The Energy Spectrum of Fronts: Time Evolution of Shocks in Burgers' Equation

JOHN P. BOYD

Department of Atmospheric, Oceanic & Space Science and Laboratory for Scientific Computation, University of Michigan, Ann Arbor, Michigan

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ABSTRACT

Andrews and Hoskins used semigeostrophic theory to argue that the energy spectrum of a front should decay like the −8/3 power of the wavenumber. They note, however, that their inviscid analysis is restricted to the very moment of breaking; that is, to the instant $t = t_B$ when the vorticity first becomes infinite. In this paper, Burgers' equation is used to investigate the postbreaking behavior of fronts. We find that for $t > t_B$, the front rapidly evolves to a jump discontinuity. Combining our analysis with the Eady wave/Burgers' study of Blumen, we find that the energy spectrum is more accurately approximated by the −8/3 power of the wavenumber, rather than by the k^{-2} energy spectrum of a discontinuity, for less than two hours after the time of breaking.

We also offer two corrections. Cai et al. improve a pseudospectral algorithm by fitting the spectrum of a jump discontinuity. This is not legitimate at $t = t_B$ because the front initially forms with a cube root singularity and its spectral coefficients decay at a different rate. Whitham claims that for $t > t_B$, the characteristic equation has two roots. We show by explicit solution that there are actually three.

1. Introduction

Based on Hoskins and Bretherton (1972), Blumen (1980) showed that one can model atmospheric frontogenesis by the one-dimensional advection equation

$$u_t + uu_x = 0. \quad (1.1)$$

Andrews and Hoskins (1978) studied the energy spectrum of fronts by invoking the method of Platzman (1964), who showed that (1.1) has the special solution:

$$u(x, t = 0) = -\sin(x) \Longrightarrow$$

$$u(x, t) = -2 \sum_{k=1}^{\infty} \frac{J_k(kt)}{kt} \sin(kx), \quad t \leqslant 1. \quad (1.2)$$

This solution breaks at the time $t_B = 1$ when

$$u(x, 1) \approx -(6x)^{1/3}, \quad |x| \ll 1. \quad (1.3)$$

This cube root singularity implies $u(x, t = 1)$ has an infinite slope at $x = 0$. Platzman showed that the cube root also implies that the Fourier sine coefficients b_k are asymptotically

$$b_k \sim -\frac{0.894615}{k^{4/3}}, \quad k \to \infty. \quad (1.4)$$

Since the spectral coefficients of the energy are proportional to b_k^2, it follows that the $k^{-4/3}$ decay of the Fourier coefficients implies an energy spectrum that is decaying as $k^{-8/3}$ (Andrews and Hoskins 1978).

This analysis has been quoted often. For example, Hoskins (1982, p. 149) states: "If mixing is weak enough so that sharp fronts form, an energy spectrum behaving like a −8/3 power of the wavenumber is predicted by semigeostrophic theory (Andrews and Hoskins 1978)." Blumen (1980) reviews Platzman's analysis while Blumen (1987) shows its relevance to finite amplitude flow over orography. Blumen (1990a) notes "a spectral decay law $n^{-8/3}$." Fu (1983) applies the Andrews–Hoskins (1978) prediction to ocean waves.

Although data scatter makes firm conclusions difficult, some numerical models clash with the −8/3 spectrum. Hollingsworth and Lonsberg (1983, p. 128) note: "The variations in the slopes of the spectra within the troposphere are of considerable interest. In the upper troposphere, they seem to follow a power law that could be between k^{-3} and k^{-4} while in the lower troposphere they are much flatter. A k^{-2} spectrum is a discontinuity spectrum and there may therefore be a suggestion that the wind forecast errors are mainly associated with the frontal structure, although Andrews and Hoskins (1978) suggest that the frontal structures may have a $k^{-8/3}$ spectrum." The uncertainty in this quote is a motivation for resolving the theoretical ambiguities with this present article.

There is less uncertainty in Gall et al. (1987, p. 2573): "We examined the energy spectra produced by the frontogenesis. Near the surface, this spectrum approaches a −2 slope, although it never reaches −2.

Corresponding author address: Dr. John Boyd, Dept. of Atmospheric and Oceanic Science, University of Michigan, 2455 Hayward Avenue, Ann Arbor, MI 48109.

Away from the surface, the slope of the spectrum is considerably greater than at the surface. The slope of the vertically integrated energy spectrum is more negative than the slope of the spectrum at the surface, but is still considerably greater than the $-8/3$ spectrum predicted by Andrews and Hoskins (1978) for the surface." (See their Fig. 11.)

Platzman (1964) was careful to point out that his solution was only valid up to and including the time of breaking, $t_B = 1$. Indeed, for $t > 1$, he shows that his Fourier series converges to a rather peculiar single-valued function with two cusps, which has no known relation to anything physical. Similarly, Andrews and Hoskins (1978) write: "for later times . . . the solution has no physical meaning." Blumen (1980) notes that for "$t > t_B$, . . . [the model (1.1)] is invalid because the solution is multivalued. Before this occurs viscosity and small-scale mixing would be expected to become important and to alter the frontogenetical processes."

Nevertheless, in spite of these qualifications that it is only valid at the moment of breaking, we have already seen that the $-8/3$ spectrum has been widely quoted. It is clearly important to fill the void left by these earlier analyses to ask: What happens to the energy spectrum for $t > t_B$?

To remove this restriction to $t \leqslant t_B$, we shall generalize the one-dimensional advection equation by interpreting it as the zero-viscosity limit of Burgers' equation:

$$u_t + uu_x - \nu u_{xx} = 0, \qquad (1.5)$$

where ν is the viscosity coefficient. There are two obvious limitations to our strategy. The first is that the zero viscosity limit is inappropriate if viscosity and mixing are very strong. However, the mere fact that sharp, well-defined fronts do form in the atmosphere suggests that this limitation is not serious except in the frontal zone where the infinite gradients of the inviscid model are smeared out over a zone of finite thickness. Using the method of matched asymptotic expansions, we shall relax the assumption of vanishing viscosity and discuss the frontal zone itself in section 6.

