NOTES AND CORRESPONDENCE

Kelvin-Helmholtz Instability in Severe Downslope Wind Flow

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ABSTRACT

To test the idea that observed oscillations in severe downslope winds are due to Kelvin–Helmholtz instability, the eigenvalues for the Taylor–Goldstein equation are found for a family of shear flows arising from local hydraulic theory. These two theories, local hydraulic theory and linear Kelvin–Helmholtz theory, provide a reasonable prediction of the period and speed of movement of the wind oscillations but underestimate their growth. The rule-of-thumb critical Richardson number of 0.25 agrees better than the linear theory value found here, 0.1, possibly indicating a nonlinear subcritical instability.

1. Introduction

Both the Boulder downslope windstorms and the Yugoslavian Bora exhibit quasi-periodic oscillations in strength that are not easily ascribed to boundary layer eddies. A standard explanation for such unsteadiness in orographic flow is the occurrence of Kelvin–Helmholtz (K–H) instability. (Long 1953, 1955; Lilly 1975; Mahrt 1987). In setting forth the local hydraulic theory (LHT) of downslope windstorms, the author predicted the point of onset of K–H instability in an idealized case (Smith 1985, henceforth S85). This prediction has gone untested until now as no idealized numerical or laboratory experiments have captured the oscillation phenomenon. In a recent paper, Scinnoccio and Peltier (1989, henceforth SP89) present a numerical simulation of an idealized severe downslope wind state that exhibits the oscillation and thus provides an opportunity to check the prediction of local hydraulic theory.

A second objective of this note is to clarify the nature of Kelvin–Helmholtz instability in the severe wind environment. In particular we wish to know whether K–H instability is linear or nonlinear (i.e., subcritical) in the severe wind environment.

2. Local hydraulic theory

The local hydraulic theory of mountain airflow is based on the idea that wavebreaking over the mountain will create a large region of mixed slow moving fluid aloft. This “dead” region of fluid decouples the lower flow from conditions aloft and allows it to take on a hydraulic character, even though the incoming flow does not contain evident layering. Layering had previously been thought necessary for hydraulic behavior. To carry out the theoretical calculation, one need only suppose that the dead region exists; its size, position, and other aspects of the flow can then be easily computed. The predicted flow corresponding to the SP89 simulation with constant N and U is shown in Fig. 1.

The parameters taken from SP89 and used to construct Fig. 1 are

\[ U_o = 5 \text{ m s}^{-1} \]  
\[ N_o = 0.01 \text{ s}^{-1} \]  
\[ h_m = 500 \text{ m} \]  
\[ a = 5 \text{ km} \]

and the hill shape is given by

\[ h(x) = \frac{h_m}{1 + (x/a)^2}. \]

These values give the nondimensional mountain height

\[ \hat{h}_m = \frac{Nh_m}{U_o} = 1, \]

and (using the notation in S85) the nondimensional dividing streamline height for the SP89 simulation must be

\[ \hat{H}_o = \frac{3\pi}{2}. \]

Once \( \hat{H}_o \) is known, the downward deflection of the dividing streamline \( \hat{\delta}_e \) is given implicitly by

\[ \hat{h} = \hat{\delta}_e \cos(\hat{H}_o + \hat{\delta}_e - \hat{h}) \]

and the wind speed is

\[ U(x, z) = U_o(1 + \hat{A} \sin \hat{z} - \hat{B} \cos \hat{z}) \]

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where
\[ \hat{A}(x) = \hat{\delta_c} \cos(\hat{H}_o + \hat{\delta}_c) \] (10)
and
\[ \hat{B}(x) = \hat{\delta_c} \sin(\hat{H}_o + \hat{\delta}_c). \] (11)
Values of \( \hat{\delta}_c, \hat{A}, \) and \( \hat{B} \) for the case \( \hat{H}_o = 3\pi/2 \) were given for this case in the appendix of S85, so the construction of Fig. 1 required no further computation. The LHT predictions can be checked by overlaying Fig. 1 on Fig. 11 of SP89. The predicted shape and location of the dead region are quite good, as expected from the earlier work of Durrnan and Klemp (1987), Bacmeister and Pierrehumbert (1988), and Rottman and Smith (1989), with the exception of the sharp leading edge. The theory also fails to capture the weak residual gravity wave above the dividing streamline.

