WKBJ Approximation of the Stability of a Frontal Mean State

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ABSTRACT

The baroclinic instability of a frontal mean state is investigated using the WKBJ approximation. The results are compared with numerical calculations performed on the same mean state. Excellent agreement (within 5%) is found for jets whose half-width is as small as a Rossby radius of deformation. For jets 20% broader, the agreement is almost perfect.

1. Introduction

With the help of the WKBJ approximation Ioannou and Lindzen (1986, hereafter IL) investigated the spectrum and structure of waves growing on two-dimensional jets. Although the method presented was general, it was applied to the stability of symmetric barotropic jets superimposed, as perturbations, on the model introduced by Eady (1949) and the model introduced by Charney (1947). With this method it was possible, with great economy, to determine the growth rate of the first three meridional modes (first symmetric, first antisymmetric, second symmetric) as well as their back effect on the mean state. It was found that the inclusion of a jet confined the instabilities meridionally, thus internally determining the meridional wave scale. Once this internally determined meridional scale is taken into consideration, the stability results correspond plausibly to the classical results without a jet.

The calculations performed by Lin and Pierrehumbert (1988), using a fully numerical scheme confirmed that the accuracy of the method presented is not limited to the narrow range of asymptotic validity. The comparisons were made with the nonseparable Charney problem.

Recently, Moore and Peltier (1989, hereafter MP), in the context of investigations of the stability of frontal mean states, developed a numerical scheme that makes it possible to quantitatively establish the ranges of utility of simpler asymptotic techniques. Specifically, they investigated techniques developed by McIntyre (1970) and IL. They found good asymptotic accuracy with the results of McIntyre. On the other hand, MP claimed that IL had grossly overestimated the maximal growth rates. This claim is, in large measure, due to a misprint in section 3 of IL. The expression for the Y-dependence of the basic flow should have been

$$u^2(Y) = (1 + qY^2)^{-1},$$

and *not*

$$u(Y) = (1 + qY^2)^{-1}.$$

Moore and Peltier (1989) assumed that we had used the second expression, which they then employed for their numerical analysis. Thus, they were comparing their results with our results for a different profile. In addition, MP did not use the results of full WKBJ calculations for their comparison; rather, they compared their numerical results with results we obtained with a cruder heuristic approximation that we had also introduced. In this note, we present results for the WKBJ approximation developed in IL applied to the same profile used by MP, and compare these results for growth rates with those computed by MP.[1] It will be seen that for jets as broad as a Rossby radius of deformation, the two approaches yield results within 5% of each other. Indeed, for jets 20% broader than this the results are essentially indistinguishable. For narrower jets, the WKBJ results become progressively less accurate. From the data analyses in Newell et al. (1972), it appears that the ratio of the Rossby radius of deformation to the jet half-width (Ros) is characteristically

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[1] To normalize MP's Eqs. 3.1 and 3.3, we must multiply their Eq. 3.3 by $^9/_{16}$.

between 0.5 and 1.0. For these widths, there should be no problem in using the WKBJ results.

2. Comparison of the calculations

The normal mode stability of the zonal (x-direction) velocity profile given by

$$\bar{V}(y, z) = \frac{\Delta V}{H}\bar{u}(Y)\left(z - \frac{H}{2}\right),$$

$$\bar{u}(Y) = (1 + Y^2)^{-1}, \quad Y = \mathrm{Ros}\frac{y}{NH/f}, \quad (1)$$

to harmonic disturbances of the form $\exp[ik(x - ct)]$ (k is the zonal wavenumber, c the eigenvalue to be determined), is investigated with rigid horizontal surfaces bounding the flow at heights $z = 0$ and $z = H$. H is taken to be the typical tropopause height (8 km); y measures distances in the meridional direction from the center of the jet; N is the Brunt–Väisälä frequency, taken to have the constant value of 10^{-2} s^{-1}; and f is the Coriolis parameter ($\sim 10^{-4}$ s^{-1}). Finally, Ros, the parameter that describes the tightness of the jet, is simply the ratio of the Rossby radius (NH/f) to the half-width of the jet. For typical terrestrial atmospheric applications, Ros is between 0.5 and 1.4. ΔV represents the difference of the zonal wind between the tropopause ($z = H$) and the earth's surface ($z = 0$). The WKBJ calculations performed by IL are formally valid in the limit of Ros approaching 0. In their paper it is shown (referring to section 3 of IL's paper—dealing exclusively with the Eady problem) that the spectrum can be obtained by solving the following quantization condition:

$$\frac{2}{\mathrm{Ros}}\int_0^{Y_0} l(Y)dY = \left(n + \frac{1}{2}\right)\pi, \quad (2)$$

where $n = 0$ for the first symmetric mode, $n = 1$ for the first antisymmetric, etc.

