A Theoretical Study of Cold Air Damming

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ABSTRACT

The dynamics of cold air damming are examined analytically with a two-layer steady state model. The upper layer is a warm and saturated cross-mountain (easterly or southeasteasterly onshore) flow. The lower layer is a cold mountain-parallel (northerly) jet trapped on the windward (eastern) side of the mountain. The interface between the two layers represents a coastal front—a sloping inversion layer coupling the trapped cold dome with the warm onshore flow above through pressure continuity.

An analytical expression is obtained for the inviscid upper-layer flow with hydrostatic and moist adiabatic approximations. Blackadar's PBL parameterization of eddy viscosity is used in the lower-layer equations. Solutions for the mountain-parallel jet and its associated secondary transverse circulation are obtained by expanding asymptotically upon a small parameter proportional to the square root of the inertial aspect ratio—the ratio between the mountain height and the radius of inertial oscillation. The geometric shape of the sloping interface is solved numerically from a differential-integral equation derived from the pressure continuity condition imposed at the interface.

The observed flow structures and force balances of cold air damming events are produced qualitatively by the model. In the cold dome the mountain-parallel jet is controlled by the competition between the mountain-parallel pressure gradient and friction: the jet is stronger with smoother surfaces, higher mountains, and faster mountain-normal geostrophic winds. In the mountain-normal direction the vertically averaged force balance in the cold dome is nearly geostrophic and controls the geometric shape of the cold dome. The basic mountain-normal pressure gradient generated in the cold dome by the negative buoyancy distribution tends to flatten the sloping interface and expand the cold dome upstream against the mountain-normal pressure gradient (produced by the upper-layer onshore wind) and Coriolis force (induced by the lower-layer mountain-parallel jet). It is found that the interface slope increases and the cold dome shrinks as the Froude number and/or upstream mountain-parallel geostrophic wind increase, or as the Rossby number, upper-layer depth, and/or surface roughness length decrease, and vice versa. The cold dome will either vanish or not be in a steady state if the Froude number is large enough or the roughness length gets too small. The theoretical findings are explained physically based on detailed analyses of the force balance along the inversion interface.

1. Introduction

The phenomenon of cold air damming along the slopes of mountain ranges is referred to as cold air damming (Richwien 1980), or, mountain barrier jet (Schwerdtfeger 1975). In the events of Appalachian Mountain cold air damming (see, for example, Forbes et al. 1987; Bell and Bosart 1988), the cold air is in a form of a dome, identified by a pressure ridge in the sea level isobar pattern. The temperature difference can exceed 20°C over the approximate 150 km between the dammed region and the coast. The formation of the cold dome can be a critical factor in distinguishing a rain event from a sleet or freezing rain event as unfrozen precipitation passes through the cold dome and freezes or becomes supercooled. Freezing rain and freezing drizzle are dangerous forms of winter weather. However, because of the shallow and mesoscale nature of these events there have been many operational problems associated with forecasting cold air dammings. Inadequate resolution of the observation, analysis, and forecasting system, as well as our very limited understanding of this phenomenon seem to be the factors that may have caused poor forecasts of cold air damming events (Stauffer and Warner 1987). Cold air damming events are also identified in association with one type of winter storm along Colorado's Front Range, which displays great variability in the distribution of heavy snow and causes significant economic and human consequences for the urban corridor from Denver to Fort Collins (Dunn 1987). In view of these episodes and the fact that cold air dammings also occur on other mountain slopes around the world (Schwerdtfeger 1975; Lackmann and Overland 1989), we are motivated to explore their common dynamics by using a simple analytical model. This will be the focus of this paper.

The observational model of Appalachian cold air damming (see Fig. 21 of Forbes et al. 1987) contains three layers: an upper-layer cross-mountain flow, a lower-layer cold dome of northerly mountain-parallel
jet, and an inversion layer in the middle that couples the upper and lower layers through pressure continuity and a limited extent of turbulent mixing. Theoretically, turbulent mixing will develop and make the inversion layer deep until the local Richardson number $Ri$ is larger than 0.25 (in which case, turbulent mixing is suppressed by the density stratification). With a typical $10^\circ C$ temperature increase and 20 m s$^{-1}$ wind speed difference across the inversion layer, the estimated thickness for an inversion layer with $Ri \approx \frac{1}{4}$ is about 300 m, which is the same order of magnitude as that revealed by the observations. Since the inversion layer is relatively thin and strong wind shear does not seem to spread far above the inversion during the mature stage of cold air damming (see Figs. 12–13 of Bell and Bosart 1988), we may assume that the turbulent mixing in the inversion layer is a secondary feature, though the internal friction along the inversion layer may be important for the force balance at the front nose as we will see later. In this paper, the inversion layer is treated as a free-slip interface that couples the inviscid upper layer and viscous lower layer only through pressure continuity in our analytical model (see Fig. 1). Clearly, the cold air dammings are mesoscale phenomena and occur only under certain favorable synoptic conditions. When the favorable synoptic conditions persist, the cold air mass is often observed to remain in a quasi-steady state. Our interest in this paper is in the steady state dynamics of cold air damming.

According to the observational analysis of Bell and Bosart (1988), the mountain-parallel jet in a cold dome is maintained mainly by the force balance between the pressure gradient and friction, while the force balance in the mountain-normal direction is largely between the Coriolis force and pressure gradient generated by the cold air mass accumulation against the mountain. This observational fact allows us to speculate that the (eddy viscous) friction is crucial for the persistence of a cold air damming event. When the cold air is blocked by the mountain, the Coriolis force vanishes in the mountain-parallel direction and the mountain-parallel jet is developed (due to the unbalanced pressure gradient) and intensified continuously until it is equally decelerated by friction. If the friction is small enough, then the mountain-parallel jet will become so strong that the mountain-normal Coriolis force will overcome the negative buoyancy and push the cold air over the mountain. Thus, without friction the damming cannot persist.

Using a numerical hydrostatic model, Pierrehumbert and Wyman (1985) examined the upstream effects of mesoscale mountains. Because their model had a free-slip lower boundary and no explicit diffusion (but numerical diffusion could exist), the results for the rotational cases in their Figs. 10–16 showed that the strong damming (stagnant flow) lasted only for a time period shorter than a half-pendulum day after the flow was impulsively started. This time scale may characterize a rapid formation of damming under a favorable synoptic condition, but it does not explain the observed persistence of cold air damming in the mature stage. Based on an analysis of the equation of motion, Pierrehumbert and Wyman (1985) explained the transient nature of inviscid blocking of rotational flows. Specifically, they argued that there can be neither permanent modification of the far upstream flow nor permanent stagnation of the cross-mountain flow in a finite volume if the flow is inviscid and rotational. The transient nature of inviscid cold air damming was also clearly illustrated by Shutts (1987) with a semigeostrophic element model. His model showed that the cold air was only decelerated on the windward side of the mountain. The flow never became steady or stagnant.

As explained above, in order to produce a steady state cold air damming in our two-layer model (Fig. 1), we have to consider the effect of eddy viscosity and develop a PBL model for the surface-based cold dome. On the other hand, since turbulent eddies are largely confined below the inversion layer, we have assumed that the interface is free-slip and that the upper-layer flow is inviscid. As we will see, such a two-layer model, though very crude and restricted to two-dimensional flows, does allow us to examine the quasi-steady dynamics of cold air damming and, particularly, to quantify the effect of several important meteorological parameters on the structure of dammed flow.

In the following section we formulate and solve for the upper-layer flow over arbitrary “topography” that is equivalent to a cold dome within the restriction of the assumed free-slip interface condition. Section 3 develops the lower-layer PBL equations and solves for the leading order solutions of the mountain-parallel jet and associated transverse circulation. Section 4 derives a differential-integral equation for the geometric shape of the interface based on the condition of pressure continuity across the interface. Section 5 examines the structures of the solutions over broad ranges of parameters and explains the results physically. Conclusions are drawn in section 6.

