A Nonacceleration Theorem for Transient Quasi-geostrophic Eddies on a Three-Dimensional Time-Mean Flow

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ABSTRACT

A nonacceleration theorem is derived for small-amplitude, transient quasi-geostrophic eddies on a three-dimensional time-mean flow. This theorem states that the divergence of the eddy potential vorticity flux—and hence the forcing by the eddies of the mean geostrophic flow—vanishes to leading order under conditions that (i) the eddies *and* mean flow are conservative, (ii) the eddy enstrophy density and the quantity $F_u/|\bar{\mathbf{u}}|$ (where F_u is the component of the eddy potential vorticity flux in the direction of the time-mean flow $\bar{\mathbf{u}}$) are constant along time-mean streamlines, and (iii) the boundary conditions on the mean geostrophic flow are independent of the eddies.

The requirement of downstream-constant eddy amplitudes parallels that of steadiness of eddy amplitudes in the equivalent theorem for eddies on a zonal-mean flow. In general, when this condition is not met, the divergence of the transient eddy flux of potential vorticity is nonzero. Thus, unlike in the zonal-mean problem, small-amplitude, conservative, transient eddies propagating on a steady, three-dimensional mean flow will, in some if not in most cases, influence the mean flow in a nontrivial way, even though their amplitudes are steady in time. There are, however, some constraints on the nature of this interaction; conservative eddies do not impact on the global time-mean enstrophy budget, while small-amplitude conservative eddies on a conservative mean flow make no explicit contribution to the global budget of time-mean energy.

1. Introduction: the time-mean budgets of potential vorticity, momentum, and heat

Our understanding of the interaction between waves and zonal flows has advanced profoundly in recent years. The foundations for this advance have been provided by fundamental theorems concerning the propagation of eddies on general zonal flows; in particular, by laws describing the conservation of "wave activity" (Bretherton and Garrett 1968; Hayes 1977; Andrews and McIntyre 1978a,c) and parallel laws governing the properties of the interaction of eddies with the mean zonal flow (Andrews and McIntyre 1976, 1978a,b; Boyd 1976). Among other things, these developments have made us appreciate the dependence of eddy transport on certain basic properties of the eddies, viz., transience (i.e., time-dependence of eddy amplitudes), nonconservative, and nonlinear effects.

Nevertheless, there is a large class of eddy transport problems in both atmosphere and ocean for which a zonally averaged perspective is not appropriate. In many such problems (for example the three-dimensional atmospheric climatology) a separation into time-mean and "transient" components is more natural. However, progress in the resolution of such problems is hindered by the lack of a comprehensive body of theory such as exists for the zonal-mean framework. A number of conservation laws—some, but not all, involving approximations—have been derived for transient eddies on nonparallel time-mean flows (Andrews and McIntyre 1978c; Rhines and Holland 1979; Andrews 1983; Plumb 1985, 1986; McIntyre and Shepherd 1987). Several techniques have been developed for *quantifying* transport by transient eddies (see Holopainen 1984, for a discussion of some of these); while some partial links between the characteristics of the propagation of transient eddies and their interaction with the mean state (as expressed by the divergence of the eddy flux of vorticity or potential vorticity) have been established (Hoskins et al. 1983; Plumb 1986; hereafter I), there is still no simple understanding of the fundamentals of the transport process. For example, is there a parallel in the time-averaged problem of the "nonacceleration theorem" of zonal-mean theory and, if so, under what conditions? There are reasons to believe, for example, that conservative, statistically steady transient eddies will interact locally with a nonuniform background flow, even though zonal-mean nonacceleration conditions are satisfied. Consider, for example, the situations depicted in Fig. 1.

In case (a), we suppose there is a source of transient waves located at large negative y, such that in the region of interest a statistically steady conservative packet of transient waves is propagating through a flow that would, in the absence of waves, be zonal. We know

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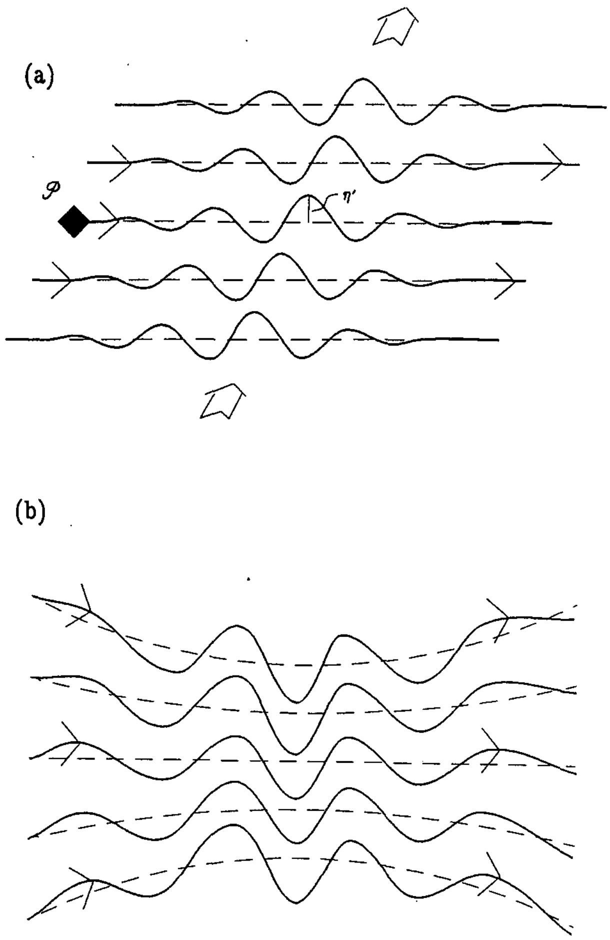

FIG. 1. Schematic figure of two barotropic examples of statistically steady, conservative, transient wave packets in which there is nontrivial local interaction with the time-mean state. (a) A packet of transient Rossby waves propagating from large negative y across an otherwise uniform westerly flow; (b) a packet propagating downstream on a nonuniform westerly flow. Instantaneous streamlines solid; basic state streamlines dashed. See text for discussion.

from zonal-mean theory that, if the source is sufficiently far (many Rossby radii) away, nonacceleration conditions are locally satisfied and there is no sustained forcing of the local *zonal-mean* flow by the wave packet. However, consider the fluid parcel $\mathcal{P}$ being carried into the wave packet by the zonal flow. As it enters the packet, it will suffer an oscillatory displacement η'; we may anticipate (e.g., Rhines and Holland 1979) that the increase in amplitude of this displacement as $\mathcal{P}$ penetrates deeper into the wave packet will be associated with nontrivial transport of potential vorticity and therefore with a wave-induced modification to the time-mean flow. This modification may be reversible in the sense that equal and opposite transport occurs as the parcel leaves the wave packet downstream; nevertheless, the local modification of the time-mean flow seems unlikely to be zero, even though the wave packet is "nonaccelerating" in the zonal-mean sense.

A second example (one that we shall consider in more detail in section 5) is depicted in Fig. 1b. In this case, the source of transient waves is at large negative x; the conservative statistically steady waves, of positive group velocity, propagate downstream through a prevailing westerly flow, in which is embedded a stationary wave. Again, zonal-mean nonacceleration conditions are satisfied but the distortion of the transients as the wave packet propagates through the nonuniform flow might be expected to induce local interaction between the transients and the stationary wave. We shall see in section 5 that this is indeed the case, although (because the transient wave is conservative) the interaction is such as to make the stationary wave translate without change of amplitude.

With such situations in mind, we ask here if any analogue to the zonal-mean nonacceleration theorem exists for transient eddies on a time-mean flow. It will be shown that, at least in principle, such a theorem can be demonstrated for transient eddies under quasi-geostrophic assumptions. Identification of the conditions under which such a theorem holds allows us to define those processes (viz., those that violate these conditions) which will in general lead to nonzero eddy transport of potential vorticity, momentum, and heat. [See Andrews (1990) for a more general exposition of these issues.] In the examples cited above, the theorem is violated because of the nonsteadiness of eddy amplitudes following the time-mean flow (i.e., downstream spatial variability) which is the analogue for the present case of "wave transience" in the zonal-mean problem.