Here $l(Y)$ (the meridional wavenumber) is determined so that the full wavenumber (meridional and zonal) d (where $d^2 = k^2 + l^2$), when introduced to the Eady dispersion relation

$$c^2 = \bar{u}^2(Y)d^{-2}[d/2 - \coth(d/2)][d/2 - \tanh(d/2)], \quad (3)$$

yields the same growth rate, Im(c), at all values of y. Finally, Y_0 is the latitude where the meridional wavenumber, $l(Y)$, becomes zero on its way to becoming purely imaginary. This is required by disturbances that are latitudinally contained.

The solution of (2), as described above, determines the WKBJ approximation. To be sure, when (3) is used, closed form solutions to (2) cannot be obtained, and one must turn to elementary numerical methods. The results are presented in Fig. 1. In this figure the curve of maximum growth rate is plotted against Ros. This curve is compared with the curve obtained by fully numerical techniques by MP. The percentage difference between these two curves is shown in Fig. 2. The agreement is very good (maximum deviation 5%) up to Ros = 1.0 (and virtually perfect for Ros < 0.8).

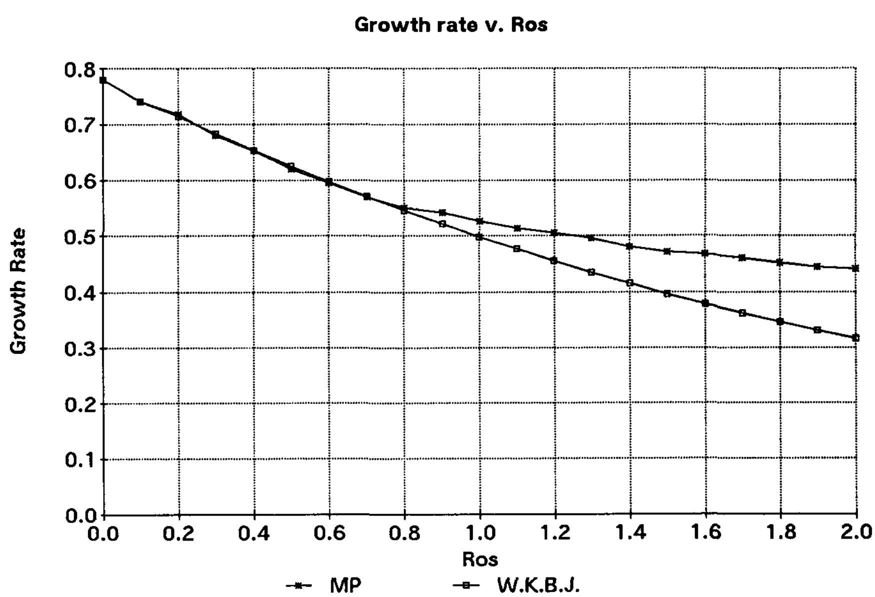

FIG. 1. The maximum growth rate vs. Ros (an inverse measure of the jet width). One curve is the result of the WKBJ calculation; the other is the numerical calculation performed by Moore and Peltier (1989). The jet varies in latitude as $\bar{u}(Y) = (1 + Y^2)^{-1}$ where $y = 0$ is taken at the jet center.

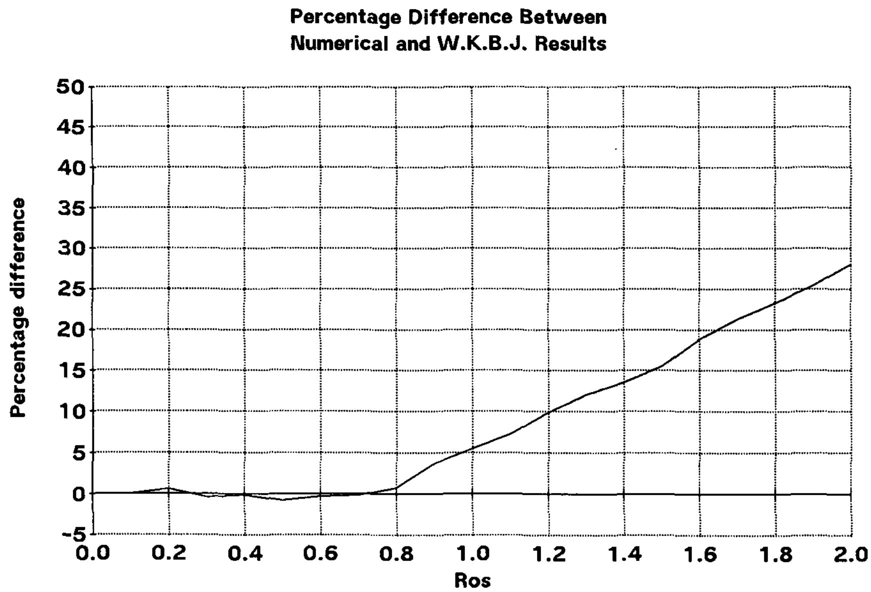

FIG. 2. The percentage difference between WKBJ growth rates and those numerically determined in MP.