The second limitation is that Burgers' equation is much simpler than the three-dimensional atmosphere. However, Blumen (1980, 1990b) has derived Burgers' equation as a consistent model for frontogenesis in semigeostrophic Eady waves. The same equation has been derived as a model for frontogenesis in a wide variety of other hydrodynamic flows. Van Dyke (1964), Lesser and Chrighton (1975), and Kevorkian and Cole (1981) offer examples. It follows that we are in safe ground in assuming that atmospheric frontogenesis is very like Burgers' shocks. We cannot emphasize too strongly, however, that the one-dimensional model sloughs over the niceties of Ekman pumping, radiation, convection, etc., which are important in real fronts.

Nevertheless, understanding the postbreaking behavior of a simple model is a logical first step to understanding the postbreaking behavior of atmospheric fronts.

Figure 1 shows how the slope S of the Fourier coefficients (not energy) evolves with time for Platzman's sinusoidal initial condition and vanishing ν where the coefficient slope is defined by

$$S = \lim_{k \to \infty} \{\log|b_k|/\log(k)\} \text{ (Coefficient slope)}. \qquad (1.6)$$

As predicted by Platzman, $S = -4/3$, but only for the single instant for which he made this prediction: $t = 1$. For larger times, the slope rapidly grows and then asymptotes to the -1 slope of a jump discontinuity.

If we define the duration of the $-8/3$ law as the time interval $T_{-8/3}$ when $|S - (-4/3)| < 1/6$—this definition implies that after breaking, the slope is nearer to $-4/3$ than to the -1 slope of a jump discontinuity—then Fig. 1 shows that for Platzman's initially sinusoidal solution

$$\left|S - \left(-\frac{4}{3}\right)\right| \leqslant \frac{1}{6} \quad \text{if} \quad t \in [0.984, 1.030] \Rightarrow T_{-8/3} \equiv 0.046. \qquad (1.7)$$

Thus, the $-8/3$ energy spectrum ($-4/3$ coefficient spectrum) fails as even a crude approximation except for a fleeting interval that is less than 5% of the time required for the sine wave to break.

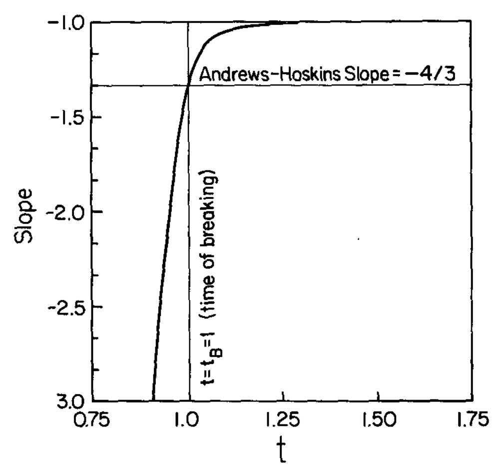

FIG. 1. Slope of the Fourier coefficients; that is, the exponent r where the Fourier coefficients are asymptotically of the form $b_k \sim k^r$ for large k. The slope was estimated by numerically calculating the Fourier coefficients from the analytical solution presented in section 2 and taking the difference between b_{25} and b_{100}. The vertical dividing line marks the time of breaking while the horizontal line marks the slope predicted by Andrews and Hoskins. The fact that the two dividing lines intersect on the slope curve confirms Platzman's prediction. For larger times, however, the slope is significantly greater than predicted by Andrews and Hoskins (1978).

In the next section, the exact zero-viscosity Burgers' solution for a general initial condition is given as Theorem 1.

Section 3 performs a local analysis of the time-and-space neighborhood of the developing front to show that the initial front has a cube root singularity, and then quickly evolves to a step function discontinuity. We shall also illustrate the time evolution of $u(x, t)$ and the development of its jump discontinuity at $x = 0$. In this section, we also correct a (minor) error in Whitham (1974), who wrote that the characteristic equation has two roots after breaking. (There are really three roots.)

We also give a much more general topological argument, requiring only the mild assumptions of single-valuedness and continuity, which gives the same conclusion: at the instant of breaking, the wave has a cube root singularity at the point of infinite slope. This topological argument applies to many wave equations besides Burgers' equation.

Section 4 is a graphic illustration of $u(x, t)$ and the time evolution of the Fourier coefficients for all t, not just the interval described by the local analysis of section 3. We find that the front rapidly evolves away from the cube root singularity. This explains the viewpoint, typical among aerodynamicists, that shocks are jump discontinuities. Cai et al. (1989) is a good illustration of this "aero view" because their Fourier numerical method incorporates a special, discontinuity-fitting basis function. Even though the front has a different cube root singularity at the moment of breaking so that their Fourier/discontinuity series converges poorly at that instant, this algorithmic realization of the "aero view" gives fairly good results. The reason is that Cai et al.'s explicit assumption that the front is a discontinuity is valid for *most* of the life of the front.

Using Blumen's (1990b) analysis, we show that atmospheric fronts, too, are discontinuities over most of their life cycle. With Blumen's length and time scale, the interval $T_{-8/3}$ is only about 2 h long for a Rossby number of 0.2!

Andrews and Hoskins (1978) actually analyzed a slightly more complex model than (1.1), but their prediction for the asymptotic Fourier coefficients at the moment of breaking is identical to (1.4), even to the time of breaking. In section 5, we solve exactly a generalized Burgers' equation with linear instability to explain why the Andrews and Hoskins spectrum agrees with Platzman's.

In section 6, we apply the method of matched asymptotic expansions to generalize Burgers' equation to more general forms of dissipation. The damping smooths the shock so that the jump in $u(x, t)$ occurs over a narrow frontal zone instead of as a discontinuity. The *form* of the damping controls the structure of the frontal zone, but the solution outside this zone is *independent* of the form of the damping, and identical with that of Burgers' equation.

The final section is a summary. The local analysis of section 3 shows that at the instant of breaking, the singularity is *always* a cube root as in (1.3), implying that the energy spectrum always follows a $-8/3$ power law at $t = t_B$. Platzman's solution is not atypical in any way: the front that develops from the sine wave displays the generic behavior of all the zero-viscosity-limit fronts in Burgers' equation.

Nevertheless, the suggestion that "an approximate $k^{-8/3}$ law may hold for a substantial portion of the life of a front" (Andrews and Hoskins 1978) is not supported by Fig. 1 unless "substantial" is defined to mean "less than two hours" for baroclinic instability. The energy spectrum does not freeze at the values it assumes at the instant of breaking, but rather changes rapidly. The cube root singularity is but a transient phase as the flow evolves from smoothness to a step-function–like discontinuity.

