On the Accuracy of the WKBJ Approximation to the Nonseparable Quasi-geostrophic Baroclinic Instability Problem

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8 January 1990 and 8 May 1990

ABSTRACT

The accuracy of the WKBJ approximation to the nonseparable quasi-geostrophic baroclinic instability problem is investigated. By direct comparison with an effectively exact solution, we demonstrate for a particular class of frontal zones that the range of parameters for which agreement is attained is rather limited. Most importantly, the WKBJ solution is shown to be significantly in error for frontal zones representative of typical atmospheric conditions.

1. Introduction

The stability of realistic two-dimensional baroclinic mean states to three-dimensional small-amplitude perturbations is an important topic in atmospheric dynamics. Even within the context of quasi-geostrophic theory, the technical difficulties involved in solving this problem have lead to the development of various approximate solutions (e.g., McIntyre 1970; Ioannou and Lindzen 1986). Each of these schemes relies upon simplifications of the underlying basic state that occur in some special asymptotic limit. As is typical of such asymptotic solutions, their range of validity is often significantly larger than one would, on the basis of the formal analysis, have any reason to expect (e.g., Bender and Orszag 1978). Of itself this "usual" circumstance provides no reason for confidence in applying such methods. The **only** rigorous way to establish the regime in which a particular approximate solution is valid is by means of a comparison with the exact solution of the same problem. Recently, methodologies have been developed that allow one to solve either the nonseparable quasi-geostrophic Charney (Lin and Pierrehumbert 1988) or Eady (Moore and Peltier 1989) problems without approximation. As a result, it is now possible to determine the validity of the various approximate solutions to the nonseparable quasi-geostrophic baroclinic instability problem that have been proposed.

Indeed this has already been attempted. With regards to McIntyre's (1970) solution that expressed the growth rate of the unstable waves as a power series in μ, the parameter in his solution that specified the strength of the meridional variation in the zonal wind field, Moore and Peltier (1989) were able to show that McIntyre's scheme was indeed valid for small values of μ. The range of validity of his solution was however limited by the existence of a pole associated with the rapid growth of a subharmonic mode through the barotropic instability mechanism. The existence of this pole was not, of course, predicted by McIntyre's asymptotic analysis.

In contrast, Moore and Peltier (1989) found large differences between the stability of a frontal mean state according to their full solution and that predicted by the W.K.B.J. approximation of Ioannou and Lindzen (1986). Indeed, the WKBJ solution of Ioannou and Lindzen (1986) was apparently valid only in the asymptotic limit in which the parameter in their solution (Ros), that is inversely proportional to the width of the frontal zone, approaches zero. If this were correct then it would clearly imply a fundamental difference in the applicability of the asymptotic analysis of McIntyre as compared to that of Ioannou and Lindzen. This result would be rather unexpected and somewhat counter intuitive. It has recently come to our attention, however, that a typographical error in the definition of the along-front wind field provided by Ioannou and Lindzen (1986) led Moore and Peltier (1989) to compare the stability characteristics predicted by the full and approximate analyses of *two different mean states.* In this note we have redone the WKBJ analysis in order to compare the stability characteristics predicted by the full and approximate analyses for *the same mean state.*

In doing so, we will show that agreement between the two solutions does indeed extend to small but nonzero values of Ros. However, this agreement does not encompass values of Ros that are representative of typical midlatitude conditions, for which the WKBJ so-

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lution **underpredicts** the growth rate of the most unstable baroclinic wave by approximately 20%.

2. Comparison of the full and approximate solutions

We wish to investigate the linear stability of the two-dimensional nonseparable frontal zones whose along-front wind and potential temperature fields are given by:

$$\bar{V}(x, z) = \frac{\Delta V}{H} \frac{(z - H/2)}{(1 + [\mathrm{Ros}^2 x^2/(NH/f)^2])}$$

and

$$\bar{\Theta}(x, z) = \Delta\Theta\left[\frac{z}{H} + \frac{\Delta V}{NH\,\mathrm{Ros}} \tan^{-1}\left(\frac{\mathrm{Ros}x}{(NH/f)}\right)\right]. \quad (1)$$

The coordinate system used is the same as that in Moore and Peltier (1989) and has the x axis oriented in the along-front or meridional direction. The parameters ΔV and $\Delta\Theta$ represent the differences in the along-front wind and potential temperature fields between the rigid tropopause ($z = H$) and the surface ($z = 0$). For the purpose of this note we have chosen $\Delta V = 20$ m s^{-1}, $\Delta\Theta = 24°$C, and $H = 8$ km. The Brunt–Väisällä frequency N and Coriolis parameter f are assumed to be constant and to have the values 10^{-2} s^{-1} and 10^{-4} s^{-1}, respectively. The parameter Ros controls the width of the frontal zone. In the limit Ros $\rightarrow 0$, one regains the Eady (1949) mean state. For frontal zones representative of typical midlatitude conditions, Ros is in the range of 0.8 to 1.5. An example of a frontal zone defined by (1) is displayed in Fig. 2 of Moore and Peltier (1989).