2. Equations and solutions for the upper-layer flow

The upper-layer flow is assumed to be saturated and moist adiabatic. The flow overruns the composite topography $H(x) = H_0(x) + \Delta h(x)$ where $H_0(x)$ is the mountain profile normalized by its maximum height $H$, and $\Delta h(x)$ is the normalized depth of the cold dome (see Fig. 1). On the upstream side ($x \to \infty$), the flow ($-U, V$) is uniform and geostrophic. The moist-adiabatic approximation is based on the following observational facts and theoretical considerations. As shown in Figs. 12–13 of Bell and Bosart (1988), during the mature phase of cold air damming the temperature and dewpoint temperature soundings are very close to each other and nearly parallel to the moist-adiabatic curve above and below the inversion layer.
which suggests that the upper-layer flow is largely saturated and the moist stratification is nearly neutral. Specifically, the four soundings in the above referred figures show that the increase of potential temperature is less than 1°K km$^{-1}$ in the upper (between 850 and 750 mb) and lower layers, but exceeds 10°K across the inversion layer. Clearly, it is the inversion that largely accounts for the strong overall stratification (the net increase of potential temperature over the total depth of the upper and lower layers), so the effect of stratification on the cold air damming is largely associated with the intensity of the inversion. As demonstrated in the Appendix, the cross-sectional area of the cold dome is seen to reduce by only about 2% (or 10%) as the upper-layer moist stratification $N_1^2$ increases from 0.0 to $10^{-2}$ s$^{-2}$ (or $10^{-4}$ s$^{-2}$). Note that the upper-layer stratification estimated from the above observation is as small as $N_1^2 < 0.3 \times 10^{-4}$ s$^{-2}$, so the effect of upper-layer stratification on the dynamics of cold air damming is small and the moist-adiabatic assumption (i.e., $N_1^2 = 0$) should be valid.

According to the observations of Forbes et al. (1987) and Bell and Bosart (1988), we may assume that

(i) $U \approx O(10$ m s$^{-1}$),
(ii) $H \approx O(1$ km),
(iii) the cold dome width $L$ is between 100 km $\leq L \leq 500$ km and
(iv) the upper boundary of the cross-mountain moist flow is at $z = HH_2$ with $1.5 \leq h_2 \leq 5.0$ (see Fig. 1).

Here we choose $L$ empirically as an a priori horizontal scale for the cold dome and expect that the nondimensional width of the cold dome will be $O(1)$ in the solutions. Strictly speaking, $L$ is an internal length scale and should be determined together with the solution of the total flow, while the remaining three parameters ($U, H, HH_2$) are external. However, in this section we only consider the upper-layer flow running over a given topography $HH(x)$, so the above four parameters are all "external." Consequently, we have the following three "independent" nondimensional parameters:

$$Ro = U/L \approx 0.2-1.0,$$
$$h_2 \approx 1.5-5.0,$$
and

$$a = H/L = W/U \approx (0.2-1.0) \times 10^{-2}, \quad (2.1)$$

where $Ro$ is the Rossby number and $f$ the Coriolis parameter. The aspect ratio $a$ is also chosen as the ratio between the vertical velocity scale $W$ and horizontal velocity scale $U$. The nondimensional variables for space ($x, z$), velocity ($u, v, w$), and pressure perturbation $p$ are obtained as

$$(x, z) \leftarrow (x/L, z/H),$$
$$(u, v, w) \leftarrow (u/U, v/U, w/W),$$
$$p \leftarrow p/(fLU\rho_1), \quad (2.2)$$

where $\rho_1$ is the constant reference density for the upper layer. Because of the moist-adiabatic approximation there is no buoyancy perturbation term in the vertical equation of motion. The small aspect ratio $a = H/L \leq 1$ estimated in (2.1) then allows the hydrostatic approximation.

The nondimensional hydrostatic equations of motion ($\partial_x = 0$) are

$$RoJ(\psi, u) - v = -\partial_z p, \quad (2.3a)$$
$$RoJ(\psi, v) + u = u_\xi = -1, \quad (2.3b)$$
$$0 = \partial_z p \quad (2.3c)$$

and

$$(-\partial_z \psi, \partial_x \psi) = (u, w), \quad (2.3d)$$

where $J(\psi, \psi) = \partial(\psi, \psi)/\partial(x, z)$. In the streamline coordinates ($x, \psi$), the above equations (2.3a–c) take the following forms:

$$Ro(u\partial_x u) - v = -\partial_x p, \quad (2.4a)$$
$$Ro(u\partial_x v) + u = -1, \quad (2.4b)$$

and

$$0 = \partial p/\partial \psi, \quad (2.4c)$$

where $\partial_x = \partial/\partial x|_\psi$. Clearly, (2.4a–c) indicate that ($u, v, p$) are independent of $\psi$. This also means that ($u, v, p$) are independent of $z$ in the ($x, z$) coordinates.

Because $u$ is independent of $\psi$ or $z$, $u$ is obtained immediately from the mass continuity as follows.
\[ u = -h_2/(h_2 - h). \]  
(2.5a)

Substituting (2.5a) into (2.4b) and integrating gives

\[ v = (h_2 \, \text{Ro})^{-1} \int_x^\infty h(x')dx' + v_\infty, \]  
(2.5b)

where \( v_\infty = V/U \) is the nondimensional upstream \((x \to \infty)\) mountain-parallel wind (scaled by \(U\)). Integrating (2.4a) gives the pressure perturbation \(p\):

\[ p = \int_x^\infty v \, dx' - Ro(u/2) + \text{constant}. \]  
(2.5c)

Obviously, (2.5c) is the Bernoulli equation under the hydrostatic and moist-adiabatic approximations. It will be used to obtain the interface equation in section 4.

3. Equations and solutions for the lower-layer flow

According to the observations of Forbes et al. (1987), Dunn (1987), and Bell and Bosart (1988), the cold dome is shallow and the air in the cold dome is largely saturated by precipitation (the temperature lapse rate is very close to moist adiabatic), so we assume that the cold dome has a constant equivalent potential temperature or a constant density \(\rho_0 (>\rho_1)\). The scaling here is similar to (2.2) except that \(\rho_1\) is replaced by \(\rho_0\). Thus, the lower-layer mountain-parallel jet can be described by the following nondimensional hydrostatic steady state viscous equations

\[ \text{Ro} J(\psi, u) = v - \partial_x p + Du, \]  
(3.1a)

\[ \text{Ro} J(\psi, v) + u = -1 + Du, \]  
(3.1b)

\[ 0 = \partial_x p, \]  
(3.1c)

\[ (-\partial_x \psi, \partial_x \psi) = (u, w), \]  
(3.1d)

where \( D = a^2 \partial_x E_k \partial_x \partial_x \partial_x + \partial_x E_k \partial_x \partial_x \partial_x \partial_x E_k \partial_x \partial_x \partial_x = v/((fH)^2) \) is the Ekman number, and \( v \) is the coefficient of eddy viscosity.

As mentioned in the Introduction, the mountain-parallel jet cannot be steady unless the effects of eddy viscosity and surface friction are taken into consideration. Now we can show this precisely with Eqs. (3.1a–e). Integrating (3.1a–b) over the cold dome with the free-slip (interface) boundary condition on \(z = h(x)\) gives

\[ \{\tau_x\} + \{[v]\} = \{[\partial_x p]\}, \]  
(3.2a)

\[ \{\tau_y\} = -\{[1]\}, \]  
(3.2b)

where \( \{\tau\} = \int \tau \, dx, \{[v]\} = \int ([v]) \, dx \), \(\tau_x, \tau_y = E_k \partial_x (u, v)\) is the surface stress vector. Physically, (3.2b) means that the volume-averaged force balance in the mountain-parallel direction is between the surface stress \(\{\tau_y\}\) and pressure gradient \(-\{[1]\}\) associated with the upper-layer cross-mountain geostrophic flow, while (3.2a) means that the volume-averaged force balance in the mountain-normal direction is between the pressure gradient \(\{[\partial_x p]\}\), surface stress \(\{\tau_x\}\), and Coriolis force \(\{[v]\}\) associated with the mountain-parallel jet \(v\). Clearly, in the absence of eddy viscosity or surface friction, the pressure gradient force \(-\{[1]\}\) will accelerate the mountain-parallel jet; i.e., increase the magnitude of \(\{[v]\}\) with time. In this case, the mountain-normal Coriolis force will increase without bound, pushing more and more cold air over the mountain (against the pressure gradient \(\{[\partial_x p]\}\) generated by the negative buoyancy of the cold dome).

As a result, the cold dome should shrink with time and eventually vanish on the windward side of the mountain. This kind of transient nature for inviscid cold air damming has been explored by Shutts (1987) with a semigeostrophic Lagrangian-element model.