2. The analytical solution of Burgers' equation

Burgers' equation was actually first derived by Bateman (1915) and then rediscovered several times thereafter. Benton and Platzman (1972) give a comprehensive review of both its history and its special solutions. Some of the analysis below is borrowed from Whitham (1974).

Theorem 1. Exact zero-viscosity solution to Burgers' equation

In the limit that $\nu \to 0$, the solution to Burgers' equation,

$$u_t + uu_x = \nu u_{xx} \quad u(x, 0) = Q(x) \qquad (2.1)$$

is given without error by

$$u(x, t) = Q(y[x, t]), \qquad (2.2)$$

where $y(x, t)$ is the solution of

$$x = y + Q(y)t \quad \text{(characteristic equation)}. \qquad (2.3)$$

The solution "breaks" (and develops an infinite first x derivative) when

$$t_B = -1/Q'(y_B) \quad \text{(time of breaking)}, \qquad (2.4)$$

where Q' is the first derivative of $Q(y)$ and where y_B is the argument where the derivative of $Q(y)$ achieves its largest negative value.

For $t > t_B$, however, the characteristic equation will have three or more roots for some range of x. At such points, (2.2) still solves Burgers' equation provided that $y(x, t)$ is that root of the characteristic equation that gives the smallest value of the function

$$G(y, t) \equiv \int_0^y Q(w)dw + Q^2(y)t/2. \qquad (2.5)$$

The proof is given in Whitham (1974).

As noted in the abstract and proved both through Fig. 2 and an explicit, analytical approximation in section 3, the characteristic equation has *three* roots for some x when $t > t_B$. Whitham (1974) asserts there are only two. However, because the contribution from the missing root is exponentially small in the viscosity coefficient, relative to that of the dominant root, his slip is *irrelevant* to the zero-viscosity solution. (The third root *does* matter, albeit only as a tiny correction, when the viscosity is small but nonzero.)

Nevertheless, the cube root singularity, which is the cause of the $-8/3$ energy spectrum at the instant of breaking, clearly requires three roots. We cannot form a consistent concept of frontogenesis while enslaved to an error.

We have labeled (2.3) the "characteristic equation" because this same relation arises in solving the one-dimensional advection equation (1.1) by the method of characteristics. This is no surprise: in the limit of zero viscosity, the solutions of the advection equation (1.1) and Burgers' equation (1.5) are indistinguishable until the time of breaking. What is remarkable is that Theorem 1 shows that the inviscid solution still applies even *after* frontogenesis. The difference is that for $t > t_B$, only one of the multiple roots of the characteristic equation is relevant to Burgers' equation. We must apply the test of minimizing the function defined by (2.5) at each (x, t) pair where the characteristic equation has multiple real roots. [For the advection equation (1.1), the three roots of the characteristic equation all define branches of a genuinely triple-valued solution.]

This mathematical test for which root to insert into $Q[y(x, t)]$ can be replaced by an equivalent criterion based on physical reasoning. Suppose, purely for the sake of simplicity, that the front is at $x_B = 0$. This is true of Platzman's solution and can be made true for arbitrary fronts merely by shifting the coordinate. Similarly, suppose that $u < 0$ when $x > 0$. Then when $x > 0$, two of the three stationary points will be negative and one will be positive.

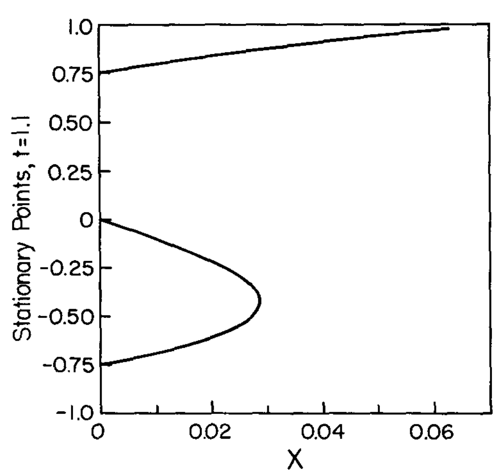

FIG. 2. The stationary points of the solution to Burgers' equation for $t = 1.1$ and $x \geqslant 0$. The initial condition is $u(x, 0) = -\sin(x)$. Because the solution is antisymmetric with respect to the origin for all t, it suffices to show the stationary points only for positive x.

Whitham (1974) and Boyd (1980) gives the physical interpretation of $y(x, t)$ as the "characteristic": the fluid particle that was at $x = y$ at $t = 0$ has been advected to coordinate x at time t. Thus, $y(x, t)$ gives the original location of the fluid particle (at time $t = 0$) which is now at the point (x, t) in the time–space plane.

It follows that the two negative roots of the characteristic equation correspond to the upper and lower surfaces of a tongue of fluid that was advected from $x < 0$ into the front where the tongue was smoothed by viscosity in the frontal zone. The lone positive root of the characteristic equation for $x > 0$ corresponds to fluid that began to the right of the front, has been advected leftwards, but has still not reached the front. Consequently, this fluid (in the limit of zero viscosity) still conserves its initial amplitude. This root is the proper root to insert into (2.15). In summary, the correct choice of characteristic is defined by

$$\operatorname{sgn}(y[x, t]) = \operatorname{sgn}(x) \quad x_B = 0, \tag{2.6}$$

where $\operatorname{sgn}(x)$ denotes the sign of its argument. In words, only the upstream root is physical.

3. Local analysis of frontogenesis

The key to an approximate but illuminating local treatment is to expand $Q(y)$ in a power series about an arbitrary point $y = y_0$. Locally, (2.3) becomes

$$x - x_0 = \{1 + tQ'(y_0)\}(y - y_0) + t\frac{Q''(y_0)}{2}(y - y_0)^2 + t\frac{Q'''(y_0)}{6}(y - y_0)^3. \tag{3.1}$$

When $t < t_B$, the coefficient of the term that is linear in $(y - y_0)$ does not vanish for any y_0. When $t = t_B$, however, the linear term vanishes when $y_0 = y_B$, where y_B is that argument for which $Q'(y)$ achieves its largest negative value. This is the characteristic where the solution breaks; the point of breaking in physical space is

$$x_B \equiv y_B + Q(y_B)t. \tag{3.2}$$

To prove that the wave breaks at $x = x_B$, we solve the local model (3.1) for $y(x, t)$. First, note that because Q' has a minimum at $y_0 = y_B$, $Q''(y_B) = 0$ and $Q'''(y_B) > 0$. Therefore (3.1) simplifies to

$$x - x_B = t_B\frac{Q'''(y_B)}{6}(y - y_B)^3, \tag{3.3}$$

which has the single real solution

$$y(x, t_B) = y_B + \left(\frac{6(x - x_B)}{t_B Q'''(y_B)}\right)^{1/3}. \tag{3.4}$$