Following Moore and Peltier (1989), we can state that the perturbation geopotential φ' on such a mean state must have the following functional form:

$$\varphi'(x, y, z, t) = \mathrm{Re}[\varphi^+(x, z)e^{iby}e^{ist}], \quad (2)$$

where b is the along-front wavenumber and s is the complex valued growth rate. In the methodology of Moore and Peltier (1989), one employs a Galerkin decomposition for φ^+ to reduce the problem of determining the stability of the mean state of interest to the solution of a large matrix eigenvalue problem. The complex valued growth rate s is the eigenvalue, while the projection of φ^+ onto the chosen basis functions is the eigenvector. As discussed in Moore and Peltier (1989), this approach leads to a determination of the linear stability of the mean state that is free of any approximation provided that the Galerkin expansion is truncated at sufficiently high order.

In their approximate solution, Ioannou and Lindzen (1986) assume that the length scale over which the mean state varies in the meridional direction is large compared to the Rossby radius of deformation, i.e., Ros $\ll 1$. Given this assumption, they introduce a "slow" variable $X = \mathrm{Ros}^* x$ and assume that $\varphi^+(X, z) = e^{ia(X)}f(z)$. Making the usual WKBJ approximation (Bender and Orszag 1978), leads to the following system of equations that must be satisfied by $a(X)$ and s (Ioannou and Lindzen 1986):

$$\frac{2}{\mathrm{Ros}} \int_0^{X_0} a(X)dX = \pi/2 \quad (3)$$

and

$$s^2 = b^2\bar{v}(X)^2[\coth(d/2) - d/2] \times [d/2 - \tanh(d/2)]/d^2, \quad (4)$$

where d is the full wavenumber [$d^2 = a(X)^2 + b^2$] and X_0 is the zero of $a(X)$ along the positive X axis. In the above, $\bar{v}$ is the variation of the zonal wind field with respect to the slow variable X and is given by:

$$\bar{v}(X) = \frac{1}{(1 + X^2)}. \quad (5)$$

The implicit definition of $a(X)$ in the transcendental Eady dispersion relation (4) makes an analytic solution impossible. As a result, (3) and (4) were solved using an iterative scheme.

In Fig. 1, we present our comparison of the growth rate of the most unstable wave as a function of Ros as predicted by the full and approximate methods that have been described above. As can be clearly seen from this figure, there is excellent agreement between the two solutions for values of Ros less than 0.5. This includes a common asymptote as Ros $\rightarrow 0$. Indeed the largest error in this regime was on the order of 0.25%. For Ros larger than 0.5, the approximate solution begins to inexorably diverge from the full solution. In

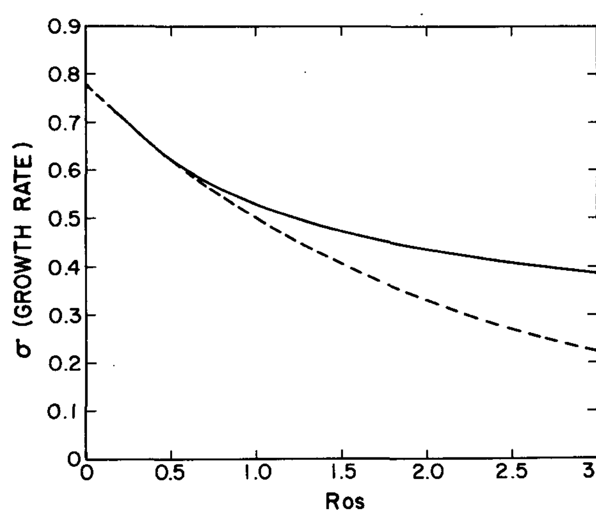

FIG. 1. The nondimensional growth rate of the most unstable wave as a function of Ros. The solid curve is the result of the solution of the nonseparable quasi-geostrophic stability problem without approximation. The dashed curve is the result of WKBJ solution of the same problem.

this regime, the approximate solution systematically under predicts the growth rate of the most unstable wave. For values of Ro*s* representative of typical atmospheric conditions, the deviation is on the order of 20%.

3. Conclusions

The stability of a nonseparable baroclinic mean state to quasi-geostrophic disturbances as predicted by WKBJ theory has been compared to that in which no such approximation is made. The approximate solution is formally valid in the limit in which Ro*s*, the parameter that defines the width of the baroclinic zone, tends to zero. It has been shown that, as expected, the regime in which the WKBJ solution is accurate extends to finite vales of Ro*s*. In particular, for Ro*s* < 0.5 (that is for very broad baroclinic zones), the maximum deviation between the two solutions is on the order of 0.25%. However, for values of Ro*s* representative of typical atmospheric frontal zones, there is a considerable discrepancy, on the order of 20%, between the full and approximate solutions.

Acknowledgments: The authors would like to thank P. Ioannou and R. S. Lindzen for pointing out the typographical error in their publication (Ioannou and Lindzen 1986).

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