The LHT also predicts the total drag according to
\[ D = \frac{\rho N^2}{6} (H_o - H_1)^3 \] (12)
where \( H_1 \) is the final height of the dividing streamline. For \( H_o = 3\pi/2 \), the nondimensional final height is \( \hat{H}_1 = \pi/2 \). Choosing an average density of \( \rho = 1.0 \, \text{kg m}^{-3} \), (12) gives
\[ D = 6.4 \times 10^4 \, \text{N m}^{-1}. \] (13)
The drag from SP89’s simulation is approximately
\[ D \sim 6 \times 10^4 \, \text{N m}^{-1}. \] (14)
It thus appears that LHT gives a satisfactory description of the SP89 simulation results.

3. The Richardson number

According to local hydraulic theory, the smallest value of Richardson’s number
\[ \text{Ri} = \frac{N^2}{U_z^2} \] (15)
is found at or just below the dividing streamline and has the value
\[ \text{Ri} = \hat{\delta}_c^{-2} \] (16)
This inverse relationship between displacement \( \hat{\delta}_c \) and \( \text{Ri} \) arises in the following way. In steady state Boussinesq flow, the streamlines and density isolines coincide. The tilt of the streamlines is a measure of how rapidly air is descending while the tilt of the density lines is proportional to the rate of baroclinic vorticity production. Thus the vorticity, in the form of shear \( (U_z) \), grows in proportion to \( \hat{\delta} \) as the air descends. The static stability \( N^2 \) is constant along the dividing streamline.

It follows from (16) that the critical value of \( \text{Ri} \) for Kelvin–Helmholtz instability
\[ \text{Ri} = \frac{1}{4} \] (17)
will be reached at the point on the descending dividing streamline where
\[ |\hat{\delta}_c| = 2. \] (18)
This point is marked with a large \( X \) on Fig. 1. Point \( X \) seems to lie just at the leading edge of the turbulent region found by SP89.

We conclude that if \( \text{Ri} = 1/4 \) is taken as the instability criterion, the local hydraulic theory does very well in predicting the onset of K–H waves. In the next section however, we find that \( \text{Ri} = 1/4 \) is not the critical Richardson number for this flow according to linear theory.

4. Linear Kelvin–Helmholtz instability theory

There is a danger in using a generic Richardson’s number criterion alone to predict Kelvin–Helmholtz instability. We would prefer to know the critical \( \text{Ri} \) for the particular flow in question and to know that the growth rates are large enough to explain the simulated disturbance amplitude. It is also useful to have some idea of the horizontal wavelength and vertical structure of the disturbance. This can be achieved using linear theory and the Taylor–Goldstein equation
\[ \frac{d^2 \hat{w}}{dz^2} + \left[ \frac{N^2(z)}{(U(z) - c)^2} - \frac{d^2 U/dz^2}{U(z) - c} - k^2 \right] \hat{w} = 0 \] (19)
where the vertical velocity in the disturbance is represented as
\[ \hat{w}(x, z) = \hat{w}(z) e^{ik(x-c_t)} \] (20)
(e.g., Hazel 1972). At each location across the ridge, the shear flow is assumed to be horizontally homogeneous. The formulation of a linear stability analysis of the severe wind based on (19) is simplified by having \( U(z) \) and \( N(z) \) expressed as simple functions. Terms \( U(z) \) and \( U_{zz} \) are available from (9), while \( N^2(z) \) is given by
\[ N^2 = N_o^2 \frac{U}{U_o}. \] (21)
These quantities are defined from \( z = h \) to \( z = H_o - \hat{\delta} \) at each \( x \) position. We define a new vertical coordinate \( y = z - h \), which runs from \( y = 0 \) to \( y = D = H_o + \hat{\delta} - h \). The natural boundary conditions for our problem are then
\[ \hat{w} = 0 \] (22)
at \( y = 0 \), and
\[ \hat{w}_y = -k \hat{w} \] (23)
at \( y = D \). Condition (23) arises from applying (19) in the mixed slow moving dead region, whereupon
\[ \hat{w}(y) \sim e^{-k(y-D)} \] (24)
for \( y > D \).
Due to the lack of symmetry in the family of shear layers (9), (21) and the boundary conditions (22), (23) the rule of “exchange of stabilities” used by Hazel (1972) does not apply. We proceed numerically using a shooting method, guided primarily by the semicircle theorem (Miles 1961; Howard 1961). For each value of $\tilde{h}$ (or $\text{Re}$) and $k$, and for a variety of $c$ values, (19) is integrated upwards with (22) and the artificial condition $\tilde{w}_x = 1$ at $y = 0$. The magnitude of the complex error in (23) is contoured on the upper half of the complex $c$-plane. In this way, the eigenvalues of the problem can be reliably found wherever they appear.