Equation (3.4) shows explicitly that a cube root singularity occurs at the instant of breaking. Lighthill (1958) showed that the Fourier coefficients of a function with a cube root singularity will always asymptotically decay as the $-4/3$ power of degree (unless the function also has an even stronger singularity, such as a discontinuity). Recalling that the slope of the energy spectrum is the square of that of the Fourier coefficients, as follows from substituting the Fourier series into the integral of $u(x, t)^2/2$ and using the orthogonality of the trigonometric functions, we conclude that the $-8/3$ energy spectrum is *generic.*

At the instant of breaking, $u(x, t)$ is still single valued, but substitution of (3.4) into (1.1) and differentiation with respect to x shows that the $u_x(x, t)$ is *infinite* at $x = x_B$. When we refer to "the moment of breaking," we shall always mean that moment when the solution first develops an infinite spatial derivative.

For slightly larger times (and small $|x - x_B|$), the cubic polynomial (3.1) has the approximate roots

$$y - y_B = \begin{cases} \left(\dfrac{-6(t - t_B)Q'(y_B)}{tQ'''(y_B)}\right)^{1/2} \\ \dfrac{x - x_B}{(t - t_B)Q'(y_B)} \\ -\left(\dfrac{-6(t - t_B)Q'(y_B)}{tQ'''(y_B)}\right)^{1/2} \end{cases} \quad (3.5)$$

for $|x - x_B| \ll 1$ and $0 < (t - t_B) \ll 1$.

Note that the condition that $Q'(y)$ has its largest maximum value at the breaking characteristic $y = y_B$ implies that its curvature, $Q'''(y_B)$, must be positive. This in turn implies that the argument of square root is always real valued for $t > t_B$.

For Platzman's solution where $u(x, 0) = -\sin(x)$, these formulas simplify to

$$t_B = 1, \quad x_B = y_B = 0,$$

$$y = \begin{cases} \sqrt{6(1 - 1/t)} \\ -\dfrac{x}{(t-1)} \quad 0 < (t - 1) \ll 1 \\ -\sqrt{6(1 - 1/t)} \end{cases}$$

$$u(x, t) \approx -\mathrm{sgn}(x) \sin(\sqrt{6(1 - 1/t)}),$$

$$|x| \ll 1 \quad \text{and} \quad 0 < (t - 1) \ll 1. \quad (3.6)$$

For larger times and points distant from the front, the Taylor series approximation behind (3.5) and (3.6) fails. It is easy, however, to compute the roots using Newton's iteration and continuation in x, starting from these approximate values. Figure 2 shows that there are clearly three stationary points for a finite range of x.

Whitham (1974, p. 99) states: "The explanation is that when this stage [post-breaking] is reached there are two stationary points that satisfy [the characteristic equation, (2.3)]." Although most of the analysis in section 2 is borrowed directly from Whitham's lucid and readable account, this one statement is erroneous: there are actually three roots after the time of breaking.

The generic nature of the cube root singularity at the instant of breaking can alternatively be established by topological arguments quite independent of the one-dimensional advection equation (1.1). Define "breaking" to mean that at $t = t_B$, $du/dx = \infty$ at $x = x_B$, consistent with the rest of this article. The inverse function $x(u)$ is not singular at the moment of breaking and can be approximation by a low-degree polynomial in u around the point of breaking. At $u = u_B$, $dx/du = 0$, so the term linear in $(u - u_B)$ is missing from the polynomial approximant to $x(u)$. However, a quadratic local approximation, that is,

$$x(u) - x_B \approx q(u - u_B)^2 + O([u - u_B]^3),$$
$$|u - u_B| \ll 1 \quad (3.7)$$

where q is a constant, clearly implies that $u(x)$ is *double valued,* as shown in Fig. 3. Thus, if the nonlinear wave

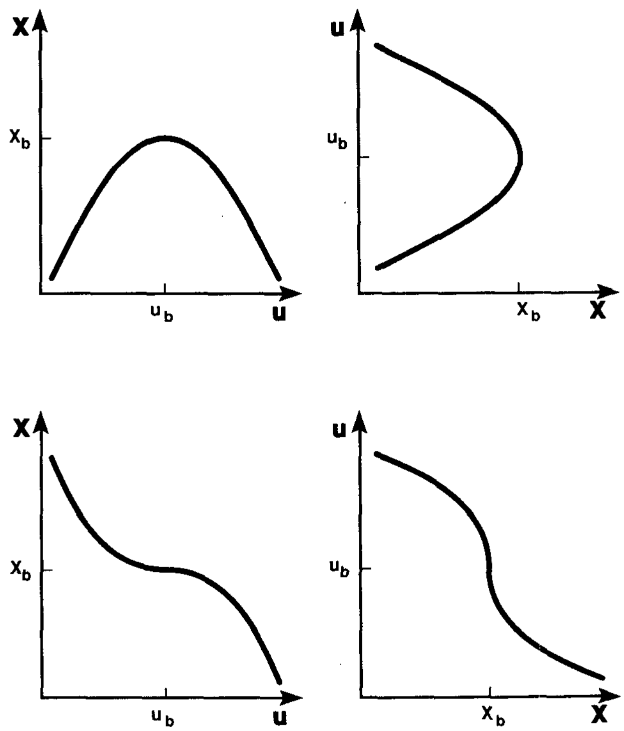

FIG. 3. (a) Illustration of $x(u)$ (left) and $u(x)$ (right) in the neighborhood of a "square root" shock; that is, a point of infinite du/dx where $x(u)$ may be locally approximated by a quadratic polynomial as in (3.6). The function $u(x)$ is double valued. (b) Same as (a) but for a "cube root" shock: the function $x(u)$ is locally approximated by a cubic polynomial in u (with linear and quadratic terms missing), implying that $u(x)$ has a cube root singularity at the point of breaking, that is, at the point where $du/dx = \infty$.

is to be continuous and single valued up to and including the instant of breaking, then the quadratic coefficient q in (3.7) must be zero so that the lowest nontrivial approximation is the cubic

$$x(u) - x_B \approx \chi(u - u_B)^3 + O([u - u_B]^4), \quad |u - u_B| \ll 1 \quad \text{at} \quad t = t_B \qquad (3.8)$$

where χ is a constant. Inverting (3.8) then shows that $u(x, t_B)$ has a cube root branch point at $x = x_B$.