2. The transient eddy potential vorticity flux and its divergence

The quasi-geostrophic potential vorticity equation is

$$\frac{\partial q}{\partial t} + \mathbf{u}\cdot\nabla q = S \tag{2.1}$$

where

$$\mathbf{u} = \mathbf{k} \wedge \nabla\psi \tag{2.2}$$

is the geostrophic velocity, ψ the geostrophic streamfunction,

$$q = f + \Delta^2\psi \equiv f + \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{f^2}{p}\frac{\partial}{\partial z}\left(\frac{p}{N^2}\frac{\partial\psi}{\partial z}\right) \tag{2.3}$$

(thus defining the operator Δ^2) is the quasi-geostrophic potential vorticity and S expresses the (nonconservative) sources and/or sinks of potential vorticity. In (2.3), f is the Coriolis parameter, p pressure, $z = -H\ln p$ where H is a constant scale-height and N is the buoyancy frequency. If the time-average is denoted by an overbar and transient deviations from this by a prime, the time-mean budget may be written

$$\frac{D\bar{q}}{Dt} = \bar{S} - \nabla\cdot(\overline{u'q'}) \qquad (2.4)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla \qquad (2.5)$$

is the total derivative following the mean geostrophic flow. (Of course the partial derivative of a time-averaged quantity with respect to time is zero by definition, but this form is retained here to remind us of the meaning of D/Dt, in particular of the parallel—which will become apparent in what follows—with "transience" in zonally averaged theory).

Following Holopainen et al. (1982) and Hoskins (1983), the time-mean momentum and heat budgets on an f-plane may be written

$$\frac{D\bar{\mathbf{u}}}{Dt} + f\mathbf{k}\wedge\bar{\mathbf{u}}_* = \mathbf{G} + \bar{\mathbf{X}} \qquad (2.6)$$

and

$$\frac{D\bar{\theta}}{Dt} + \bar{w}_*\frac{d\bar{\theta}}{dz} = \bar{Q} \qquad (2.7)$$

respectively, where θ is potential temperature, $\mathbf{X}$ the frictional or other force per unit mass, and Q is proportional to the diabatic heating. Here $\bar{\mathbf{u}}_*$ is a three-dimensional, nondivergent "residual" mean ageostrophic velocity defined in (4.4) and (4.5) of I and the quantity $\mathbf{G}$, the effective eddy-induced force per unit mass acting on the mean flow, is in this representation given by

$$\mathbf{G} = -\mathbf{k}\wedge\overline{\mathbf{u}'\mathbf{q}'}. \qquad (2.8)$$

In the zonal-mean analogues of (2.4)–(2.7) for disturbances on a zonal-mean flow [e.g., see Eq. (2.3) of Edmon et al. (1980)], only the northward component of the eddy potential vorticity flux enters the problem as an eddy forcing of the zonal-mean flow. As is now well known (Bretherton 1966) this northward flux can be expressed as the divergence of the quasi-geostrophic Eliassen–Palm flux, which can in turn be related to eddy transience and nonconservative and/or nonlinear effects. If these effects are absent, then the eddy forcing of mean potential vorticity vanishes; if the boundary conditions of the problem are similarly independent of eddy effects, the mean potential vorticity, zonal flow, and temperature are independent of the eddies—the celebrated "nonacceleration theorem" (Andrews and McIntyre 1976, 1978a,b; Boyd 1976).

Returning now to the time-mean problem (2.4), it was shown in I [and earlier by Rhines and Holland (1979), Illari and Marshall (1983), among others] that the downgradient component of the eddy potential vorticity flux can be related to what might be called "nonacceleration conditions" for this problem. The eddy potential enstrophy equation [taking (2.1) minus its time-mean, multiplying by q' and averaging] is

$$\frac{D}{Dt}\left(\frac{1}{2}\overline{q'^2}\right) + \overline{\mathbf{u}'q'}\cdot\nabla\bar{q} + \overline{\mathbf{u}'q\cdot\nabla q'} = \overline{S'q'}. \qquad (2.9)$$

Therefore, under assumptions that:

(C1) the eddies are of small amplitude (cubic and higher order nonlinearities negligible),

(C2a) the eddies enstrophy is constant following the time-mean streamlines (i.e., $D\overline{q'^2}/Dt = 0$), and

(C3) the eddies are conservative,

the downgradient flux vanishes (Rhines 1977), since the first, third, and fourth terms in (2.9) vanish under conditions (C2a), (C1), and (C3) respectively. Conversely, a nonzero downgradient component of the eddy potential vorticity flux requires the violation of one or more of conditions (C1)–(C3).

The above developments parallel analogous results in zonal-mean theory. Further, it was shown in I that for a mean flow that is sufficiently slowly varying in space,

$$\overline{\mathbf{u}'q'}\cdot\mathbf{n} \approx \nabla\cdot\mathbf{M}_R \qquad (2.10)$$

where $\mathbf{n} = \nabla\bar{q}/|\nabla\bar{q}|$ is the unit vector in the direction of the mean potential vorticity gradient and where, for small-amplitude waves, $\mathbf{M}_R$ is a flux of eddy activity which for almost-plane waves is parallel to the group velocity *relative to* the time-mean flow. Thus, under these assumptions, $\mathbf{M}_R$ parallels the Eliassen–Palm flux of zonal-mean theory (and its y- and z-components reduce to that flux on zonal averaging). The parallel extends to the mean momentum budget as expressed by (2.6), in which that component of the eddy-induced "force" $-\mathbf{k}\wedge\overline{\mathbf{u}'q'}$ in the direction $\mathbf{s} = \mathbf{k}\wedge\mathbf{n}$ (the unit vector *along* the mean potential vorticity contours) is just $-\nabla\cdot\mathbf{M}_R$; therefore $\mathbf{M}_R$ may be regarded as an eddy flux of $\mathbf{s}$-momentum. In the zonal-mean problem, where the basic-state potential vorticity gradient is of necessity purely in the latitudinal direction and where only the zonal component of the effective eddy force is of any consequence for the mean geostrophic flow,[1] these effects, together with possible boundary effects, entirely describe the geostrophic interaction.

The same is not true in the three-dimensional problem. In this case, the $\mathbf{n}$-component of the effective eddy force is in principle just as significant as the $\mathbf{s}$-component. This is equivalent to saying that it is the full two-dimensional flux of potential vorticity, including that component *along* the $\bar{q}$ contours, which impacts on the mean potential vorticity budget. As was discussed in I, the $\mathbf{s}$-component of the potential vorticity flux does not appear to be linked to simple concepts of eddy propagation or to conservation of "eddy activity" and it is therefore not clear to what extent the interaction between transient eddies and the time-mean

[1] The y-component of the effective force does enter the ageostrophic problem, however.

flow is governed by "nonacceleration conditions" analogous to those now well known for the zonal-mean case. In fact, it appears that the s-component of the eddy potential vorticity flux does not necessarily vanish under conditions (C1)–(C3). However, it was noted in I that for small-amplitude eddies on a conservative mean flow its divergence

$$\nabla \cdot [\mathbf{s}(\mathbf{s} \cdot \overline{\mathbf{u}'q'})]$$

is a "downstream transience" effect; i.e., it depends on the downstream variation of eddy amplitudes in some sense. Here we derive this result more explicitly and discuss its implications for the interaction problem as a whole.