Equations (3.1a–d) can be solved analytically by using the method of asymptotic expansion. To find an appropriate expansion parameter, we need first to estimate the coefficient of eddy viscosity \(v\). For a neutrally stratified boundary layer, \(v\) can be estimated by (see Blackadar 1962)

\[ v = \kappa_0 v_\epsilon \varepsilon = \kappa_0 v_\epsilon (z' + z_0)(1 + \kappa_0 z'/z_0)^{-1}, \]  
(3.3a)

where \(z' = z - Hh_0(x)\) is the (dimensional) height above the topography; \(\kappa_0 \approx 0.4\) is the von Kármán constant; \(\epsilon\) is the Prandtl mixing length that characterizes the size of the energy containing eddies; \(v_\epsilon\) is the frictional velocity defined by the (dimensional) surface stress \(\tau_y = \tau_y^* (\gg \tau_x); z_0\) is the surface roughness length, and

\[ z_0 \approx 0.01 v_\epsilon / f \]  
(3.3b)

measures the depth of the surface layer. Note that \(\tau_y = \nu \partial_x v\) at \(z' = 0\) and \(\nu \partial_x v = 0\) at \(z' = H\Delta H\) (the free-slip upper boundary condition). Integrating the leading order approximation equation \(\partial_x (\partial_x v) = fU\) [see (3.6a)] of the dimensional form of (3.1b) gives

\[ v_\epsilon^* = \tau_y = \left| \nu \partial_x v \right|_{z=0} = fU H \Delta H. \]  
(3.3c)

Substituting (3.3b–c) into (3.3a)/(\(fH^2\Delta h^2\)) gives (nondimensional form)

\[ \text{Ek}/\Delta h^2 = \mu(\xi, \xi_0, \xi_\zeta)/\epsilon, \]  
(3.4a)

where

\[ \mu(\xi, \xi_0, \xi_\zeta) = (\xi + \xi_0)(1 + \xi/\xi_\zeta)^{-1}, \]  
(3.4b)

\[ \epsilon = (a_0 \Delta h)^{1/2} \kappa_0^{-1} \ll 1, \]  
(3.4c)

\[ \xi_\zeta = 0.01[(a_0 \Delta h)^{1/2}]^{-1} \approx O(\Delta h^{-1/2}), \]  
(3.4d)

\( (\xi, \xi_0, \xi_\zeta) \leftarrow (z', z_0, z_0/\kappa_0)(\Delta h H)^{-1}, \)  
(3.4e)

where \(a_0 = a/\text{Ro} = fH/U\) is the inertial aspect ratio (i.e., the ratio between \(H\) and the radius of inertial oscillation \(L_0 = U/f\)). Here dimensional variables are used on the right-hand side of (3.4e) to show the scaling. The estimations in (3.4c–d) are based on (2.1).

After the above preparation, we can rescale the vertical coordinate by \(z \leftarrow z'/(\Delta h H)\) instead of the original
$z \leftarrow z/H$ in (3.1). The controlling nondimensional parameter $E_k$ in (3.1) is now replaced by $\mu/\epsilon$ in the rescaled equations. Note that $\epsilon \ll 1$ in (3.4c), so we may introduce the following asymptotic expansion

$$(\psi, u, v, p) = \sum_{i=1}^{\infty} c_i(\Delta \psi_i, u_i, v_i, p_i) \quad \text{with}$$

$$u_i = -\partial_x \psi_i.$$  \hfill (3.5)

Substituting (3.5) into the rescaled (3.1), we obtain the following equations and boundary conditions for the leading order variables ($i = 1, 2$):

$$\psi_1 = \frac{1}{\epsilon}, \quad \psi_2 = \frac{1}{\epsilon^2}, \quad v_1 = 0, \quad \psi_2 = \frac{1}{\epsilon^2}, \quad \psi_2 = \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \quad \text{at} \quad \xi = 0,$$

$$\psi_2 = \frac{1}{\epsilon^2}, \quad \psi_2 = \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \quad \text{at} \quad \xi = 1,$$

$$\psi_2 = \frac{1}{\epsilon^2}, \quad \psi_2 = \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \quad \text{at} \quad \xi = 1.$$

(3.6a-c)

$$v_1 = 0 \quad \text{at} \quad \xi = 0, \quad \psi_2 = \frac{1}{\epsilon^2}, \quad \psi_2 = \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \quad \text{at} \quad \xi = 1;$$  \hfill (3.6b)

$$v_1 = 0 \quad \text{at} \quad \xi = 0, \quad \psi_2 = \frac{1}{\epsilon^2}, \quad \psi_2 = \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \quad \text{at} \quad \xi = 1;$$  \hfill (3.6c)

$$\psi_2 = \frac{1}{\epsilon^2}, \quad \psi_2 = \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \quad \text{at} \quad \xi = 1;$$  \hfill (3.6d)

where $\mu = \mu(\xi, \zeta_0, \zeta_s)$ is given in (3.4b). The force balance in (3.6c,a) contains the same physical terms as the volume-integrated force balance in (3.2a–b). This means that the local leading order force balance is dominated by the same terms as in the volume-integrated force balance despite the height-dependence of the local force balance and the middle-level sign change of the internal mountain-normal viscous force as shown later by the solutions.

The surface roughness length $z_0$ over land may vary from 0.1 m to 2.0 m, so that $z_0 = z_0/(\Delta h H) \approx 0.1$ as long as $\Delta h H \approx 20$. When the depth of the cold air $\Delta h H$ becomes shallower than $10z_0 \approx 20$, we may consider that the inversion interface meets the ground and a surface front is actually formed. Although the flow regime in this small scale surface front region may be highly nonlinear in the real atmosphere and beyond the description of the PBL parameterization described above, the dynamical impact of this small scale flow on the mesoscale is believed to be small and local. Thus, we may ignore this small scale complication and simply assume that

$$\zeta_0 = \min[z_0/(\Delta h H), 0.1].$$  \hfill (3.7)

Physically, (3.7) assumes that within a very narrow zone $(\Delta h H < 10z_0 \approx 20)$ immediately behind the surface front, the surface roughness gradually decreases to zero at the frontal nose (i.e., $z_0 \approx 0.1 \Delta h H \approx 0$ as $\Delta h \approx 0$) and, thus, consistently matches the smooth surface assumed for the inviscid upper-layer flow on the upstream side of the front.

Equations (3.6a–d) can be integrated analytically and the solutions are (with the approximation $1 + \zeta_0 \approx 1$)

$$v_1 = (\xi^2/2 - \zeta + \zeta_0^2)/\zeta_0 - \ln(\zeta/\zeta_0 + 1),$$  \hfill (3.8a)

$$\partial p_1/\partial x = -\Phi = -\left\{17/6 - 29/(36\xi_0) + 2/(15\xi_0^2) - G[3 - 2/(3\xi_0) - G] \right\}$$

$$\times [1/(3\xi_0^2) - 1.5 + G]^{-1},$$  \hfill (3.8b)

$$\psi_2 = \xi/(120\xi_0^2) + \xi^2(4 - 3/\xi_0)/(72\xi_0^2) + (\xi^3/6) \{0.5 + [4/3 + \Phi - \ln(\zeta/\zeta_0) + 1]/\xi_0 + 1)\}$$

$$+ (\xi^2/2)[5/2 + \Phi - (1.5 + \Phi - G)/\xi_0 + 1/(3\xi_0^2) - \ln(\zeta/\zeta_0 + 1)] - \xi[1/(3\xi_0^2) - 1.5 + G - \Phi][1 - \ln(\zeta/\zeta_0 + 1)],$$  \hfill (3.8c)

$$G = \ln(1 + 1/\xi_0).$$

These solutions are plotted in Figs. 2a–b for different parameter settings of cold dome depth $\Delta h = 1.0, 0.5, 0.2, 0.1, 0.05$ with fixed surface roughness length $z_0$/$H = 10^{-3}$ and inertial aspect ratio $\omega_0 = 0.01$. Note that $v_1 = 0$ in (3.6a), so the $\partial_x v_1$ profiles in Fig. 2a are vertical lines (independent of $\xi$). The log-quadratic $v_1$ profiles are due to the force balance between the pressure gradient and eddy viscous force in mountain-parallel direction. Physically, Fig. 2a shows that the shallower the cold dome (measured by $\Delta h$), the weaker the jet and the smaller the mountain-normal pressure gradient. It is also seen that the mountain-normal Coriolis force $(-v_1)$ and pressure gradient are nearly in balance in the middle levels but significantly off-balance in the upper and lower levels. The unbalanced part manifests the eddy viscous force—positive in the upper levels and negative in the lower levels. Corresponding to each fixed $\Delta h$ in Fig. 2a, the area to the left of the $v_1$ profile is about the same as the area to the left of the $\partial_x v_1$ profile, so that the vertical integrated mountain-normal Coriolis force $[\psi_2]$ is largely in balance with the vertically integrated pressure gradient $[\partial_x p]$ $= \Delta h \partial_x p$ and the surface friction $\tau_s$ is very small due to a large cancellation between the upper-level and lower-level internal eddy viscous forces. Thus, although the internal eddy viscous force is significant for the local force balance in the mountain-normal direction, the mountain-normal surface friction is negligible (see Fig. 12).