This argument is a second, independent proof that the cubic singularity at the instant of breaking is generic behavior. This conclusion is not limited to Platzman's solution, which is special in that the initial condition is a sine wave, nor is it restricted to the one-dimensional advection equation (1.1) or to the zero viscosity limit of Burgers' equation (1.5). Continuity and single valuedness at $t = t_B$ are all we need to establish that the wave has a cube root branch point at the point of infinite slope. In turn, as shown in Boyd (1989) and Lighthill (1958), the existence of the cube root singularity generically implies that the Fourier coefficients of $u(x, t)$ must asymptotically decay proportional to $1/k^{4/3}$.

4. The time evolution of the Burgers' solution and its spectrum

The local analysis of section 3 is valid only for times very close to the time of breaking, but it is easy to see what happens for larger and smaller times by simply evaluating the analytical solution of section 2. Figure 4 shows $u(x, t)$ for various times for the particular case of a sinusoidal initial condition. For most x, the solution does not change rapidly with time. In the frontal zone, however, as shown in Fig. 4b, the flow evolves from a smooth, finite slope at $t = 0.9$ to an infinite slope (cube root singularity) at $t = t_B = 1$ to a step-function discontinuity for larger times.

The square roots in (3.5) and (3.6) imply that the magnitude of the jump grows as the inverse square root of $(t - t_B)$—an infinite rate of growth at the moment of breaking. Figure 4 confirms that the strength of the discontinuity grows very rapidly at first and then only gradually levels off. The jump reaches its maximum of 2 at $t = \pi/2$ when the crest and trough of the initial sine wave are finally advected into the front at $x = 0$.

Figure 5 shows how the Fourier coefficients change with time. Consistent with Fig. 1, the slope changes with time. The $-4/3$ slope of the solution at the time of breaking rapidly alters to the $1/k$ slope of the coefficients of a discontinuous function. As shown in Fig. 4b, the change in slope implies that the higher-degree Fourier coefficients change much more rapidly with time than b_1 and b_2 and the other low-degree coefficients, which change very little. All the postbreaking curves are asymptotically parallel to one another, differing only in magnitude. The magnitude of the high-degree Fourier coefficients becomes larger as t increases because the jump in $u(x, t)$ at $x = 0$ is increasing in strength.

We have already seen in the Introduction that in geophysics, most discussions of energy spectra are limited to the $-8/3$ spectrum, which applies at the moment of breaking; we may dub this the "geophysical" view. In complete contrast, aerodynamicists, who must deal with shock waves in the design of every transonic aircraft and every missile, take the opposite perspective. In the "aero" view, fronts are jump discontinuities and the different, cube root behavior at $t = t_B$ is ignored.

A typical illustration of the "aero" view is Cai et al. (1989). This article is a particularly good example be-

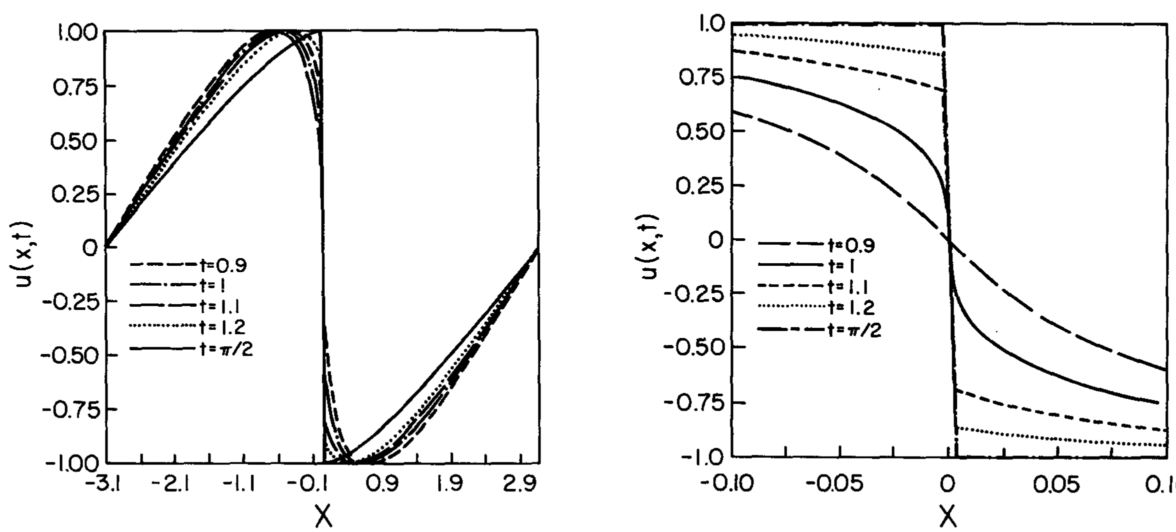

FIG. 4. The solution of Burgers' equations for the initial condition $u(x, 0) = -\sin(x)$ at the indicated times. The two panels are identical except that (b) is a "blow-up" of the region around the front.