First **n** is resolved into components along and across the mean flow.[2]

$$\mathbf{n} = \sum \hat{\mathbf{u}} + (1 - \sum^2)^{1/2} \mathbf{k} \wedge \hat{\mathbf{u}} \qquad (2.11)$$

where $\hat{\mathbf{u}} = \bar{\mathbf{u}}/|\bar{\mathbf{u}}|$. Then, from (2.4) and (2.5),

$$\sum = \hat{\mathbf{u}} \cdot \mathbf{n} = \frac{S - \nabla \cdot \overline{\mathbf{u}'q'}}{|\bar{\mathbf{u}}|\,|\nabla \bar{q}|}. \qquad (2.12)$$

It will be assumed at this stage that $|\bar{\mathbf{u}}|$ and $|\nabla \bar{q}|$ are nonzero (implications of zero $|\bar{\mathbf{u}}|$ are addressed in appendix A). Then $\sum$ is the sum of nonconservative effects and a term of $O(\epsilon^2)$, where ϵ is a sensible dimensionless measure of the transient eddy amplitude. Now,

$$\mathbf{s} = \mathbf{k} \wedge \mathbf{n} = \sum \mathbf{k} \wedge \hat{\mathbf{u}} - (1 - \sum^2)^{1/2} \hat{\mathbf{u}} = -\hat{\mathbf{u}} + \sigma, \qquad (2.13)$$

say, where

$$\sigma(\sum) = [1 - (1 - \sum^2)^{1/2}]\hat{\mathbf{u}} + \sum \mathbf{k} \wedge \hat{\mathbf{u}}, \qquad (2.14)$$

and therefore

$$\mathbf{s}(\overline{\mathbf{u}'q'} \cdot \mathbf{s}) = \frac{\bar{\mathbf{u}}(\overline{\mathbf{u}'q'} \cdot \bar{\mathbf{u}})}{|\bar{\mathbf{u}}|^2} + \sigma(\overline{\mathbf{u}'q'} \cdot \mathbf{s}) - \hat{\mathbf{u}}(\overline{\mathbf{u}'q'} \cdot \sigma). \qquad (2.15)$$

The contribution of the s-component of the flux to the total flux divergence is therefore

$$\nabla \cdot [\mathbf{s}(\overline{\mathbf{u}'q'} \cdot \mathbf{s})] = \bar{\mathbf{u}} \cdot \nabla \left[\frac{\overline{\mathbf{u}'q'} \cdot \bar{\mathbf{u}}}{|\bar{\mathbf{u}}|^2}\right] + \nabla \cdot [\sigma(\overline{\mathbf{u}'q'} \cdot \mathbf{s}) - \hat{\mathbf{u}}(\overline{\mathbf{u}'q'} \cdot \sigma)]. \qquad (2.16)$$

The first of these two terms vanishes under the condition:

(C2b) The quantity $\overline{\mathbf{u}'q'} \cdot \bar{\mathbf{u}}/|\mathbf{u}|^2$ is constant along mean streamlines.

[2] The sign of the second term in (2.11) is chosen on the assumption that the mean flow is "pseudowestward"; i.e., with low potential vorticity to the left of the flow. This is done for simplicity of presentation and does not influence the conclusions of this discussion.

The second term in (2.16) vanishes when $|\sigma| = 0$; from (2.14), this occurs whenever $\sum = 0$. From (2.12), $\sum$ depends on $\bar{S}$ and on the eddy potential vorticity flux divergence. The first of these contributions vanishes when:

(C4) The *mean flow* is conservative, by which it is meant that $\bar{S} = 0$.

The second contribution to $\sum$ (and therefore to σ) is formally of $O(\epsilon^2)$ and its impact on (2.16) is therefore negligible for small-amplitude eddies, being $O(\epsilon^4)$ at most. Therefore the divergence (2.16) of the *s*-component of the eddy potential vorticity flux vanishes under assumptions (C1), (C2b), and (C4).

This result differs from that obtained for the vanishing of the downgradient component in that the *n*-component itself vanishes under conditions (C1)–(C3), whereas what has been demonstrated here is a criterion for the vanishing of the *divergence* of the s-flux and not of the flux itself. In fact, this latter result is a straightforward one that derives solely from the properties of the *mean flow* and the smallness of eddy amplitudes. Under conditions (C1) and (C4) $\sum = O(\epsilon^2)$, whence, to leading order, the time-mean flow $\bar{\mathbf{u}}$ is orthogonal to **n** and therefore parallel (or antiparallel) to **s**. Then, *whatever the structure of the eddies,* the s-component of the flux is necessarily parallel to $\bar{\mathbf{u}}$, whence its divergence must be in the form of a downstream derivative.

This analysis has assumed that the separation of the potential vorticity flux along and across $\bar{q}$ contours is a sensible thing to do (i.e., that $\nabla \bar{q}$ is finite) and that **u** is nonzero. It is demonstrated in appendix A that (2.16) also vanishes when these conditions are not met.

3. A nonacceleration theorem for transient eddies on a time-mean flow

It has been established that, under conditions (C1)–(C4), $\overline{\mathbf{u}'q'} \cdot \mathbf{n} = 0$ and $\nabla \cdot [\mathbf{s}(\overline{\mathbf{u}'q'} \cdot \mathbf{s})] = 0$, whence

$$\nabla \cdot \overline{\mathbf{u}'q'} = 0; \qquad (3.1)$$

the forcing of the time-mean flow therefore vanishes *locally* wherever (C1)–(C4) are satisfied. If these conditions are satisfied *everywhere,* (2.4) becomes

$$\frac{D\bar{q}}{Dt} = J(\bar{\psi}, f + \Delta^2 \bar{\psi}) = 0, \qquad (3.2)$$

where $J(\ ,\ )$ is the Jacobian. This nonlinear equation for the time-mean geostrophic streamfunction is independent of explicit eddy-related terms (and, because we have had to invoke (C4), of mean nonconservative effects also). In general, eddies could still influence $\bar{\psi}$ through boundary conditions (recall that Δ^2 is an elliptic operator). However, we introduce the further assumptions that:

(C2c) both $\overline{\theta'^2}$ and $\overline{\mathbf{u}'\theta'} \cdot \bar{\mathbf{u}}/|\mathbf{u}|^2$ are constant along streamlines on horizontal boundaries, and

(C5) the boundary conditions for the time-mean geostrophic flow are independent of eddy effects.

The boundary conditions on the time-mean flow are discussed in appendix B, where it is shown that (C5) frequently follows from (C1)–(C4). Then, if conditions (C1)–(C5) are satisfied *everywhere* [since $\bar{S} = 0$ by (C4)] *the equations governing the time-mean flow are independent of the eddies.* We have thus arrived at a nonacceleration theorem for the transient eddies on a time-mean flow and identified (C1)–(C5) as comprising "nonacceleration conditions" for this problem. There is a caveat to this statement, however. The mean potential vorticity equation (3.2) is degenerate whenever conditions (C1)–(C4) are satisfied, since then any flow with $\bar{\psi} = \bar{\psi}(\bar{q})$ is a solution. Therefore, no formal proof has been established that a given, steady, conservative flow is unaffected by the introduction of transient eddies. If we consider the response of such a flow to the gradual introduction of transient eddies then, during the period of time in which the eddies are being introduced, the statistical steadiness assumed here does not apply (and zonal-mean theory leads us to expect at least a temporary impact on the "mean" flow). There is no guarantee that the final steady flow under such circumstances would be the same as that at the outset. What *has* been established is that there is no sustained forcing of the time-mean state by the eddies once statistical steadiness has been reached and that the eddies have no effect on the mean potential vorticity budget of the final state.