The eigenvalues are of two types: left and right moving gravity waves ($c_l = 0, c_r < U_{\text{min}}$, or $c_r > U_{\text{max}}$) and growing instabilities ($c_l \neq 0, U_{\text{min}} < c_r < U_{\text{max}}$). The gravity waves are not of interest here. Their phase speeds can be useful, however, in interpreting the sub- or supercritical nature of the flow (Durran and Klemp 1987).

Eigenvalues corresponding to unstable modes do not emerge from the wave eigenvalues but rather appear spontaneously along the real axis of the $c$-plane when $\tilde{h} \approx 0.0$ (i.e., $\tilde{\delta} \approx 0.0$ (i.e., $\tilde{\delta} \approx -\pi$, $\text{Re} \approx 0.1 \pm 0.002$), and $kD \approx 1.8$. For smaller values of $\text{Re}$ (i.e., $\tilde{h} < 0$) the range of $k$ for which disturbances can grow increases rapidly as shown in Fig. 2. Data on the fastest mode is found below.

Two aspects of these results deserve mention. First, the critical Richardson number ($\text{Re}_{\text{crit}} = 0.1$) is much smaller than the $\text{Re} = 1/4$ “rule-of-thumb” value, but not inconsistent with earlier results. Hazel (1972), for example, found even smaller values of $\text{Re}_{\text{crit}}$ when a solid boundary was brought close to a “tanh” shear layer. The presence of the solid boundary (22) may be a reasonable explanation for the low $\text{Re}_{\text{crit}}$ found here as well but we have not attempted to prove this. Other relevant factors include the particular shear and static stability profiles and the free reflective condition at the top of the layer (23).

A second point of interest is the onset of instability near $\tilde{h} = 0$; that is, when the terrain height has just returned to zero. This marginally stable stratified shear flow is the conjugate of the upstream uniform flow (i.e., the steady flow with the same mass flux and Bernoulli constant on each streamline). It is not clear why the conjugate state of uniform stratified flow should correspond to the linear theory stability boundary. It may be a numerical accident.

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**FIG. 1.** The flow field predicted by local hydraulic theory for the case discussed by Scinnocca and Peltier (1989). Solid lines are isolasts. Locations where the Richardson number decreases to $1/4$ and $1/\pi^2$ are indicated. Both the dimensional and nondimensional height scales are given. Vertical exaggeration is 18:1.

**FIG. 2.** The regime diagram for K–H instability of the stratified shear flows predicted by local hydraulic theory. $\text{Re}_{\text{min}}$ is the minimum Richardson number occurring in the layer; $kD$ is the nondimensional horizontal wavenumber. The stability boundary (solid) and the wavenumber of the most rapidly growing mode (dashed) are shown. On the right, a vertical scale indicates the nondimensional terrain height corresponding to each shear flow.
5. Discussion

The properties of the fastest growing mode at several lee-side positions are given in Table 1. All quantities in Table 1 are nondimensional and are accurate to about 5 percent. The first five rows describe the stratified shear flow whose stability is being examined. Further information of this type is given in S. The second five rows describe the most rapidly growing mode, according to linear theory.

As an example, let us consider a flow with $U_o = 5$ m/s and $N_o = 0.01$ sec$^{-1}$. Critical hydraulic conditions are achieved when the mountain height $h_m = (U_o/N_o)$ (0.98) = 490 m, and the critical streamline will be located at an altitude of $z = (U_o/N_o)(3\pi/2)$ = 2355 ms.