Note further from Figs. 2a–b that $(v_1, \partial_x p_1) \approx O(1)$ and $\psi_2 \approx O(1)$, so $v_1$ and $\partial p_1 \approx \psi_2$ and the solutions are consistent with the asymptotic expansion (3.5). We can further show that the higher order solutions $(v_2, \partial_x p_2) \approx O(1)$, $(\psi_2 \approx O(1), (v_3, \partial_x p_3) \approx O(v_1)$, $\psi_4 \approx O(1)$ and, thus, the leading order solutions (3.8a–b) remain valid even when the surface roughness length $z_0$ (or $\zeta_0$) is very small and $(v_1, \partial_x p_1) \gg O(1)$. In the limit of $\zeta_0 \approx 0$, the leading order solutions (3.8a–b) become increasingly dominant, or, say specifically,
Fig. 2. Vertical distributions of (a) $v_1(t)$ and $\partial_x p_1$, and (b) $\phi_2(t)$ for different settings of $\Delta h = 0.05, 0.1, 0.2, 0.5, 1.0$, with fixed $a_0 = fH / U = 0.01$ (H = 1 km, $U = 10$ m s$^{-1}$, and $f = 10^{-4}$ s$^{-1}$) and $a_0/H = 10^{-3}$ ($c_0 = 1$ m); or, equivalently, $(c_0, \xi) = (0.001, 0.25), (0.002, 0.35), (0.005, 0.56), (0.01, 0.8), (0.02, 1.1)$ [see (3.4e)].
\( v_1, \partial_x p_i \rightarrow \ln \xi_0(1, 1) \rightarrow -\infty \) but
\[
\psi_2 \approx O(1) \quad \text{as} \quad \xi_0 \rightarrow 0. \quad (3.9)
\]

Clearly, the smaller the surface friction, the stronger the jet flow and the larger the mountain-normal pressure gradient force needed to hold the mountain-parallel jet on the windward side of the mountain against the jet-induced mountain-normal Coriolis force. However, with all the external parameters fixed, the mountain-normal pressure gradient can be increased only by piling up the cold dome more steeply against the mountain and, inevitably, squeezing out more cold air and pushing it over the mountain. Therefore, as explained earlier, the cold dome will shrink and eventually vanish if \( z_0 \) decreases and becomes extremely small. This property is shown quantitatively by the solutions in section 5.

Note that \((\epsilon, \xi_0, \xi)\) in (3.7) and (3.4c-e) are functions of \((\Delta h, a_0, z_0)\), so \(\partial_x p = \epsilon \partial_x p_i = -\epsilon \Phi(\xi_0, \xi)\) is also a function of \((\Delta h, a_0, z_0)\), which is the final product of the lower-layer parameterization that will be used in the interface equation [see (4.3)]. The functional dependence of \(\partial_x p\) on \(\Delta h(x)\) for different settings of \(a_0\) and a fixed \(z_0\) is shown in Fig. 3 and the physical meaning is explained as follows. As the cold dome becomes deeper (\(\Delta h\) increases), the vertically integrated mountain-parallel jet is less resisted by the surface friction and becomes stronger, so that a larger negative mountain-normal pressure gradient \(\partial_x p\) is needed to hold the jet on the windward side of the mountain against the jet-induced mountain-normal Coriolis force.

In order to examine the sensitivity of the solution to the distribution of \(\nu\), we replace the function \(\mu(\xi)\) in (3.4b) with an intermediate constant value
\[
\tilde{\mu} = \alpha(1 + 1/\xi_0)^{-1}, \quad (3.10)
\]
where \(\xi_0 < \alpha < 1\) and \(\min[\mu(\xi)] < \tilde{\mu} < \max[\mu(\xi)].\) In this case, instead of (3.8a-c), we have the following simplified solutions:
\[
\begin{align*}
v_1 &= (\xi^2/2 - \xi)/\tilde{\mu} \approx O(1)/\alpha, \tag{3.11a} \\
\partial p_i/\partial x &= -0.4/\tilde{\mu} \approx O(1)/\alpha, \tag{3.11b} \\
\psi_2/\Delta h &= (\xi - 1)\xi^2(2 - \xi)^2(120\tilde{\mu}^2)^{-1} < O(1)\alpha^{-2}. \tag{3.11c}
\end{align*}
\]

Note that the eddy viscosity in (3.3a) increases with height, so the surface friction is enhanced when \(\mu\) is replaced by \(\tilde{\mu}\) even for \(\alpha = 0.35\). But, for \(\alpha = 0.35\), the internal viscous effect is reduced and consequently, as
implied by the comparison between Figs. 3 and 4, the total viscous effect remains roughly the same. The functional dependence of \(\partial_x p\) on \(\Delta h\) in Fig. 3 or 4 controls the net impact of the lower-layer dynamics to the cold dome geometry, so the geometric shape of the cold dome should remain largely the same when \(\mu\) is replaced by \(\bar{\mu}\) with \(\alpha = 0.35\). This speculation is verified by the interface solutions in section 5.

### 4. Equation for the interface geometric shape

In this section we use the condition of pressure continuity to derive a differential-integral equation for the interface geometric shape. The pressure continuity across the interface can be expressed as

\[
\Delta P + \Delta p = 0 \quad \text{along} \quad z = h(x),
\]

where \(\Delta (\quad)\) represents the jump of \((\quad)\) across the interface from the cold dome to the upper layer, \(\Delta P\) is the basic hydrostatic pressure jump associated with the reference density difference \(\Delta \rho = \rho_0 - \rho_1 (>0)\) or temperature difference \(\Delta \theta = \theta_0 - \theta_1 (<0)\) between the lower and upper layers, and \(\Delta p\) is the perturbation pressure jump. Here, for simplicity, we have used the Boussinesq approximation and, thus, neglected the factor \(\rho_0 / \rho_1\) for the lower-layer nondimensional \(p\) that is obtained in section 3 and now rescaled by \(\rho_1 f LU\). The nondimensional hydrostatic equation for \(\Delta P\) is

\[
\Delta P = [1 - h(x)] \text{Ro(Fr)}^{-2},
\]

where \(\text{Fr} = U/(\theta_0 h)^{1/2}\) is the Froude number and \(\theta_0 = g \Delta \rho / \rho_1 \approx -g \Delta \theta / \theta_1 (>0)\) is the basic negative buoyancy.

Substituting (4.2), (3.8b), and (2.5a–c) into \(\partial_x h\) from (4.1), we obtain the following equation for \(h\):

\[
\text{Ro}[1/\text{Fr}^2 - h_2^2/(h_2 - h_3)^3] \partial_x h + v + \epsilon \Phi = 0,
\]

\[
- (i) \quad (ii) \quad (iii) \quad - (iv)
\]

\[
\text{with} \quad v = v_0 - (h_2 \text{Ro})^{-1} \int_{x_0}^{x} hdx,
\]

where \(\epsilon = (a_0 \Delta h)^{1/2} / \kappa_0\) is defined in (3.4c); \(\Phi = \Phi(\xi_0, \xi_1)\) is defined in (3.8a); \(\gamma_2\) and \(\gamma_0\) and \(\Phi\) are functions of \(\Delta h = h - h_0\) as shown in (3.4d) and (3.7); and \(v_0 = v(x_0)\) is the mountain-parallel wind at the top point of the cold dome, where \(x = x_0\) and \(z = h(x_0) = h_0(x_0)\). [But \(h(x) > h_0(x)\) as \(x > x_0\).] The relationship between \(v_0\) and \(v\) in (2.5b) is

\[
\frac{v_v}{(h_2 \text{Ro})^{-1} \int_{x_0}^{x} hdx} = v_v = V / U = \tan \beta, \quad (4.4)
\]

![Graph](image_url)

**Fig. 4.** As Fig. 3, but for \(\partial_x p = -0.4 \epsilon / \bar{\mu}\) in (3.11b) with \(\alpha = 0.35\) in (3.10).
where $\beta$ is the upstream inflow angle; i.e., the angle between the inflow ($-U$, $V$) and the negative $x$-direction.