4. Interpretation of the nonacceleration theorem

a. The eddy-induced force under nonacceleration conditions

We have seen that when conditions (C1)–(C4) are satisfied the eddy potential vorticity flux is nondivergent but the *s*-component remains nonzero. The eddy-induced force $\mathbf{G}$ defined in (2.6) therefore also remains nonzero, in fact being

$$\mathbf{G} = -\mathbf{k} \wedge \overline{\mathbf{u}'q'} = \mathbf{n}(\overline{\mathbf{u}'q'} \cdot \mathbf{s}) \qquad (4.1)$$

under these conditions. Since $\mathbf{n} \cdot \bar{\mathbf{u}} = 0$ under conditions (C2) and (C4),

$$\mathbf{G} \cdot \bar{\mathbf{u}} = 0 \qquad (4.2)$$

i.e., the effective force is everywhere "workless," being directed cross-stream. This in itself does not imply that the eddies have no impact on the mean geostrophic flow; e.g., large-scale topography may generate stationary waves without doing any work on the flow.[3] A further property of $\mathbf{G}$, however, derives from the fact that the eddy potential vorticity flux is nondivergent whence, since

$$\mathbf{k} \cdot \nabla \wedge \mathbf{G} = \nabla \cdot \overline{\mathbf{u}'q'}, \qquad (4.3)$$

the horizontal force $\mathbf{G}$ is irrotational. Therefore

$$\mathbf{G} = \nabla g \qquad (4.4)$$

where g is a scalar function of space. It is then easy, from the mean momentum and heat equations (2.6) and (2.7), to establish consistency with the result obtained from the mean potential vorticity budget (viz., that the geostrophic flow is independent of the eddies). In general, any irrotational component of $\mathbf{G}$ (which, in this case, is all of it) may be absorbed into a redefinition of the ageostrophic circulation[4] (cf. Hoskins et al. 1983) to give

$$\frac{D\bar{\mathbf{u}}}{Dt} + f\mathbf{k} \wedge \bar{\mathbf{u}}_{**} = \bar{\mathbf{X}}$$

and

$$\frac{D\bar{\theta}}{Dt} + \bar{w}_{**}N^2 = \bar{Q}, \qquad (4.5)$$

where

$$\bar{\mathbf{u}}_{**} = \bar{\mathbf{u}}_{*} + \mathbf{k} \wedge \nabla\left(\frac{g}{f}\right). \qquad (4.6)$$

Then (4.5) presents a problem for the geostrophic flow and the transformed ageostrophic flow $\bar{\mathbf{u}}_{**}$ that is independent of eddy forcing.[5] The ageostrophic flow $\bar{\mathbf{u}}_a$, however, is equal to the sum of $\bar{\mathbf{u}}_{**}$ and terms [the negative of the second term on the rhs of (4.6) and terms given in (4.4) and (4.5) of I] which do not necessarily vanish. As in the zonal mean problem (Andrews and McIntyre 1976), therefore, while the geostrophic mean flow is independent of the eddies in the nonacceleration limit, the same is not generally true of the ageostrophic mean flow.

b. The applicability of nonacceleration conditions in a three-dimensional flow

Even for small-amplitude, conservative eddies propagating on a conservative time-mean flow, the nonacceleration theorem requires that (C2a, b, c) be satisfied. Specifically, it is required that $\overline{q'^2}$ and $\overline{\mathbf{u}'q'} \cdot \bar{\mathbf{u}}/|\bar{\mathbf{u}}|^2$ [and, on a nonisentropic lower boundary, $\overline{\mathbf{u}'\theta'} \cdot \bar{\mathbf{u}}/|\bar{\mathbf{u}}|^2$, from (A2.4)] be constant following the

[3] M. E. McIntyre (personal communication). In fact whether or not stationary topographic forcing appears as "workless" depends on how the energetics is analyzed; it is indeed workless in the zonally averaged, transformed Eulerian-mean formulation (Plumb 1983).

[4] This is equivalent, through (1.8), to the statement that the rotational component of the eddy potential vorticity flux is irrelevant for the geostrophic flow (cf. Holopainen et al. 1982). The irrelevance of the irrotational part of $\mathbf{G}$ has been noted in a more general context by Haynes and McIntyre (1987).

[5] Note that, since $\bar{w}_{**} = \bar{w}_{*}$, the transformation (4.6) does not introduce any hidden effects through the lower boundary condition.

flow. Now, if the mean flow and/or mean potential vorticity gradient vary downstream, the transient eddies will usually alter in structure in response to their changing environment. Although no formal proof will be offered here, it seems reasonable to assert—except perhaps in the most contrived circumstances—that not all of these quantities will be constant downstream in the presence of the nonuniform flow. It appears, therefore, that nonacceleration conditions can be fully satisfied only when the flow is everywhere constant downstream (and therefore probably zonal) and the eddy statistics are zonally uniform—in which case a zonally-averaged approach would be a more natural and simpler view of the eddy mean-flow interaction. That is not to say, however, that the nonacceleration limit discussed here is without significance; just as in the zonally averaged problem, the strength of the nonacceleration theorem relies not on the fact that the limit can ever be exactly satisfied in real flows, but on the identification of those processes, viz., violation of one or more of assumptions (C1)–(C5), which give rise to meaningful interaction.

5. Conservative eddies

If the transient eddy field has downstream structure [whether because of local sources or sinks of eddy activity or the presence of a stationary wave component in the time-mean flow that modulates the transients as in the examples of Figs. 1a and 1b respectively] there is no reason to believe that the eddy forcing of the mean flow will vanish. As we shall see, it is easy to find examples of such cases in which the eddies do modify the geostrophic mean state. To some extent, this makes the three-dimensional, time-averaged problem more complex than the zonally averaged case; in the latter, it is well established that conservative waves of steady amplitude satisfy nonacceleration conditions and, in the present problem, eddy amplitudes are by definition steady in time. However, the close parallel between the two problems becomes clear once it is recognized, as noted above, that "transience" (in time) in the zonally averaged case is analogous to "downstream transience" (in space) in the time-averaged problem and that, in this sense, the eddies are never "steady" unless their structure and amplitudes are constant downstream in the sense required by (C2a, b, c).

Nevertheless, there are some constraints on the extent to which conservative transients can interact with the mean state. The globally integrated time-mean enstrophy budget is, from (2.4),

$$\{\overline{\mathbf{u}'q'}\cdot\nabla\bar{q}\} + \{\bar{S}\bar{q}\} = 0 \qquad (5.1)$$

where the braces denote the area integral over a closed domain through whose boundaries there is no mean or eddy flow. In deriving (5.1), use has been made of the relations

$$\{\bar{\mathbf{u}}\cdot\nabla(\bar{q}^2)\} = \{\nabla\cdot(\bar{\mathbf{u}}\bar{q}^2)\} = 0$$

and

$$\{\bar{q}\nabla\cdot\overline{\mathbf{u}'q'}\} = -\{\overline{\mathbf{u}'q'}\cdot\nabla\bar{q}\}.$$

Equation (5.1) expresses the contribution of transient eddy transport of potential vorticity in maintaining the enstrophy of the mean state. However, from the global integral of (2.10),

$$\{\overline{\mathbf{u}'q'}\cdot\nabla\bar{q}\} = \{\overline{S'q'}\} \qquad (5.2)$$

since

$$\left\{\frac{D}{Dt}\left(\frac{1}{2}\overline{q'^2}\right)\right\} = \frac{1}{2}\{\nabla\cdot(\bar{\mathbf{u}}\overline{q'^2})\} = 0$$

and

$$\{\overline{\mathbf{u}'q'\cdot\nabla q'}\} = \frac{1}{2}\{\nabla\cdot(\overline{\mathbf{u}'q'^2})\} = 0.$$

From (5.2) therefore, the contribution of the transient eddies to the global mean enstrophy budget vanishes if the eddies are conservative.[6]

If, moreover, the eddies are of small amplitude as well as conservative and the mean flow is conservative, the eddy forcing of the mean flow is globally (though not necessarily locally) workless. Since, under assumptions (C1) and (C4), $\mathbf{s} = -\bar{\mathbf{u}}/|\bar{\mathbf{u}}|$[7] and $\bar{\mathbf{u}}\cdot\mathbf{n} = 0$,

$$\bar{\mathbf{u}}\cdot\mathbf{G} = -\bar{\mathbf{u}}\cdot\mathbf{s}(\overline{\mathbf{u}'q'}\cdot\mathbf{n})$$
$$= -\bar{\mathbf{u}}\cdot(\mathbf{k}\wedge\nabla\bar{q})\overline{\mathbf{u}'q'}\cdot\nabla\bar{q}/|\nabla\bar{q}|^2. \qquad (5.3)$$