Now suppose we are at a position downstream of the mountain where the terrain has dropped to $z = (U_o/N_o)(-0.1) = -50$ m. At this location the dividing streamline will have descended $(U_o/N_o)(-3.27) = 1635$ m, and layer thickness is $D = (0.326)(2356) = 770$ m. The maximum speed $(4.26)(5) = 21.3$ m s$^{-1}$ occurs at the bottom of the layer and the minimum wind speed, 5 m s$^{-1}$, and the minimum Richardson number, $R_i_{\text{min}} = 0.0094$, occur at the top. This shear flow is linearly unstable.

The most rapidly growing wave has a wavelength $\lambda = (2\pi/1.69)(770) = 2855$ m. This growing wave moves downstream at a speed $C_o = (2.09)(5) = 10.4$ m s$^{-1}$. The period of time between wind pulses from these K–H waves will be $T = \lambda/C_o = 273$ sec = 4.5 min. The e-fold growth time of the wave $\tau = (kC_o)^{-1} = (1.69)^{-1}(0.020)^{-1}(768/5) = 4544$ sec = 75 min. The group velocity of the waves is directed downstream $C_g = (5)(1.52) = 7.6$ m s$^{-1}$. The Richardson number at the critical level for the waves [i.e., where $C_r = U(z)$] is $R_i(z = z_{\text{cr}}) = 0.218$. This is less than 0.25, as required.

In their numerical simulation of this case, SP89 found a phase speed of about 6 m s$^{-1}$ and a range of periods from 5 to 20 min, giving a range of wavelengths from 2 to 7 km. We conclude that the LHT and linear K–H theory provide reasonable estimates of these quantities. The growth rate and the onset of instability on the other hand are clearly underpredicted by LHT and linear K–H theory. From the e-fold time we can estimate the distance a wavepacket would move while experiencing significant growth; $L_e = C_o\tau = (7.6) (4544) = 34534$ m = 35 km. Thus, the Kelvin–Helmholtz waves would only be found well downstream of the ridge.

The vertical structure of a K–H disturbance is shown in Fig. 3 for the case $R_i_{\text{min}} = 0.066$; the last case in Table 1. The total horizontal wind speed is computed from

$$u(x, z) = U(z) + F \cdot \text{Re} \left( \frac{d\psi}{dz} e^{ikx} \right)$$

where $F$ is an arbitrary factor chosen so that the winds at the lower boundary vary by about a factor of two about the mean speed of 24 m s$^{-1}$. The pattern shifts downstream at 12 m s$^{-1}$.

Several features are evident in Fig. 3. The rightward phase shift with height allows disturbance energy to be drawn from the negative shear in the mean flow. Because of the low growth rate, large spatial gradients are found near the critical level. Regions with reversed flow exist above the critical level. With the chosen disturbance amplitude, the dynamics would be highly nonlinear in these regions.

6. Conclusions

In some ways, the qualitative predictions of local hydraulic theory (LHT) and linear K–H theory are quite good. The wavelength of the disturbance does seem to be controlled by the depth of the accelerating layer as these theories suggest. The speed of the drifting disturbance is much less than the low-level wind speed indicating that the disturbance is rooted in the slower winds aloft rather than being advected by the low-level wind. The drift speed lies between the predicted flow speed at the top and bottom of the layer indicating that the disturbances are not propagating gravity waves but rather growing K–H waves satisfying the semicircle theorem.

Regarding the onset and growth rate of the disturbance, we are left with a confused picture. If we accept $R_i_{\text{cr}} = 1/4$, as did S85, the prediction of K–H onset in severe wind flow by LHT is remarkably good. Kel-