Note that (4.3a) becomes singular when the coefficient of $\partial_x h$ vanishes. This singularity should not occur in the solution except for the top (or nose) point where the interface meets a vertical wall of the mountain tangentially (or meets the ground vertically). The coefficient of $\partial_x h$ should remain positive (negative) along the interface if $Fr^2 < h_2$ ($Fr^2 \geq h_2$). As we will see later, with respect to a uniform variation of interface slope, the interface solution is stable for the low Froude number flow ($Fr^2 < h_2$), but unstable for the high Froude number flow ($Fr^2 \geq h_2$). Thus, we will only examine the low Froude number flow in detail.

By examining the behavior of (4.3a) at the top point $[x_0, h(x_0)]$ of the cold dome, we can show that the cold dome can reach the mountain top [i.e., $h(x_0) = 1$] only if

$$1/ Fr^2 \geq \lambda_1^2 = \bar{\lambda}_1^2 + (k_0 h_2 \, Ro^2)^{-1}$$

$$\times \min_{Fr} \left[ \int_0^\infty \Delta h dx | h(x_0) = 1 \right] , \quad (4.5a)$$

where

$$\bar{\lambda}_1^2 = h_2^2 (h_2 - 1)^{-3} + v_\infty (k_0 \, Ro)^{-1}$$

$$+ (k_0 h_2 \, Ro^2)^{-1} \int_0^\infty h_0 dx \approx \lambda_1^2 , \quad (4.5b)$$

and $k_0 = -\partial_x h_0 (> 0)$ is the windward-side slope of the (triangle) mountain in the nondimensional space ($x$, $z$). Here (4.5a–b) are derived from (4.3a) based on the following facts: (i) $v \rightarrow v_\infty + (h_2 \, Ro)^{-1} \int_0^\infty h dx$ and $(\Delta h, e\Phi) \rightarrow (0, 0)$ as $(x, z) \rightarrow (0, 1)$, (ii) $-\partial_x h |_{x=0} \rightarrow k_0$ as $Fr$ increases towards 1/$\lambda_1$. Under the condition $h(x_0) = 1$ (i.e., that the cold dome reaches the mountain top), the integral in the last term of (4.5a) reaches its minimum as $Fr \rightarrow 1/\lambda_1$, so $\lambda_1$ is defined implicitly in (4.5a) and only can be searched together with the marginal solution that gives the minimum integral in (4.5a). However, because $\Delta h(x)$ shrinks into a very small volume as $Fr \rightarrow 1/\lambda_1$ (see Fig. 5), $\lambda_1$ can be estimated explicitly by $\bar{\lambda}_1$ as in (4.5b).

When $Fr > 1/\lambda_1$, the cold dome becomes shallower than the mountain; i.e., $h_0(x_0) < 1$ and $x_0 > 0$. In this case we can show that $x_0$ satisfies

$$1/ Fr^2 - 1/[h_2 (1 - h_0(x_0)/h_2)] = v(x_0)(k_0 \, Ro)^{-1} , \quad (4.6a)$$

where

$$v(x_0) = v_\infty + (h_2 \, Ro)^{-1} \int_0^\infty h dx$$

$$\approx v_\infty + (h_2 \, Ro)^{-1} \int_0^\infty h_0 dx . \quad (4.6b)$$

The derivation here is similar to that for (4.5a–b). It is easy to see from (4.6a) that the existence of the cold dome $[h_0(x_0) > 0]$ requires

$$1/ Fr^2 > \lambda_2 = \max \{1/ h_2, 1/ h_2 + v_\infty (k_0 \, Ro)\} , \quad (4.7)$$

where the condition $Fr^2 < h_2$ is used because, as mentioned earlier, we only consider the low Froude number flow. Physically, (4.7) indicates that the lower-layer air should be sufficiently cold if it is to stay on the windward side of the mountain. [(4.7) may be subject to a minor modification if the windward-side slope $k_0$ of the mountain is not constant.]

---

**Fig. 5.** Interface solutions (i.e., the geometric shapes of the cold domes) for different settings of $1/ Fr = 1.0, 1.5, 2.0, 2.5, 3.0$ [or dimensional $- \partial h = 3.0, 6.75, 12.0, 18.75, 27.0$ (°K) with $g = 10$ m s$^{-2}$, $\theta_0 = 300$ K, $H = 1$ km and $U = 10$ m s$^{-1}$]. Here, $a_0 = 0.01$, $Ro = 0.25$, $v_\infty = 0.0$, $z_0/H \approx 10^{-3}$ and $h_2 = 2.5$ [or dimensional $f = 10^{-4}$ s$^{-1}$, $L = 400$ km, $V = 0.0$ m s$^{-1}$, $x_0 = 1$ m and $HH = 2.5$ km]. The mountain profile is shaded.
In the above analysis we have examined the behavior of (4.3a) at the top point of the cold dome. Similarly, we can examine the behavior of (4.3a) at the nose point (surface front) of the cold dome. As the interface approaches the nose point, we have \( h = \Delta h \to 0 \), \( v \to v_\infty \), and \( \epsilon \Phi \to 0 \) [see (2.5b) and Fig. 3]. By substituting these limits into (4.3a) we can see that, in addition to the condition (4.7), the existence of the solution (5.1) requires
\[
v_\infty \geq 0 \quad (\text{or } \beta \geq 0). \tag{4.8}
\]
Physically, (4.8) requires that the upstream environmental pressure decreases, or, at least, does not increase towards the mountain (i.e., \( \partial_x p_\alpha \geq 0 \)). This requirement is merely of mathematical interest as it results from the free-slip interface condition and does not agree with the observation of Forbes et al. (1987). More discussions about this will be given later. The differential-integral equations (4.3a–b) are solved numerically under the constraints (4.6)–(4.8) and the results are presented in the following section.

5. Results and discussions

5.1 Pressure gradient (PG) balance along the interface

Before we present the solutions and discuss the results, it is helpful to interpret the Eq. (4.3a) physically. The four terms (i)–(iv) in (4.3a) represent different components of the pressure gradient (PG) along the interface. Their physical meanings are

(i) the lower-layer basic hydrostatic PG generated by the negative buoyancy \( g_0 \) [see (4.2)] in association with the sloping interface,

(ii) the upper-layer non-geostrophic (Bernoulli) PG generated by the kinetic energy variation in the cross-mountain flow [see (2.5a)],

(iii) the upper-layer geostrophic PG induced by the mountain-normal Coriolis force associated with the upper-layer mountain-parallel flow [see (2.5b)],

(iv) the lower-layer PG induced by the mountain-normal Coriolis force and surface friction [see (3.6c) and (3.8b)].

The lower-layer components (i) and (iv) are the negatives of the first and last terms in (4.3a) [i.e., \( -(Ro/Fr^2) \partial_x h \) and \( -\epsilon \Phi \)], so it is convenient to rewrite (4.3a) into
\[
(ii) + (iii) = (i) + (iv) \tag{5.1}
\]
for (upper-layer PGs) (lower-layer PGs).

Although (5.1) is obtained from \( \partial_x (4.1) \), (5.1) is more informative than (4.1) in terms of gaining physical insights into the dynamics that control the interface slope. This is because the interface slope is determined by the variation of the pressure balance, so the pressure balance in (4.1) at a fixed point gives no information as to how the pressure balance controls the interface slope. On the other hand, the PG balance in (5.1) at a given point represents the first-order variation of the pressure balance in (4.1) over the neighborhood of that point along the interface; so a pointwise analysis of the PG balance in (5.1) can help to understand how the PG balance controls the interface slope.