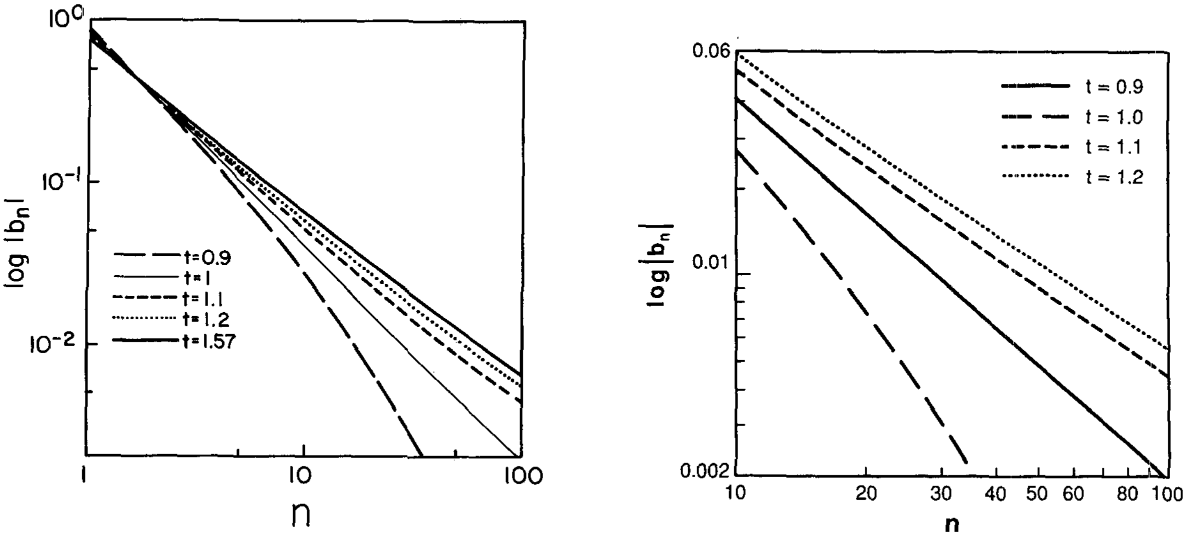

FIG. 5. The Fourier spectrum of the Burgers' solution illustrated in Fig. 4. The two panels are identical except that (b) shows a smaller range in wavenumber. The two solid curves denote the time of breaking ($t_B = 1$) and the time of maximum jump ($t = \pi/2$).

cause the shock-is-a-jump-discontinuity perspective is built explicitly into the numerical solution: Cai et al. (1989) fit the shock with a piecewise linear, discontinuous function of variable jump strength and location. The rationale is that this special "sawtooth" basis function will represent the discontinuity so that the Fourier sines are required to represent only the smooth remainder of $u(x, t)$.

Neither the "aero" nor "geophysical" views are necessarily erroneous. We have already seen that the meteorologists have noted that the $-8/3$ spectrum is valid only at the time of breaking; similarly, at least some aerodynamicists are aware that shocks form as cube root singularities. Figure 5 shows clearly, however, that these viewpoints are both incomplete. The front is both a cube root singularity ($-8/3$ spectrum) and a discontinuity (-2 spectrum) at different stages in its life cycle.

Surprisingly, Cai et al. (1989) obtain fairly good results in spite of the fact that at $t = t_B$, their shock fitting is based on a fiction and the spectrum of their basis function does not match that of the singularity it is supposed to represent. Figure 6, which illustrates the

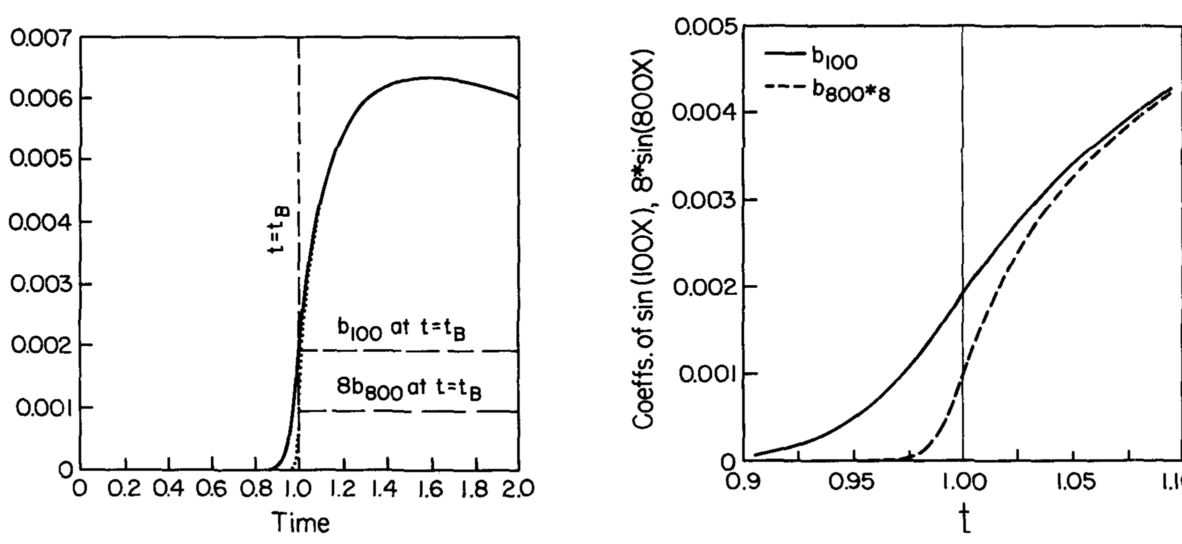

FIG. 6. Time evolution of $|b_{100}|$ (solid) and $|b_{800}|$ multiplied by a scaling factor of 8 (dashed). Because of the scaling factor, the two curves are identical for large time after the $1/k$ spectrum has fully developed. At the time of breaking, which is marked by the vertical dividing line, $|b_{100}|$ is double the rescaled value of $|b_{800}|$ because both coefficients are proportional to $1/k^{4/3}$ at this time. (a) and (b) differ only in the time interval shown.

time evolution of two Fourier coefficients, explains why. The magnitude of b_{800}, which should be eight times smaller than b_{100} if the Fourier coefficients are decreasing proportional to $1/k$, has been scaled on the graph by this expected factor of 8. The extent to which the two curves coincide is therefore a measure of the correctness of Cai et al's. assumption that the shock behaves like a step-function discontinuity. Clearly, their assumption is wrong for a neighborhood of $t = t_B$, as we showed analytically in the previous section. However, the graph also demonstrates that the width of the region of non–step-function-like behavior is very narrow. By $t = 1.1$, only 10% beyond the time of breaking, the two curves are almost indistinguishable. The transition from the $-4/3$ spectrum to the -1 spectrum happens very quickly after the moment of breaking.

Thus, Cai et al.'s (1989) step-function fitting procedure is valid over most of the life of the shock. One would expect, however, that the accuracy would be improved still further if the function fitting allowed the type of singularity to vary with time to match that of the solution.

We can put Fig. 6 in a meteorological perspective by noting that Blumen (1980, 1990b, 1990c) has shown that the growth and nonlinear evolution of unstable, initially sinusoidal Eady waves is described by Burgers' equation. His nondimensional time coordinate τ is equivalent to Platzman's time variable so that $\tau = 1$ is the time of breaking. For the length and time scales assumed by Blumen to model midlatitude baroclinic instability,

$$t = 1.87 \log(\tau) + 5.64 \text{ [days]}, \quad \text{Ro} = 0.2. \quad (4.1)$$

Substituting in $\tau = 0.984$ and $\tau = 1.030$, the limits of the interval in which the $-8/3$ energy spectrum law holds for the initially sinusoidal solution of Burgers' equation (Fig. 1), we find that

$$T_{-8/3} = 0.046 \quad \text{[nondimensional]} \Leftrightarrow 2.02 \text{ hours}. \quad (4.2)$$

This is only 1.5% of the time interval from initiation of the small-amplitude sinusoidal Eady wave until the wave develops an infinite vorticity at $\tau = 1$.