However, under these assumptions, $\bar{\psi} = \bar{\psi}(\bar{q})$ and $\bar{u} = \mathbf{k}\wedge\nabla\bar{\psi} = \Lambda\mathbf{k}\wedge\nabla\bar{q}$ where $\Lambda = d\bar{\psi}/d\bar{q}$ (Andrews 1983). Now, (2.10) gives us

$$\overline{\mathbf{u}'q'}\cdot\nabla\bar{q} = -\frac{D}{Dt}\left(\frac{1}{2}\overline{q'^2}\right)$$

in these circumstances and therefore (5.3) becomes

$$\bar{\mathbf{u}}\cdot\mathbf{G} = -\Lambda\frac{D}{Dt}\left(\frac{1}{2}\overline{q'^2}\right). \qquad (5.4)$$

But $\Lambda = \Lambda(\bar{q})$ is constant along a mean streamline whence

$$\bar{\mathbf{u}}\cdot\mathbf{G} = -\nabla\cdot\left[\frac{1}{2}\bar{\mathbf{u}}\Lambda\overline{q'^2}\right] \qquad (5.5)$$

and therefore

$$\{\bar{\mathbf{u}}\cdot\mathbf{G}\} = 0. \qquad (5.6)$$

None of this is to say, however, that conservative eddy transport is ineffective, as is evident in the following example. We consider barotropic flow on a beta-

[6] Indeed, substitution of (5.2) into (5.1) simply gives us the statement that the total (mean + eddy), globally integrated, nonconservative generation of enstrophy must vanish if the flow is steady.

[7] Again, the mean flow is here assumed to be "pseudowestward." This assumption is made for simplicity of presentation and does not influence the conclusions.

plane bounded on $y = 0$, L. As depicted in Fig. 1b, small-amplitude conservative eddies are propagating in the x-direction from minus infinity on a conservative basic state which, in $x < 0$, is simply a uniform zonal flow U with potential vorticity gradient $\partial\bar{q}/\partial y = \beta$. In $x < 0$, the eddy streamfunction is of the form

$$\psi' = \mathbb{R}e\Psi_0 e^{ik(x-ct)} \sin\pi y/L \qquad (5.7)$$

where kc is the forcing frequency with $c = U - \beta/(k^2 + \pi^2/L^2)$. In $x > 0$, the mean flow includes a stationary wave of long zonal scale (i.e., scale $\gg L$)

$$\bar{\psi} = -Uy + \Phi(x)\sin(2\pi y/L) \qquad (5.8)$$

where Φ is small (in the sense that $|\Phi| \ll UL$) and a slowly varying function of x [and $\Phi(0) = 0$]. It is assumed that the channel width has the particular value

$$L = 2\pi(U/\beta)^{1/2}$$

so that the mean potential vorticity is, to leading order (i.e., neglecting terms involving x-derivatives)

$$\bar{q} = f + \beta y + \frac{\partial^2\bar{\psi}}{\partial y^2}$$

$$= f + \beta y - \frac{4\pi^2}{L^2}\Phi\sin(2\pi y/L) = f - \frac{\beta}{U}\bar{\psi},$$

which is constant along mean streamlines. This ensures that the unforced basic state wave is stationary.

If the eddies are conservative then, in $x < 0$,

$$\overline{u'q'} = \frac{\pi}{2L}(k^2 + \pi^2/L^2)|\Psi_0|^2\sin(2\pi y/L)$$

and

$$\overline{v'q'} = 0 \qquad (5.9)$$

so that the potential vorticity flux divergence is zero in $x < 0$ where the mean flow and eddy amplitudes are constant downstream. In $x > 0$, however, the eddy structure is distorted by the stationary wave. The approximate solution for the transient eddy streamfunction (under the assumption of a slowly varying mean flow) is derived in appendix C, where it is shown that the potential vorticity flux divergence is

$$\nabla\cdot\overline{\mathbf{u}'q'} \approx |\Psi_0|^2\frac{d\Phi}{dx}$$

$$\times[(\Gamma_1 + \Gamma_2)\sin2\pi y/L - 2\Gamma_2\sin4\pi y/L], \quad (5.10)$$

where

$$\Gamma_1 = k^2l(3U - c)(k^2 - 3l^2)/c_g{}^2(k^2 + l^2)$$

and

$$\Gamma_2 = (k^2 + l^2)(k^2 - 3l^2)/8\beta l \qquad (5.11)$$

where c_g is the transient eddy group velocity. Here Γ_1 and Γ_2 each have the same sign as $(k^2 - 3l^2)$.

The pattern of potential vorticity flux divergence for one example of a flow governed by (5.8) is shown in Fig. 2. The case chosen has a "blocking"-like configuration in $x > 0$; it has been assumed for the purposes of this example that $k^2 > 3l^2$. Upstream of the block, the flux is predominantly divergent in the northern part of the channel and convergent in the southern part. This kind of pattern is to some extent characteristic of eddy fluxes upstream of blocks in observations (e.g., Illari and Marshall 1983) and in the modeling studies of Shutts (1983) and Haines and Marshall (1987). As these authors have noted, this effective downgradient flux is equivalent to a decelerative force acting on the mean flow just upstream of the block. Note, however, that in the present example the flux divergence has no projection onto $\bar{q}$, in accordance with (5.2). Moreover, the projection onto the stationary wave component of the background flow is also zero (if $\Phi \rightarrow 0$ far downstream) and therefore, in this example of conservative eddies, there is no positive feedback between the eddy fluxes and the mean flow, which would tend to *amplify* the basic state stationary wave (as appears to be happening in Shutts' calculations, for example). Rather, the tendency is to make the stationary wave *move* upstream (or downstream if $k^2 < 3l^2$) without amplification. [If basic state wave were nonstationary, this kind of interaction could oppose the tendency for it to be "blown away by the mean flow" (Illari and Marshall 1983).] This reflects the constraint that conservative transients can make no contribution to the time-mean enstrophy budget. Positive feedback in the interaction between the eddies and the stationary wave pattern requires nonconservative processes; such processes may be the end product of the breaking of transients in the deformation field in the region of a block (Shutts 1983; Hoskins et al. 1985; Haines and Marshall 1987).

6. Discussion

We have seen that the nonacceleration theorem of zonal-mean quasi-geostrophic theory carries over to the time-mean problem; the central result of this paper

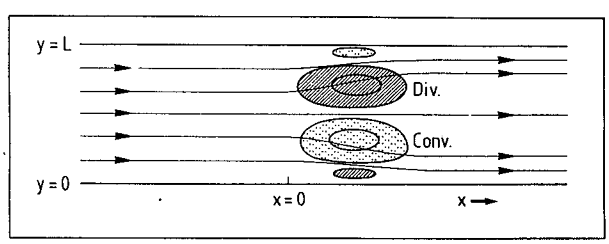

FIG. 2. Schematic figure of mean streamfunction (thin) and eddy potential vorticity flux divergence (heavy; regions of divergence shaded, of convergence stippled) for transient eddies on a flow given by (6.4) (the case depicted in Fig. 1b). Longitudinal variations are exaggerated for the sake of clarity.

is that the divergence of the transient eddy potential vorticity flux vanishes locally where the eddies are conservative, of small, constant amplitude (following the local mean streamline) and the mean flow is conservative. The condition that the mean flow (rather than just the eddies) must be conservative does not appear in the zonally averaged theorem. A similar condition on the conservation of a form of transient "eddy activity" was found by Andrews (1983), who noted that this is an indication that S' is "not the most natural measure" of eddy forcing (and dissipation). The most significant difference, however, is that "transience" of eddy amplitudes is here to be interpreted in the sense of "downstream transience, following the time-mean flow." It should also be noted that the conditions for violation of the nonacceleration theorem may not be mutually independent; e.g., nonconservative dissipation may result in spatial decay and hence "transience" of the eddies.