First, let us temporarily ignore the upper-layer effect and concentrate on the competition between (i) and (iv) in (5.1). The PG in (i) is produced by the negative buoyancy \( g_0 \) in association with the negatively (i.e., \( \partial_x h < 0 \)) sloping interface, so this term is positive and increases with \( 1/\text{Fr}^2 \). But (iv) is negative and represents a deduction to the PG generated by (i). To see this term in detail, we integrate (3.6c) with the boundary condition (3.6d) and obtain the following decomposition of (iv):
\[
-\epsilon \Phi = \epsilon \partial_x p_1 \approx (v^2 + \tau_x) \Delta h^{-1} < 0, \tag{5.2}
\]
where \( \tau_x \approx \epsilon v_1 = \epsilon \int v_1 d\zeta < 0 \) (see Fig. 2a) and \( \tau_x = E_k \partial_x h|_{x=0} \approx -\mu \Delta h \partial_x \partial_x v_2|_{x=0} < 0 \) (see Fig. 2b). Thus, (iv) contains two parts: (1) the negative mountain-normal Coriolis force associated with the vertically averaged mountain-parallel jet \( |v|/\Delta h \) and (2) the depth-averaged negative mountain-normal surface friction \( \tau_x/\Delta h \) in the cold dome. Both parts depend on the surface roughness, but in the opposite senses, and \( |v| \) is the dominant part [see (3.8a–c) and Fig. 12]. Thus, the smoother the surface, the negatively larger the PG in (iv). The competition between (i) and (iv) suggests that the interface is steeper and the cold dome is narrower as \( \text{Fr} \) is larger and/or the surface is smoother. This property is indeed shown by the numerical solutions with different settings of \( 1/\text{Fr} \) and \( z_0/h \) in Figs. 5–6. It is also found that the cold dome virtually disappears (not shown) as \( z_0/h \) becomes extremely small (\(<10^{-15}\)), which is consistent with our earlier qualitative analysis. Clearly, the above analysis explains the results in Figs. 5–6 and the competition between (i) and (iv) plays an important role in controlling the geometric shape of the cold dome. But, quantitatively, (iv) is smaller than (i) and, as we will see later, (i) is the dominant term that counters all the remaining three terms in (5.1), so the interface slope is strongly controlled by the Froude number \( Fr \) (Fig. 5) but weakly affected by the surface roughness (Fig. 6).

Now we can examine the upper-layer effect. Because \( -\partial_x h < 0 \), (ii) is positive [see (4.3a)]. This upper-layer PG component partially offsets the lower-layer positive PG in (i) and renders the interface steeper. This effect increases with \( 1/h_2 \) because (ii) increases with \( 1/h_2 \). Physically, a shallower upper-layer cross-mountain flow undergoes stronger "choke" and larger Bernoulli pressure drop above the cold dome, or, in other words, larger positive Bernoulli PG along the interface against the lower-layer positive PG produced by the negative buoyancy \( g_0 \). Thus, the cold dome shrinks down as the upper-layer \( h_2 \) becomes shallow and this is verified.
by the results in Fig. 7. Besides, as \( h_2 \) becomes shallow, the term (iii) [see (4.3b)] increases and this also causes the cold dome to shrink.

The term (iii) in (5.1) can be separated into two parts: (1) \( v - v_\infty = (h_2 \beta_0) \frac{1}{2} \int_x^\infty h(x') dx' \) [see (2.5b)] and (2) \( v_\infty \). The first part is positive and represents the PG induced by the Coriolis force associated with the perturbation part of the mountain-parallel wind \( v - v_\infty \). Like the positive PG in (ii), this PG counteracts the lower-layer PG generated by the negative buoyancy \( g_0 \). The second part represents the upstream mountain-normal PG; i.e., \( \partial_x p_\infty = v_\infty \), which is non-negative under the condition (4.8). When this PG is positive, it counteracts the lower-layer PG generated by the negative buoyancy \( g_0 \) and renders the interface steeper. This property is shown by the solutions with different settings of \( v_\infty \) in Fig. 8. It is also found that the cold dome vanishes (not shown) as \( v_\infty \) increases further to the value of \( \tan(75^\circ) = 3.73 \). Thus, in addition to the condition (4.8), the existence of the solution requires

\[
0 \leq v_\infty < v_m, \tag{5.3}
\]

where \( v_m \) is a function of the remaining independent parameters. As shown earlier in (4.8), the lower bound for \( v_\infty \) in (5.3) is derived from the PG balance at the frontal nose point where the PG generated by the negative buoyancy on the cold dome side is resisted, in the model, only by the upper-layer mountain-normal geostrophic PG. For a real atmospheric cold air damping, however, the internal friction and nonhydrostatic force exerted from the upper-layer onto the cold dome also should counteract the negative buoyancy effect, especially at the front nose, so the environmental PG in the upper layer can be negative. In this case, the lower bound for \( v_\infty \) should be a negative value.

---

**FIG. 6.** As Fig. 5, but for different settings of \( z_0/H = (0.1, 0.2, 0.5, 1.0, 2.0) \times 10^{-3} \) with \( Fr = 0.5 \).

**FIG. 7.** As Fig. 5, but for different settings of \( h_2 = 1.5, 2.0, 2.5, 3.0, 3.5, 4.0 \) with \( Fr = 0.5 \).
b. External parameters and factors that control the solutions

In the nondimensional space \((x, z)\) the solutions are determined by the following six "independent" parameters and two factors:

\[
(Fr, v_\infty, h_2, z_0/H, a_0, Ro), \quad (5.4a)
\]

\[
h_0(x), \text{ and abundance of cold air supply.} \quad (5.4b)
\]

We have assumed so far that the cold air supply is abundant. The scenario of limited cold air supply will be examined later. Note that in place of the inertial aspect ratio \(a_0 = a/Ro\) in (5.4a), one may adopt the conventional aspect ratio \(a = H/L\) as an independent parameter. But, in this case, the Rossby number \(Ro\) involves all the four terms in (4.3a) and its effect in controlling the PG balance is not analytically separable from that of \(a\). The merit for using the inertial aspect ratio \(a_0\) is the mathematical separation between \(a_0\) and \(Ro\), because \(a_0\) appears only in (iv) and \(Ro\) involves only (i)–(iii) (for fixed \(a_0\)). The remaining four parameters \(Fr, h_2, v_\infty\) and \(z_0/H\) appear only in (i), (ii)–(iii), (iii), and (iv), respectively, and their effects in controlling the PG balance and interface slope have been examined in the previous subsection. Thus, we only need to examine the effects of \(a_0\) and \(Ro\).

Our results indicate (not shown) that the cold dome shrinks as the inertial aspect ratio \(a_0\) increases. This behavior can be easily understood if the effect of \(a_0 = a/Ro\) is viewed as the combined effects of \(a\) and \(1/Ro\). If the horizontal coordinate is rescaled by the radius of inertial oscillation \(L_0 = U/f\); i.e., \(x = x/L_0\) then the Rossby number \(Ro\) in (4.3a-b) can be absorbed into the new coordinate \(x\). Thus, in the \((x, z)\) space, the flow structure is independent of \(Ro\) and only the remaining five parameters in (5.4a) are independent. This suggests that in the \((x, z)\) space both the flow structure and the mountain expand (or shrink) horizontally in proportion to the increase (or decrease) of \(Ro\) [with the remaining five parameters in (5.4a) fixed]. Thus, by relabeling the horizontal coordinate, Figs. 5–8 can be used for the solutions with different settings of \(Ro\). When \(Ro = 1\) \((L = L_0)\), the relabeled figures give the solutions in the \((x, z)\) space.

The six parameters in (5.4a) comprise seven independent dimensional external parameters:

\[
(g_0, H, f, U, V, HH_2, z_0) \quad (5.5)
\]

and one internal parameter \(L\) [recall the explanation for (2.1)]. Note that \(g_0, z_0, HH_2\) and \(V\) appear only in \(Fr, z_0/H, h_2\) and \(v_\infty\), respectively, so their controls over the solutions are as in Figs. 5–8, respectively. The dimensional solution should not depend on the internal length scale \(L\). Note that \(L\) appears only in \(Ro\) [among those in (5.4a)], so the above discussed "similarity" of solutions in the nondimensional \((x, z)\) space for different settings of \(Ro\) [with the remaining parameters in (5.4a) fixed and, thus, \(f\) and \(U\) fixed] is simply due to using different scales of \(L\) to measure the same dimensional solution. Therefore, we only need to examine the effects of the remaining three-dimensional parameters \(f, U\), and \(H\).

