properties are somewhat different from those of the "modified-anelastic system" discussed in Durran—indeed, the system proposed by Lipps and Hemler (1982) always conserves energy.

As a final minor point, I would like to clarify the description of the pseudo-incompressible and modified-anelastic approximations provided by Fiedler in connection with his Eqs. (3) and (4). It is the total derivative of the *perturbation* pressure which may be neglected in (3) to obtain the pseudo-incompressible approximation. Likewise, it is the total derivative of the *perturbation* density which is neglected in reducing (4) to the modified-anelastic continuity equation.

REFERENCES

Durran, D. R., 1989: Improving the anelastic approximation. *J. Atmos. Sci.*, **46**, 1453–1461.

Fiedler, B. H., 1990: Comments on "Improving the anelastic approximation." *J. Atmos. Sci.*, **47**, 1817–1818.

Lipps, F. B., and R. S. Hemler, 1982: A scale analysis of deep moist convection and some related numerical calculations. *J. Atmos. Sci.*, **39**, 2192–2210.

Ogura, Y., and N. A. Phillips, 1962: Scale analysis of deep and shallow convection in the atmosphere. *J. Atmos. Sci.*, **19**, 173–179.