Beyond this point there are growing differences between the results. The deviation of the results for Ros > 1 suggests that barotropic conversions are becoming significant and are contributing positively to the growth rate. Presumably, the WKBJ approximation underestimates growth rates because it ignores barotropic curvature in the basic state. Nevertheless, for characteristic values of Ros the WKBJ results seem adequate. The same robustness of WKBJ results was found by Lin and Pierrehumbert (1988) for the nonseparable Charney problem.

It might be worthwhile, at this point, to note the usefulness of an additional approximation introduced by IL. In this approximation, we replaced the exact Eady dispersion relation with a rational approximation chosen to duplicate the Eady relation's branch point

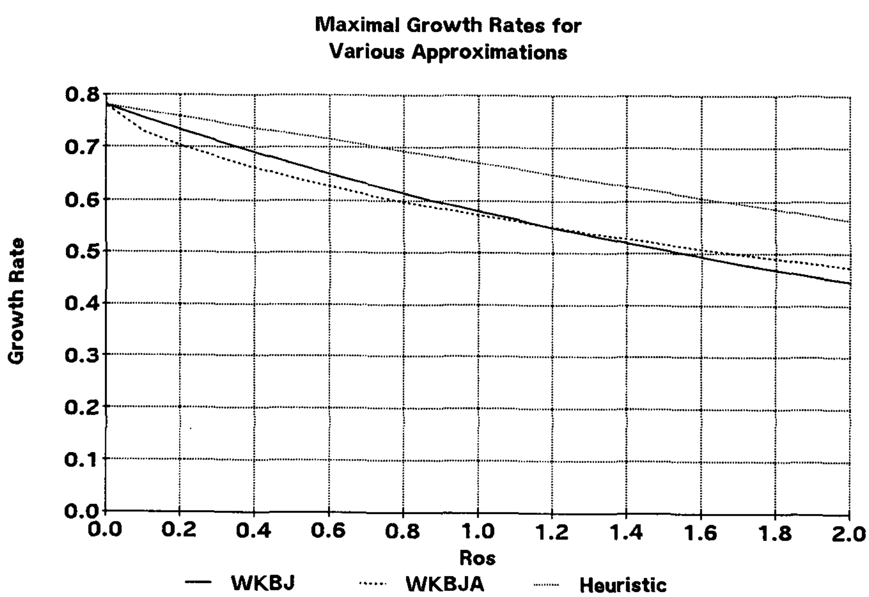

FIG. 3. Here the jet varies in latitude as $\bar{u}^2(Y) = (1 + Y^2)^{-1}$. The curves are plots of the maximum growth rate vs. Ros. The WKBJ curve is the result of the WKBJ approximation, the WKBJA curve is the result of the WKBJ calculation with the Eady dispersion relation replaced by the rational approximant (4). Finally the curve marked "heuristic" is the result of the heuristic calculation for the Eady problem (described in the text).

and singularity nearest the origin in the complex wavenumber space:

$$4c^2 = \frac{9}{16}\bar{u}^2(Y)\left(1 - \frac{\pi^2 + d_c^2}{\pi^2 + d^2}\right), \qquad (4)$$

where $d_c = 2.3994$. This enables the closed form integration of the quantization condition. As shown in Fig. 3, the use of the rational approximation yields results very close to the full WKBJ results. The heuristic approximation in IL involves using (4) to evaluate (2), which is then further approximated by the first term in its power series expansion. As seen in Fig. 3, results from the heuristic approximation are less accurate than either the full WKBJ results or the results using the rational approximation to the Eady dispersion relation. The results of the heuristic approximation are what MP compared with their numerical results.

3. Conclusion

Results are presented that demonstrate the excellent performance of the WKBJ approximation for jets with half-widths as small as one Rossby radius; atmospheric jets characteristically fall within this range. For such jets, the WKBJ approximation can simplify atmospheric stability calculations, reducing them to relatively trivial tasks. The authors apologize for the misprint in their earlier paper which led MP to misassess this situation.

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REFERENCES

Charney, J. G., 1947: The dynamics of long waves in a baroclinic westerly current. *J. Meteor.*, **4**, 135–162.
Eady, E. T., 1949: Long waves and cyclone waves. *Tellus*, **1**, 33–52.
Ioannou, P., and R. S. Lindzen, 1986: Baroclinic instability in the presence of barotropic jets. *J. Atmos. Sci.*, **43**, 2999–3014.
Lin, S.-J., and R. T. Pierrehumbert, 1988: Does Ekman friction suppress baroclinic instability? *J. Atmos. Sci.*, **45**, 2920–2933.
McIntyre, M. E., 1970: On the non-separable baroclinic parallel flow instability problem. *J. Fluid Mech.*, **40**, 273–306.
Moore, G. W. K., and W. R. Peltier, 1989: Nonseparable baroclinic instability. Part 1: Quasi-geostrophic dynamics. *J. Atmos. Sci.*, **46**, 57–78.
Newell, R. E., J. W. Kidson, D. G. Vincent and G. J. Boer, 1972: *The General Circulation of the Tropical Atmosphere and Interactions with Extratropical Latitudes, Volume 1,* MIT Press, 258 pp.