If the aspect ratio \(a\) is used to replace the inertial aspect ratio \(a_0\) in (5.4a), then the Coriolis parameter \(f\) will appear only in the Rossby number \(Ro\), but \(Ro\) involves all four terms in (5.1). In this case, \(f\) and \(Ro\) have the same controlling effect on the solutions. Our results indicate (not shown) that the width of the cold dome increases from \(0.2L\) to \(4L\) as \(Ro\) varies from 0.1 to 1.0 (corresponding to a variation of \(f\) from \(6^\circ\)N to \(90^\circ\)N with \(U = 10\) m s\(^{-1}\), \(H = 1\) km, and \(L = 400\) km).

It may also be shown that the cold dome shrinks (to nearly a half of its volume) as the cross-mountain in-
flow $U$ increases (from 5 to 10 m s$^{-1}$ with fixed $-\Delta \theta = 12^\circ$K). This result suggests that neither the radius of inertial oscillation $L_0 = U/f$ nor the Rossby radius of deformation $L_R = (g_0 H)^{1/2}/f$ can be an appropriate length scale for the cold dome width, because $L_0$ increases with $U$ and $L_R$ is independent of $U$ (but the cold dome shrinks as $U$ increases). If $L_R$ is modified by the nondimensional internal wave speed $c = (g_0 H)^{1/2}/U$, then the modified Rossby radius of deformation $L_m = c L_R = g_0 H/(f U)$ varies with $g_0, f$, and $U$ in the same direction as the cold dome width. But $L_m$ increases with $H$, while the cold dome width is nearly independent of $H$ (see the next paragraph). Besides, the cold dome width depends on $V$ (see Fig. 8), but $L_m$ does not. Therefore, it seems difficult to define an appropriate length scale for the cold dome width with a simple formulation based on the external parameters in (5.5). Because of this, we have preferred to choose $L$ empirically as an internal parameter rather than to specify $L = L_0$ or $L = L_R$, or $L = L_m$, although the latter has the merit of mathematical neatness as discussed earlier.

In general, it is found that the nondimensional shape of the cold dome is not very sensitive to the variation of the mountain scale height $H$. As $H$ increases, both (i) and (ii)–(iv) increase and, therefore, the PG balance can be maintained with only a small adjustment of the interface slope in the nondimensional $(x, z)$ space. As the solution is transformed back to the dimensional space, the cold dome depth is seen roughly proportional to the mountain height $H$ (until $H > H h_0/2$), but the cold dome width is nearly independent of $H$ [with the remaining parameters in (5.5) fixed].

Figure 9 shows that if the intensity of the upstream inflow is fixed (e.g., $|V| = (U^2 + V^2)^{1/2} = 10$ m s$^{-1}$), then the cold dome shrinks as the upstream inflow angle $\beta = \arctan(V/U)$ increases from $0^\circ$ to $45^\circ$, but remains almost the same as $\beta$ varies between $45^\circ$ and $60^\circ$, and then expands slightly as $\beta$ increases from $60^\circ$ to $85^\circ$. However, if $\beta > 75^\circ$ (or $86^\circ$), then $U = |V| \times \cos \beta < 2.5$ m s$^{-1}$ (or 0.7 m s$^{-1}$), $\epsilon = (f H/U)^{1/2} \kappa_0^{-1} > 0.5$ (or 1.0), and the asymptotic expansion (3.5) becomes poor (or invalid). Here the situation is different from that in Fig. 8, where the mountain-normal component $U = 10$ m s$^{-1}$ of the upstream inflow is fixed and $\epsilon = 0.25$ for all the solutions.

As the cold air supply becomes limited, the cold dome solution shrinks down below the mountain top (not shown), even though the Froude number $Fr$ is still small enough to satisfy the condition (4.5). The solution with limited cold air supply may resemble a damming situation where the mountain is not exactly two-dimensional and the cold air passes the mountain laterally. Solutions are also obtained for different types of mountain profiles and examples are given in the next subsection. These solutions (mostly not shown) share the same qualitative properties as exhibited in Figs. 5–9 and the above physical interpretations remain generally applicable.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig9.png}
\caption{As Fig. 5, except for different settings of the upstream inflow angles $\beta = 0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ, 85^\circ$ with $-\Delta \theta = 12^\circ$K and fixed total upstream wind speed $|V| = (U^2 + V^2)^{1/2} = 10$ m s$^{-1}$. The remaining dimensional parameters are as in Fig. 5, so the nondimensional parameters $Fr = U/(g_0 H)^{1/2}$ and $Ro = U/(f L)$ have different values corresponding to the above settings for $U = |V| \cos \beta$.}
\end{figure}

c. Internal flow structure and stability

Figure 10 shows the internal flow structure for the solution with an Appalachian-type mountain profile. The parameter settings in Fig. 10 are the same as in Fig. 5 with $Fr = 0.5$. The $\psi$ and $\nu$ fields in Fig. 10 contain the major features of the observed winds (Forbes et al. 1987). The dimensional expression for the mountain-parallel jet in the cold dome is

$$e v_1 U = v_1(U f H \Delta h)^{1/2} \kappa_0^{-1} \leq \max[\psi(\xi)](U f H \Delta h_m)^{1/2} \kappa_0^{-1}, \quad (5.6)$$
Figure 12 shows the sea level pressure and force balance at the anemometer level (15 m above the ground $h_0$), and vertically-averaged force balance in the cold dome for the solution in Fig. 10. Important features include (1) the U-shaped sea level pressure pattern, (2) the dominant force balance between the pressure gradient and frictional force in the mountain-parallel direction, (3) the secondary force balance between the Coriolis force, pressure gradient, and frictional force in the mountain-normal direction, and (4) the nearly geostrophic balance between the vertically-averaged Coriolis force and pressure gradient in the mountain-normal direction. The sea level pressure contours and force balance at the anemometer level in Fig. 12 are comparable with the observations in Fig. 16c of Bell and Bosart (1988). The sea level pressure in Fig. 12 is computed by filling up the mountain volume (above the sea level) with the cold air (extended cold dome), so the resulting pressure ridge is along the mountain ridge. If we assume that the surface temperature and, thus, the temperature of the air that occupies the mountain volume increases gradually to the environmental surface temperature as the cold dome becomes shallow toward the mountain top, then the pressure ridge in Fig. 12 will shift slightly to the east of the mountain ridge and, thus, become better comparable with Fig. 16c of Bell and Bosart (1988).

We now consider the stability of the solution with respect to a uniform variation of interface slope. Assume that the cold air supply is abundant and that the interface slope decreases uniformly, or, the cold dome extends further upstream. In other words, $-\partial h_0$ de-

Figure 11 is the same as Fig. 10 except that the lower-layer flow is computed from (3.11a-c) with the constant coefficient of eddy viscosity $\mu$ and $\alpha = 0.35$ in (3.10). In this case, $\epsilon \Phi$ is replaced by $0.4 \epsilon / \mu$ in (4.3a). As explained earlier, since the variation of $-0.4 \epsilon / \mu$ with $\Delta h$ in Fig. 4 is similar to that of $-\epsilon \Phi$ in Fig. 3, the shape of the cold dome in Fig. 11 is very close to that in Fig. 10. This suggests that the overall shape of the cold dome is not very sensitive to the detailed distribution of the eddy viscosity $\nu$ provided that the total viscous effect remains approximately the same. However, the secondary transverse circulation in the cold dome, though extremely weak, is significantly affected by the distribution of $\nu$.\n
\[ [-U, V] = [-10, 0, 0] \text{ (ms}^{-1}) \]
FIG. 12. Sea level pressure (computed by assuming that the cold air occupies the mountain volume above the sea level), vertically-averaged force balance (thick arrows in the upper row) in the cold dome, and force balance (thin arrows in the lower row) at the anemometer level (15 m above the ground $H_{0}$) for the solution in Fig. 10. The pressure contours are drawn every 0.4 (dimensional 2 mb). The vertically-averaged Coriolis force, pressure gradient, and frictional force [see (5.2)] are labeled $C$, $P$, and $F$, respectively. The Coriolis force, pressure gradient, and frictional force at the anemometer level are labeled $C$, $P$, and $F$, respectively. The mountain-parallel (along $y$) components for $P$, $F$, $P$ and $F$ are all of unit length; i.e., $f U = 10^{-3}$ m s$^{-2}$. The mountain profile and cold dome are also shown for reference in the lower panel.

increases, but $h$ and $\Delta h$ increase (for fixed $x$). Consequently, the sum of the two coefficients of $\partial_{h} h$ in (4.3a) decreases, the PG in (iii) increases positively [see (2.5b)], the PG in (iv) increases negatively (see Fig. 3), and all these PG variations counteract the decrease of the interface slope and push the interface back to its original balanced position. Similarly, we can argue that an increase of the interface slope is also counteracted by its incurred PG variation. Therefore, the low Froude number flow is stable with respect to a uniform variation of the interface slope. Similarly, we can argue that the high Froude number flow ($Fr^{2} > h_{2}$) is unstable.