In general, transient eddies will interact in a nontrivial way with the mean flow whenever their amplitudes are not constant downstream (in the sense required to violate C2a, b or c), even if they are conservative. There are, however, some constraints on the nature of interaction between conservative transients and the time-mean flow. Globally, but not locally, conservative eddy transports are "workless" and make no contribution to the mean energy or enstrophy budgets. That is not to say, however, that their impact on the mean flow is trivial (a fact that should remind us of the care with which budget analyses should be interpreted). A "workless" force, whether eddy-related or otherwise, is capable of driving a nontrivial response; e.g., stationary waves. Stationary wave energy and enstrophy may, at least in a budgetary sense, be derived from the mean flow, with the "workless" forcing acting as a catalyst for this exchange [see Plumb (1983) for an analogous zonally averaged example—a transformed Eulerian mean analysis of the energetics of topographically forced stationary waves].

In the case of conservative transient eddies impinging on a preexisting stationary wave such as in the example of Fig. 2, the spatial variation of the mean flow induces nonzero eddy transport that may induce a translation of the stationary wave but which does not influence the stationary wave amplitude. This reinforces the conclusions of previous studies that nonconservative processes (such as may result from breaking of the transient eddies) are essential if the transients are to help sustain a block in the mean flow. It also cautions us that care is needed in the interpretation of observed eddy potential vorticity fluxes in such situations.

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APPENDIX A

Proof that the Potential Vorticity Flux Divergence Vanishes Under Nonacceleration Conditions in the Special Cases of Vanishing u or $\nabla\bar{q}$

a. Vanishing mean potential vorticity gradient

If $|\nabla\bar{q}| = 0$ locally the separation of section 2 into components along **n** and **s** breaks down. The region of vanishing gradient may be either a local extremum of $\bar{q}$ or an extended ridge or trough. We consider these separately in the case where conditions (C1)–(C4) are otherwise satisfied.

1) A LOCAL EXTREMUM OF $\bar{q}$

Consider a local extremum of $\bar{q}$ as depicted in Fig. A1a. By (C1) and (C4), the mean flow must be around the extremum, along the $\bar{q}$ contours. We consider the integrated eddy potential vorticity flux divergence over an area A enclosed by a $\bar{q}$ contour C:

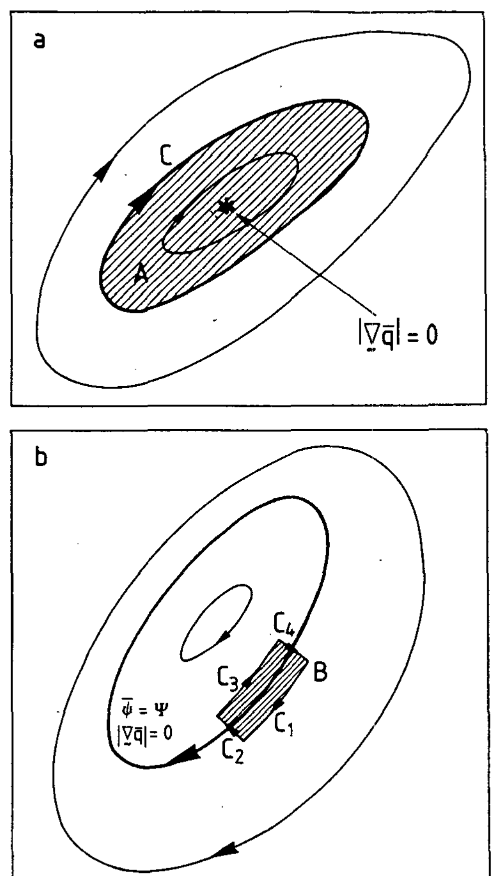

FIG. A1. Domains of integration for Eqs. (2.10) and (2.11) in the neighborhood of regions of vanishing potential vorticity gradient. (a) Local extremum of $\bar{q}$; (b) ridge or trough of $\bar{q}$ along a time-mean streamline. See text for discussion.

$$\int_A \nabla \cdot \overline{\mathbf{u}'q'}\,dxdy = \oint_C \overline{\mathbf{u}'q'} \cdot \mathbf{j}ds \qquad \text{(A1.1)}$$

where $\mathbf{j}$ is the outward unit normal and s is the element of length around the contour. Since $\mathbf{j} = \pm\mathbf{n}$, the integrand on the rhs of (A1.1) vanishes everywhere along C, by (C1), (C2a), and (C3). We are free to choose A as small as we like; therefore the potential vorticity flux divergence remains zero in the neighborhood of the extremum as A → 0.

2) AN EXTENDED RIDGE OR TROUGH IN $\bar{q}$

This situation, depicted in Fig. A1b, is of an extremum of $\bar{q}$ extending around a closed mean streamline $\bar{\psi} = \psi_0$. We consider the integrated flux divergence over the area B bounded by the streamlines $\bar{\psi} = \psi_0 \pm \delta\psi$ and two lines at right angles to the trough/ridge a distant L apart:

$$\int_B \nabla \cdot \overline{\mathbf{u}'q'}\,dxdy = \oint_C \overline{\mathbf{u}'q'} \cdot \mathbf{j}ds \qquad \text{(A1.2)}$$

where again $\mathbf{j}$ is the outward unit normal to the contour C. Now, if we divide C into segments as in Fig. A1b, with C_1 along $\psi_0 + \delta\psi$ and C_3 along $\psi_0 - \delta\psi$, with C_2 and C_4 the ends of the loop, then, by (C1), (C2a) and (C3), the contributions to (A1.2) from the segments C_1 and C_3 vanish. On C_2 and C_4, $\mathbf{j} = \pm\mathbf{s}$. Taking the limit $\delta\psi \to 0$ therefore allows us to recover the relation

$$\nabla \cdot \overline{\mathbf{u}'q'} = \nabla \cdot [\hat{\mathbf{u}}(\overline{\mathbf{u}'q'} \cdot \hat{\mathbf{u}})]$$

along the ridge or trough; this term is a downstream derivative along the streamline $\bar{\psi} = \psi_0$, and it vanishes under the conditions discussed within the text in the context of (2.16).

b. The case $\bar{\mathbf{u}} = 0$

If $|\bar{\mathbf{u}}| = 0$ the association between $\hat{\mathbf{u}}$ and $\mathbf{s}$ becomes meaningless. If the stagnant region is a local one the situation is as depicted in Fig. A1a, since the point at which $|\bar{\mathbf{u}}| = 0$ must be an extremum of $\bar{q}$ under these conditions. Therefore the arguments of (a) (1) apply. On the other hand, if the mean flow vanishes along a mean potential vorticity contour (as in Fig. A2, for example) where $|\nabla\bar{q}|$ remains finite then, under conditions (C1)–(C4), the potential vorticity flux is

$$\overline{\mathbf{u}'q'} = \mathbf{s}(\overline{\mathbf{u}'q'} \cdot \mathbf{s}). \qquad \text{(A1.3)}$$

Of course we may no longer associate $\mathbf{s}$ with the direction of the mean flow along the zero-wind line; the flux divergence vanishes there under conditions (C1)–(C4) if (C2b) is interpreted in the sense that the eddy statistics must be constant *along the line of zero flow.*

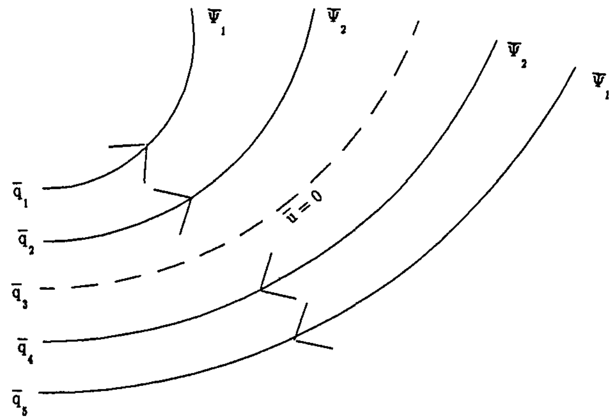

FIG. A2. A time-mean zero-wind line (dashed) in a region of nonvanishing mean potential vorticity gradient. See text for discussion.