There is one technical detail we have glossed over in deriving (4.1) and (4.2): Eady waves amplify exponentially with time whereas the solutions of the zero viscosity limit of Burgers' equation are neutrally stable. Blumen (1980) showed, however, that it is easy to solve a generalization of the one-dimensional advection equation (1.1), which has growing solutions. In the next section, we give the exact solution for a generalization of Blumen's solution that includes viscosity, too.

5. Burgers' equation with antifriction: unstable waves with advection

As noted earlier, Andrews and Hoskins (1978) did not solve the one-dimensional advection equation, but rather a slightly more complex model that incorporates the exponential growth in time of baroclinic instability. It would take us too far afield to derive their model (but see their work and Blumen 1980). However, their solution is remarkably similar to Platzman's solution of the advection equation.

To understand this similarity, we solve a generalization of Burgers' equation whose solutions grow exponentially in time. The extra term that drives this growth has the mathematical form of a linear or "Rayleigh" friction, but opposite sign—an antifriction.

If we allow the strength of the viscosity to grow exponentially with time, too, then it is possible to transform the generalized Burgers' equation into the ordinary Burgers' equation as described by the following.

Theorem 2. Burgers' equation with antifriction

The solution of the generalized Burgers' equation

$$u_t + uu_x - \mu e^{\sigma t} u_{xx} = \sigma u \quad (\text{"BAF" equation}) \quad (5.1)$$

$$u(x, 0) = Q(x) \quad (5.2)$$

is given by

$$u(x, t) = e^{\sigma t} w(x, \tau(t)) \quad (5.3)$$

$$\tau = \frac{1}{\sigma}(e^{\sigma t} - 1), \quad (5.4)$$

where $w(x, \tau)$ is the solution of the ordinary Burgers' equation

$$w_\tau + ww_x - \mu w_{xx} = 0, \quad w(x, 0) = Q(x). \quad (5.5)$$

Proof: Substitution of (5.3) into (5.1) followed by a change of variable from t to τ gives (5.5).

The "antifriction" term σu causes the solution of the BAF equation to grow exponentially with time. However, this does not modify frontogenesis except through the change in time coordinate. For small t, $\tau \approx t$ and there is no change at all from the corresponding Burger's solution. For larger times, however, the growth in wave amplitude *accelerates* the rate of frontogenesis. Then $\tau \gg t$ so that the solution of the BAF equation at time t has the same shape as the solution of Burgers' equation at the much larger time τ. Figure 1 still applies provided we interpret the time axis as a *logarithmic* scale. For a given initial condition, we see the same sequence of shapes for both the BAF equation and Burgers' equation; only the amplitude and the rate at which we flash through the shapes are different.

An exponentially growing viscosity is artificial, but this is the only case for which we have been able to find an *exact* solution for Burgers' with instability other than the inviscid case, which was solved by Blumen (1980). In the next section, however, we apply the method of matched asymptotic expansions to analyze very general damping laws, both with and without an antifriction. For weak damping, perturbation theory predicts that whether the viscosity is constant in time or exponentially increasing in lockstep with the wave amplitude is irrelevant except outside the narrow frontal zone where the damping smooths the jump discontinuity of the shock into a thin region of high but finite gradient. In the limit of vanishing viscosity, the error in the matched asymptotics series and the width of the frontal zone both tend to zero.

Another limitation of our analysis and the earlier studies of Andrews and Hoskins (1978) is that frontogenesis is idealized as a one-dimensional process. However, to quote Andrews and Hoskins (1978): "So far, we have assumed that the fronts are independent of the long-front direction y. Smooth y variations may be heuristically accounted for by multiplying the foregoing solutions by a slowly varying function of y." Similarly, van Dyke (1964), Boyd (1980), and other works show that the one-dimensional advection equation may be systematically derived via singular perturbation theory. The reason this same equation pops up as a consistent model for shock waves, equatorial Kelvin waves, and fronts is that the steep gradients perpendicular to the front imply that variations in the other coordinates will necessarily be "slowly varying." Thus, the one-dimensionality of Burgers' equation is not as artificial as it might seem. Nevertheless, an open problem is to analyze the energy spectrum by analytically or numerically integrating more realistic three-dimensional systems of equations.

6. The method of matched asymptotic expansions and damping laws

It is easy to approximately solve Burgers' equation via the method of matched asymptotic expansions. In the limit of zero viscosity, the error vanishes and the solution becomes identical with that given by Theorem 1 above.

To avoid irrelevant complexity, we shall explain matched asymptotics using the simplest case: the ordinary Burgers' equation with an initial condition restricted to be antisymmetric about the origin, so that there is only a single shock which remains at $x = 0$ for all time. At the end of this section, we shall quote a much more general theorem that incorporates both arbitrary-order viscosity and also the antifriction of the preceding section.

Our problem, as in section 2, is

$$u_t + uu_x = \nu u_{xx} \quad u(x, 0) = Q(x). \qquad (6.1)$$

Small but finite viscosity will smooth the shock so that $u(x, t)$ does not have a true jump discontinuity, but instead varies very rapidly within a narrow frontal zone. Outside this frontal zone, the effects of viscosity are negligible and $u(x, t)$ approximately satisfies the one-dimensional advection equation. In the language of matched asymptotics, the advection equation solution is the "outer solution" solving

$$u_t^0 + u^0 u_x^0 = 0 \quad \text{(outer solution)}. \qquad (6.2)$$

After the shock has formed, the general solution to the one-dimensional advection is triple valued in a neighborhood surrounding the shock. We choose the branch given in Theorem 1: the unique branch composed of particles that have not yet been advected across the front. This branch has a jump discontinuity at the shock at $x = 0$.

The frontal zone is an internal boundary layer. Defining a stretched coordinate via

$$X \equiv \frac{x}{\nu} \quad \text{(inner coordinate)} \qquad (6.3)$$

and taking the limit $\nu \to 0$ with X fixed, the so-called "inner limit," we obtain

$$u^i u_X^i - u_{XX}^i = 0 \quad \text{(inner solution)}. \qquad (6.4)$$

This nonlinear ordinary differential equation has the general solution

$$u^i(X) = -J \tanh\left[\frac{1}{2} J(X + \phi)\right], \qquad (6.5)$$

where J and ϕ are arbitrary constants. The restriction to an antisymmetric solution implies that the phase constant $\phi = 0$.