6. Summary and conclusions

It is proposed and also verified theoretically that the surface friction and PBL eddy viscosity are crucial in supporting a persistent cold air damming. The mountain-parallel jet in the cold dome is maintained by the force balance between the mountain-parallel geostrophic pressure gradient and friction; so the jet is stronger as the surface roughness length is smaller, the mountain is higher, or the cross-mountain geostrophic wind is stronger. However, in the mountain-normal direction the vertically-averaged force balance is nearly geostrophic and controls the geometric shape of the cold dome. Specifically, the negative buoyancy $g_{0}$ generates a mountain-normal pressure gradient (PG) on the sloping interface that tends to extend the cold dome further upstream. But this tendency is counteracted by the mountain-normal Coriolis force in the cold dome and by the upper-layer PG exerted on the interface. The mountain-normal Coriolis force is induced by the mountain-parallel jet, while the jet, as mentioned above, is controlled by the surface friction and PBL eddy viscosity. The upper-layer PG consists of two components: (1) the nongeostrophic (Bernoulli) PG generated by the kinetic energy variation in the cross-mountain flow and (2) the geostrophic PG induced by the Coriolis force associated with the mountain-parallel flow. Detailed analyses of the above force (or PG) balance provide a physical interpretation of the solutions and their responses to different external parameter settings.

There are six nondimensional parameters and two factors [see (5.4a–b)] that control the solution in the nondimensional space $(x, z)$ [scaled by fixed $(L, H)$]. If $x$ is scaled by the radius of inertial oscillation $L_{0} = U/f$, or Rossby radius of deformation $L_{R} = (g_{0}H)^{1/2} / f$, or modified Rossby radius of deformation $L_{m} = g_{0}H / (f U)$, then there will be only five independent nondimensional external parameters. Among these parameters and factors, the Froude number Fr, Rossby number, and nondimensional upstream mountain-parallel geostrophic wind $v_{\infty}$ (scaled by the upstream mountain-normal geostrophic wind) exert a dominant control on the nondimensional flow structure and interface slope. The effects of the upper layer depth (for $h_{2} > 2$), surface roughness (in the range of $10^{-4} \leq z_{0}/H \leq 2 \times 10^{-3}$), and inertial aspect ratio (in the range of $0.5 \times 10^{-2} \leq a_{0} \leq 4 \times 10^{-2}$) are relatively weak. The cold dome will shrink if the cold air supply is limited. This situation may occur where the mountain is not exactly two-dimensional and the cold air passes the mountain laterally.

In the dimensional space the structures of the solutions depend on seven external parameters in (5.5), in addition to the mountain profile and abundance of cold air supply. It is found that the cold dome shrinks as the cold dome becomes less cold, the surface becomes smoother, the Coriolis parameter becomes larger, the upstream total inflow speed increases or as the upstream inflow angle veers toward 45°–60° range (southeasterly wind with respect to a longitudinal mountain); and vice versa. In general, the depth of the cold dome is proportional to the mountain height, but the cold dome will be shallower than the mountain if the mountain is too high [i.e., $h_{2}$ is too small to ensure (4.5)]. Furthermore, it is shown in the Appendix that the effect of upper-layer stratification ($N_{1}^{2} < 10^{-4}$ s$^{-2}$)
on the dynamics of cold air damming is small. Since the density stratification is mostly concentrated in the inversion layer, it is also very likely that the pressure gradient generated by mountain gravity waves aloft may play only a minor role in the dynamics of cold air damming. This speculation needs to be tested against observations and more realistic model simulations.

The existence of a steady state solution requires the upstream geostrophic wind to be southeasterly or, at least, easterly \( v_\infty > 0 \) as in (4.8) or (5.3)], which means that the mountain-normal geostrophic PG should push the cold dome towards the mountain, or, at least, not pull the cold dome away from the mountain. This condition results from the free-slip interface assumption, so it may be considered only as a qualitative measure of the constructive/destructive effect of positive/negative upstream mountain-parallel geostrophic wind \( v_\infty \) on the persistence of cold air damming. The quantitative aspect of this condition does not agree with observations because persistent cold air damming events take place not only in the cases of southeasterly cross-mountain flow (Fig. 4 of Dunn 1987) but also in the cases of northeasterly cross-mountain flow (Fig. 13 of Forbes et al. 1987). For a real atmospheric cold air damming, the internal friction and/or nonhydrostatic force exerted from the upper layer onto the cold dome could be strong enough, especially at the front nose, to counteract the destructive tendency (of pulling the cold dome away from the mountain) caused by \( v_\infty < 0 \). This effect, if incorporated into our model, may bring the lower bound in (4.8) and (5.3) down to a negative value (corresponding to a northeasterly inflow). This conjecture is currently under investigation.

A stability analysis of the force (or PG) balance along the interface indicates that the low Froude number solutions \( (Fr^2 < h_0) \) are stable with respect to uniform variations of interface slope caused, presumably, by perturbations with wavelength longer than the width of the cold dome. On the other hand, as the short-wave (Kelvin–Helmholtz) instability is likely to develop along the interface, the lower-layer eddy viscous parameterization may partially account for the mixing effect of turbulent eddies developed from breaking unstable short waves. This speculation, however, is not examined in this paper. In view of the fact that mesoscale cyclogenesis are often observed along the sloping inversion layer (coastal front) during Appalachian cold air damming, it seems more interesting to examine the stability or instability of the sloping interface with respect to wave perturbations propagating in the along-front direction. This problem deserves a further investigation.

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**APPENDIX**

**The Effect of Upper-Layer Stratification**

As estimated in section 2, the upper-layer moist stratification is in the range of \( 0 \leq N^2_r < 0.3 \times 10^{-4} \) \( s^{-2} \), so the upper-layer Froude number is \( Fr_r = U/ (HN_r) > 1.7 \) if \( H = 1 \) km and \( U = 10 \) m s\(^{-1} \). In this case, we can show that (2.5a–b) remain valid approximated solutions and the upper-layer temperature perturbation is

\[
\theta = -\delta/ Fr_r,
\]

where \( \delta = h(1 - \psi/h_0) \) is the vertical displacement of a streamline and \( \psi \) is the streamfunction. This buoy-

![Fig. A1. As Fig. 5, but for different settings of \( 1/ Fr_r = 1.0, 0.3, 0.0 \) with \( Fr = 0.5 \).](image)
ancy perturbation gives a hydrostatic pressure term $-\frac{\Delta \sigma^2}{2F_r^2}$ on the right-hand side of the Bernoulli Eq. (2.5c). Note that $\delta = h$ along the interface, so the upper-layer hydrostatic PG term is $(v) = \frac{-\Delta \sigma}{F_r^2} h \frac{\partial}{\partial x} h$ and should be added to (4.3) and (5.1). This PG counteracts the lower-layer PG generated by the negative buoyancy in the cold dome [i.e., the term (i) in (5.1)], so the interface slope increases with $1/F_r^2$. Physically, a smaller $F_r$ (or larger $N_r^2$) means stronger cooling for the (moist) adiabatic lifting along the interface and larger discount of the negative buoyancy in the cold dome and, thus, steeper interface slope. However, quantitatively as shown in Fig. A1, the increase of interface slope is very small even as $1/F_r$ increases to 1.0 (or $N_r^2$ increases to $10^{-4}$ s$^{-2}$), so the effect of upper-layer stratification can be neglected.

REFERENCES