APPENDIX B

Boundaary Conditions on the Time-Mean Flow

We here assess the extent to which assumption (C5) might be satisfied in real problems. Most likely lateral boundary conditions meet this requirement; e.g., boundedness on ψ as $x \to \infty$ or $y \to \infty$ translates directly to boundedness on $\bar{\psi}$, while a condition of no normal motion on a stationary boundary gives $\mathbf{i} \wedge \nabla\bar{\psi} = 0$, where $\mathbf{i}$ is the local unit normal to the boundary.

In the vertical, boundedness as $z \to \infty$ is similarly straightforward; however, conditions appropriate to a horizontal boundary on which the time-mean temperature gradient may be nonzero require more detailed attention. On such a boundary, $w = 0$, whence the time-mean heat budget there is

$$\frac{D\bar{\theta}}{Dt} + \nabla \cdot \overline{\mathbf{u}'\theta'} = \bar{Q}. \qquad \text{(A2.1)}$$

Now, this two-dimensional budget is analogous to the mean potential vorticity budget (2.4), with θ replacing q and $\bar{Q}$ replacing $\bar{S}$. It is therefore straightforward to follow the same procedures as those in section 2, writing the eddy heat flux on the boundary as the sum of components across and along the mean isentropes:

$$\overline{\mathbf{u}'\theta'} = (\overline{\mathbf{u}'\theta'} \cdot \mathbf{n}_\theta)\mathbf{n}_\theta + (\overline{\mathbf{u}'\theta'} \cdot \mathbf{s}_\theta)\mathbf{s}_\theta \qquad \text{(A2.2)}$$

where $\mathbf{n}_\theta = \nabla_2\bar{\theta}/|\nabla_2\bar{\theta}|$ and $\mathbf{s}_\theta = \mathbf{k} \wedge \mathbf{n}_\theta$. As for the potential vorticity flux, the n_θ-flux appears in the budget of eddy entropy variance

$$\frac{D}{Dt}\left(\frac{1}{2}\overline{\theta'^2}\right) + \overline{\mathbf{u}'\theta'} \cdot \nabla_2\bar{\theta} + \overline{\mathbf{u}'\theta' \cdot \nabla_2\theta'} = \overline{Q'\theta'} \qquad \text{(A2.3)}$$

[cf. (2.9)], while the divergence of the $\mathbf{s}_\theta$-component may be written

$$\nabla\cdot[\mathbf{s}_\theta(\overline{\mathbf{u}'\theta'}\cdot\mathbf{s}_\theta)] = \bar{\mathbf{u}}\cdot\nabla\left[\frac{\overline{\mathbf{u}'\theta'}\cdot\bar{\mathbf{u}}}{|\bar{\mathbf{u}}|^2}\right] + \nabla\cdot[\sigma_\theta(\overline{\mathbf{u}'\theta'}\cdot\mathbf{s}_\theta) - \hat{\mathbf{u}}(\overline{\mathbf{u}'\theta'}\cdot\sigma_\theta)] \quad \text{(A2.4)}$$

where $\sigma_\theta = \sigma(\Sigma_\theta)$ according to (2.15) with

$$\Sigma_\theta = \bar{\mathbf{u}}\cdot\mathbf{n}_\theta = \frac{\bar{Q} - \nabla\cdot\overline{\mathbf{u}'\theta'}}{|\bar{\mathbf{u}}|\,|\nabla_2\bar{\theta}|}. \quad \text{(A2.5)}$$

Using the same arguments as in section 2, therefore, the $\mathbf{n}_\theta$-flux vanishes if conditions (C1)–(C3) are satisfied on the boundary [from (A2.3)], while, from (A2.4) and (A2.5), the divergence of the boundary $\mathbf{s}_\theta$-flux is zero under conditions (C1), (C2) and (C4) *provided* (C2) is extended to include:

(C2c) $\overline{\theta'^2}$ and $\overline{\mathbf{u}'\theta'}\cdot\bar{\mathbf{u}}/|\mathbf{u}|^2$ are constant ,along mean streamlines on the boundary.

Given conditions (C1)–(C4), therefore, the time-mean heat budget on the boundary may be written

$$J\left(\bar{\psi}, \frac{\partial\bar{\psi}}{\partial z}\right) = 0 \quad \text{(A2.6)}$$

since θ is proportional to $\partial\bar{\psi}/\partial z$, by hydrostatic balance. This is a nonlinear boundary condition for $\bar{\psi}$, independent of eddy terms. For this type of boundary conditions, therefore, conditions (C1)–(C4) ensure (C5).

APPENDIX C

Solution for Small-Amplitude, Conservative Transient Eddies Propagating through a Slowly-Varying Time-Mean Flow in a Channel

The linear barotropic vorticity equation for conservative transient waves in the flow (5.8) is

$$\left(\frac{\partial}{\partial t} + \frac{\partial\bar{\psi}}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\bar{\psi}}{\partial y}\frac{\partial}{\partial x}\right)\zeta' + \mathbf{u}'\cdot\nabla\bar{\zeta} = 0. \quad \text{(A3.1)}$$

We nondimensionalize the problem by writing $x_* = x/L$ and $y_* = y/L$ where L/π is the channel width, $t_* = tU/L$, where U is the background mean velocity; $(u_*, v_*) = (u, v)/U$; $\psi_* = \psi/LU$; and $\zeta_* = \zeta L/U$. Then (A3.1) becomes

$$\left(\frac{\partial}{\partial t_*} + \frac{\partial\bar{\psi}_*}{\partial x_*}\frac{\partial}{\partial y_*} - \frac{\partial\bar{\psi}_*}{\partial y_*}\frac{\partial}{\partial x_*}\right)\zeta'_* + \left(u'_*\frac{\partial}{\partial x_*} + v'_*\frac{\partial}{\partial y_*}\right)\bar{\zeta}_* = 0, \quad \text{(A3.2)}$$

where

$$\zeta_* = f_0 + \beta_* y_* + \left(\frac{\partial^2}{\partial x_*^2} + \frac{\partial^2}{\partial y_*^2}\right)\psi_*,$$

with $\beta_* = \beta L^2/U = 4$ for this problem [since we are choosing the case for which the long wave with one complete wavelength across the channel is stationary].