| Table 1. Properties of the fastest growing mode. |
|-------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $\hat{h}$         | 0.0         | 0.98        | 0.2         | 0.0         | -0.1        | -0.2        | -0.3        | -0.4        | -0.5        |
| $\delta$          | 0           | -1.26       | -2.87       | -3.01       | -3.14       | -3.27       | -3.40       | -3.53       | -3.65       |
| $D/H_o$           | 1           | 5.23        | 3.47        | 3.38        | 3.32        | 4.26        | 4.39        | 4.52        | 4.63        |
| $U_o/U'_o$        | 1           | 1.79        | 3.86        | 4.01        | 4.14        | 4.26        | 4.39        | 4.52        | 4.63        |
| $R_i_{\text{min}} = \delta$ | $\infty$ | 0.630       | 0.120       | 0.110       | 0.100       | 0.904       | 0.887       | 0.809       | 0.705       |
| $kD$              | ---         | ---         | ---         | 1.77        | 1.69        | 1.60        | 1.48        | 1.46        | 1.43        |
| $C_r/U_o$         | ---         | ---         | ---         | 2.01        | 2.09        | 2.16        | 2.26        | 2.30        | 2.37        |
| $10^3 c_d/U_o$    | ---         | ---         | ---         | 4.4         | 20.         | 35.         | 45.         | 65.         | 75.         |
| $C_p/U_o$         | ---         | ---         | ---         | 1.47        | 1.52        | 1.56        | 1.56        | 1.6         | 1.63        |
| $R_i(z = z_{\text{cr}})$ | --- | ---         | ---         | 2.27        | 0.218       | 0.211       | 0.207       | 0.197       | 0.191       |
| $X$ in Fig. 1     | -45         | 0           | 10          | 15          | 80          | ---         | ---         | ---         | ---         |
vin-Helmholtz instability begins in the region of strong shear just under the dead region when the dividing streamline has descended an amount \( \delta = 2 \). The agreement with SP89 is excellent.

On the other hand, if the linear theory value of \( \text{Ri}_{\text{crit}} \sim 0.1 \) is taken, K-H instability would not begin at all, at least not in flows in which the lowest streamline returns to its original level. This does not agree with the SP89 simulation.\(^1\)

There are two situations for which LHT does predict \( \text{Ri} < 0.1 \). When the downstream terrain is significantly lower than upstream, \( \dot{h} \) is negative and smaller \( \text{Ri} \) can be obtained. Equivalently, if the nondimensional hill height exceeds unity, \( \tilde{h}_m > 1 \), the upstream low level flow will be stagnant and the effective lowest streamline will drop to negative heights downstream. Neither of these conditions were present in the SP89 simulations.

There are three possible solutions to our dilemma: (i) K-H in severe wind flow may be a non-linear subcritical instability (ii) our upper boundary condition and the assumption of parallel shear may cause the linear theory to underpredict the strength of the instability, or (iii) the idealized Long's model shear flow, (9) and (21), may not represent the environment of the incipient instability to a sufficient accuracy. The second possibility allows that a more complete formulation of the linear instability problem (e.g., Laprise and Peltier 1989) might discover a stronger instability. The latter possibility cannot be ruled out, even though the agreement between Fig. 1 and the corresponding plot in SP89 is very good.

The first possibility is supported by previous work on the stability of stratified shear flow. According to Maslowe (1977), "surprisingly, . . . subcritical instability can occur, i.e., modes that would be stable on a linear basis become unstable when the initial perturbation amplitude is greater than some critical value." Certainly in the real atmosphere and in the SP89 simulations, finite initial perturbations are present, due for example to the turbulence in the overlying mixed region. Unfortunately, our knowledge of nonlinear K-H instabilities is still quite limited (Thorpe 1987; Blumen 1990).

One final bit of evidence might be mentioned in support of possibility (i). In (unpublished) laboratory simulations of severe wind flow using hot and cold water, the author has seen well formed K-H waves over the lee-side of the model hill. The K-H waves are sporadic and when they cease, they can be restarted by introducing a tiny disturbance to the shear layer over the hilltop. Such sporadic behavior is usually interpreted as a sign of a subcritical instability. The SP89 simulations apparently do not exhibit such behavior.

Our inconclusive result regarding disturbance onset and growth does not necessarily cast doubt on the idea that K-H instability is responsible for oscillating downwseam winds. The disturbance period, wavelength, and speed predicted by LHT and linear K-H theory are reasonable, but a more sophisticated approach might be required to provide a quantitative model of the amplitude of the phenomenon.

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\(^1\) Note added in proof: In a recent paper, Peltier and Scinocca (J. Atmos. Sci., 1990, Vol. 47, pp. 2853-2870) have considered the stability of severe downslope winds using profiles computed from a numerical model. Their results generally agree with ours. Their wind profiles are stable or only weakly unstable according to linear theory unless the shear is artificially enhanced in some way.

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