The time-mean state is given by

$$\bar{\psi}_*(x_*, y_*) = -y_* + \epsilon\Phi(X_*, y_*) \quad \text{(A3.3)}$$

where X_* is a slow variable such that Φ varies by O(1) over $X_* = 1$ which corresponds to $x_* = \epsilon^{-1}$. In fact we specify

$$\Phi(X_*, y_*) = \Phi_0(X_*)\sin 2y_* \quad \text{(A3.4)}$$

The perturbation on the zonal flow in (A3.3) is thus a small-amplitude long wave; the choice of ϵ as a measure both of wave amplitude and of the slow variations in x is for convenience in this example. Therefore

$$\frac{\partial\bar{\psi}_*}{\partial x_*} = \epsilon^2\frac{\partial\Phi}{\partial X_*}; \quad \frac{\partial\bar{\psi}_*}{\partial y_*} = -1 + \epsilon\frac{\partial\Phi}{\partial y_*},$$

and

$$\bar{\zeta}_* = f_0 + \beta_* y_* + \epsilon\frac{\partial^2\Phi}{\partial y_*^2} + \epsilon^3\frac{\partial^2\Phi}{\partial X_*^2}. \quad \text{(A3.5)}$$

We shall need to retain terms in (A3.2) up to order ϵ^2; dropping the asterisks for simplicity, we write this as

$$\left[\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) + \epsilon\left(\frac{\partial}{\partial X} - \frac{\partial\Phi}{\partial y}\frac{\partial}{\partial x}\right) + \epsilon^2\left(\frac{\partial\Phi}{\partial X}\frac{\partial}{\partial y} - \frac{\partial\Phi}{\partial y}\frac{\partial}{\partial X}\right)\right] \times\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2\epsilon\frac{\partial}{\partial x}\frac{\partial}{\partial X} + \epsilon^2\frac{\partial^2}{\partial X^2}\right)\psi' + \left(4 + \epsilon\frac{\partial^3\Phi}{\partial y^3}\right)\left(\frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial X}\right)\psi' - \epsilon^2\frac{\partial^3\Phi}{\partial y^2\partial X}\frac{\partial\psi'}{\partial y} = 0. \quad \text{(A3.6)}$$

We solve this equation by expanding

$$\psi' = \psi_0(x, X, y, t) + \epsilon\psi_1(x, X, y, t) + \epsilon^2\psi_2(x, X, y, t) + O(\epsilon^3) \quad \text{(A3.7)}$$

To $O(\epsilon^0)$, (A3.6) gives us simply the usual linear wave equation on a zonal flow

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi_0 + 4\frac{\partial\psi_0}{\partial x} = 0,$$

which has solutions

$$\psi_0 = \mathbb{R}e A_0(X)e^{i(kx-\omega t)}\sin ly,$$

where

$$(k-\omega)(k^2+l^2)-4k=0. \qquad \text{(A3.8)}$$

This of course represents a propagating wave; we choose $l = 1$ (half-sine across the channel) and assume that the wave has eastward group velocity.

At $O(\epsilon)$, (A3.6) gives, after some manipulation,

$$i(k-\omega)\left(\frac{\partial^2}{\partial y^2}+1\right)\psi_1 = \left[\kappa^2 c_g \frac{\partial}{\partial X} - ik\left(\kappa^2 \frac{\partial \Phi}{\partial y} + \frac{\partial^3 \Phi}{\partial y^3}\right)\right]\psi_0 \qquad \text{(A3.9)}$$

where $\kappa^2 = k^2 + 1$ and $c_g = [2k(k-\omega) + \kappa^2 - 4]/\kappa^2$ is the zonal group velocity of the $O(\epsilon^0)$ solution. With (A3.4) and A3.8) the rhs of (A3.9) may be written $\sum_n R_n \sin ny$, where

$$R_n = \kappa^2 c_g \frac{dA_0}{dX}\delta_{n-1} - 2ik(k^2-3)\gamma_n \Phi(X) A_0(X) \qquad \text{(A3.10)}$$

with $\gamma_n = (\delta_{3-n} - \delta_{1-n})/2$. We suppress the secular forcing terms in (A3.9)—thus ensuring the validity of (A3.6)—by demanding that the rhs of (A3.9) have no projection onto $\sin y$. Thus we demand that $R_1 = 0$; i.e.,

$$\kappa^2 c_g \frac{dA_0}{dX} = -ik(k^2-3)\Phi(X)A_0(X). \qquad \text{(A3.11)}$$

Therefore

$$A_0(X) = A_{00} \exp\left[i \int K(X) dX\right]$$

where

$$K = -\frac{k(k^2-3)}{\kappa^2 c_g}\Phi(X). \qquad \text{(A3.12)}$$

Since K is real, this is purely a slow wavenumber correction to the $O(\epsilon^0)$ solution in the presence of a nonzero long wave.

We shall need the $O(\epsilon)$ solution; if we expand

$$\psi_1 = \sum_n A_{1n}(X) e^{i(kx-\omega t)} \sin ny, \qquad \text{(A3.13)}$$

then all A_{1n} are zero except A_{11} and A_{13}. The former is arbitrary at this stage (we shall define it at next order) while the latter, from (A3.9), is given by

$$A_{13}(X) = \frac{\kappa^2}{32}(k^2-3)A_0(X)\Phi(X). \qquad \text{(A3.14)}$$

The solution of (A3.6) at $O(\epsilon^2)$ proceeds in similar fashion; the $O(\epsilon)$ solution is completed by suppressing secular forcing of the $O(\epsilon^2)$ solution. The result of doing so is

$$-\kappa^2 c_g \frac{dA_{11}}{dX} + 2ik(k^2-3)\sum_n \gamma_n A_{1n}\Phi(X) + i(3k-\omega)\frac{d^2A_0}{dX^2} - 3(k^2-1)\frac{dA_0}{dX}\Phi(X) - \frac{1}{2}(k^2-3)A_0\frac{d\Phi}{dX} = 0. \qquad \text{(A3.15)}$$

Using (A3.12) and (A3.14), this fixes A_{11}, via

$$\frac{dA_{11}}{dX} - iKA_{11} = i\alpha_1 A_0 \Phi^2 - i\alpha_2 A_0 \frac{d\Phi}{dX} \qquad \text{(A3.16)}$$

where

$$\alpha_1 = \frac{k}{32 c_g}(k^2-3)^2 + \frac{k^2}{\kappa^4 c_g^2}(k^2-3)^2(3k-\omega) - \frac{3K}{\kappa^2 c_g}(k^2-3)(k^2-1)$$

and

$$\alpha_2 = \frac{1}{2\kappa^2 c_g}(k^2-3)\left[1 + \frac{2k}{\kappa^2 c_g}(3k-\omega)\right]. \qquad \text{(A3.17)}$$

The solution for A_{11} now follows:

$$A_{11}(X) = \left(i\alpha_1 A_{00}\int \Phi^2 dX - \alpha_2 A_{00}\Phi\right)\exp\left(i\int K(X)dX\right). \qquad \text{(A3.18)}$$

Given the $O(\epsilon^0)$ solution (A3.12), the first term in (A3.18) represents an $O(\epsilon)$ phase shift of the solution while the second is an amplitude shift, reversible in the sense that it vanishes where $\Phi(X) = 0$.

Given (A3.12) and (A3.18) it is straightforward to determine the vorticity flux for the transient eddies. The northward component is

$$\overline{v'\zeta'} = \frac{1}{2}\epsilon^2(k^2-1)|A_{00}|^2 \frac{d\Phi_0}{dX} \times (\alpha_3 \sin y \sin 3y - \alpha_2 \sin^2 y) + O(\epsilon^3)$$

where

$$\alpha_3 = \frac{\kappa^2}{32}(k^2-3). \qquad \text{(A3.19)}$$

The eastward component is

$$\overline{u'\zeta'} = \frac{1}{2}\epsilon^2\kappa^2|A_{00}|^2\Phi_0 \frac{d}{dy}(\alpha_3 \sin y \sin 3y - [\alpha_2 - \alpha_4]\sin^2 y) + O(\epsilon^3)$$

where

$$\alpha_4 = \frac{k^2(k^2-3)}{\kappa^4 c_g}. \qquad \text{(A3.20)}$$

The flux divergence is therefore, to $O(\epsilon^2)$,

$$\nabla \cdot \overline{\mathbf{u}'q'} = \frac{1}{2}\epsilon^2 |A_{00}|^2 \frac{d\Phi_0}{dX}(\gamma_1 \sin 4y + \gamma_2 \sin 2y) \tag{A3.21}$$

where $\gamma_1 = 4\alpha_3$ and $\gamma_2 = \alpha_2 - 2\alpha_2 - 2\alpha_3$. Note that only the term involving γ_2 projects onto the preexisting long wave (A3.4) and that the vorticity flux divergence is in quadrature in X with $\Phi(X, y)$ and therefore with the vorticity of the long wave.

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