The final step is to match the inner solution to the outer solution. The matching requirement is that the outer solution, rewritten in terms of the inner variable X and expanded for small ν, must match the outer limit of the inner solution; that is, the limit of the inner solution after first rewriting it in terms of the outer variable x. The result is

$$\lim_{\substack{\nu \to 0 \\ [X \text{ fixed}]}} u^0(\nu X, t) = -\text{sgn}(x) J = \lim_{\substack{\nu \to 0 \\ [x \text{ fixed}]}} u^i\left(\frac{x}{\nu}\right). \qquad (6.6)$$

In words, the constant J in the inner solution must vary with time so that it matches the magnitude of the jump in the outer, inviscid solution. Because we have assumed an antisymmetric initial condition, the jump across the shock is antisymmetric and a single function $J(t)$ is sufficient to match both to the left and to the right of the shock.

The matched asymptotics approximation is accurate for all $\nu \ll 1$. In the limit of vanishing viscosity, the approximation becomes exact and the analysis above is merely an alternative derivation of Theorem 1.

The matched asymptotics treatment of a Burgers' shock is so simple it would make an excellent textbook example and/or homework problem. However, the only prior treatment we have been able to find (in a somewhat cursory literature search after already completing the analysis above) is Kevorkian and Cole (1981), who restrict the outer solution to a constant. A reviewer (W. Blumen) noted a similar treatment in Lesser and Chrighton (1975) and his own then-unpublished work (Blumen 1990b,c).

By using the same methodology, we can extend the analysis to the following.

Theorem 3. Matched asymptotics analysis of a generalized Burgers' equation with variable-order viscosity and antifriction

The problem is

$$u_t + uu_x + (-1)^j \nu^{2j-1} u_{2jx} = \sigma u, \quad u(x, 0) = Q(x) \tag{6.7}$$

where the subscript "$2jx$" denotes the $(2j)$th derivative with respect to x and where $\nu > 0$. Introduce new independent and dependent variables via

$$u(x, t) \equiv e^{\sigma t} w(x, \tau) \tag{6.8}$$

$$\tau = \frac{1}{\sigma}(e^{\sigma t} - 1). \tag{6.9}$$

Then outside a narrow frontal layer of width $O(\nu)$, $w(x, \tau(t))$ solves

$$w_\tau + ww_x = 0, \; w(x, 0) = Q(x) \text{ (outer problem).} \tag{6.10}$$

The solution of the outer problem is given by Theorem 1.

The solution in the frontal zone is given by

$$w(x, \tau) \sim s + Jv(J^{1/(2j-1)}X), \quad |X| \sim O(1) \tag{6.11}$$

where $s(\tau)$ is the speed of the shock as given by the solution of the outer problem, that is,

$$s(\tau) = \frac{1}{2}\{w(0^-, \tau) + w(0^+, \tau)\}, \tag{6.12}$$

where $w(0^-, \tau)$ and $w(0^+, \tau)$ denote the values of the outer solution on either side of the shock, and

$$J(\tau) = \frac{1}{2}\{w(0^-, \tau) - w(0^+, \tau)\}, \tag{6.13}$$

where $v(X)$ is the unique solution of the parameter-free problem

$$(-1)^j v_{2jx} + vv_x = 0, \quad x \in [-\infty, \infty]$$

$$\lim_{|x| \to \infty} v(X) = \begin{cases} 1, & X < 0 \\ -1, & X > 0 \end{cases}, \tag{6.14}$$

where the inner coordinate is

$$X = \frac{x - \int^\tau s(\tau')d\tau' + \phi}{\nu \exp[-\sigma t/(2j-1)]}, \tag{6.15}$$

where ϕ is a phase factor determined to lowest order by the requirement that $X = 0$ at that value of x where the shock first forms in the outer solution.

In the limit that $\nu \to 0$, the width of the frontal zone is zero and the solution is *identical* to that of Theorem 1 (ordinary Burgers' equation) for all orders of dissipation j, provided that the antifriction coefficient $\sigma = 0$. When σ is nonzero, Theorem 1 still applies except that, as in section 2, we must solve the one-dimensional advection equation in the logarithmic time variable τ and then convert back to the original time variable via (6.9) and, last, multiply Burgers' solution by $\exp(\sigma t)$ to obtain $u(x, t)$ as demanded by (6.8).

To render the local approximations spatially uniform, that is, valid both inside and outside the shock zone, it suffices to apply the inner approximation with $J(\tau)$ allowed to vary slowly with x:

$$J(\tau) = \frac{1}{2}\{w(x, \tau) - w(0^+, \tau)\}, \; x < s(\tau)$$

$$J(\tau) = \frac{1}{2}\{w(0^-, \tau) - w(x, \tau)\}, \; x > s(\tau) \tag{6.16}$$

$$s(\tau) = \frac{1}{2}\{w(x, \tau) + w(0^+, \tau)\}, \; x < s(\tau)$$

$$s(\tau) = \frac{1}{2}\{w(0^-, \tau) + w(x, \tau)\}, \; x > s(\tau). \tag{6.17}$$

Proof: Application of the method of matched asymptotic expansions as above. The major difference from the treatment of antisymmetric shocks above is that when the shock is not antisymmetric, the shock will propagate at a phase speed $s(\tau)$. The integral in the definition of the inner coordinate X allows for the cumulative movement of the shock. In the language of multiple scales, we must incorporate this shock movement into the definition of the inner coordinate because this is time variability on a "fast," $O(1/\nu)$ time scale. [Note that this fast time scale is the time required for the shock to propagate an $O(1)$ distance in the inner coordinate X.] The temporal changes in $s(\tau)$ and $J(\tau)$ themselves are "slow" variability on a time scale of $O(1)$, and thus this time variability appears only parametrically in the inner problem.

Kevorkian and Cole (1981), Lesser and Chrighton (1975), and Blumen (1990b) all give more detailed treatments of Burgers' equation for various boundary conditions via matched asymptotics.

In the special case $j = 1$ (ordinary Burgers' equation), (6.14) has the analytical solution (6.5). For the case $j = 2$ (damping proportional to the fourth derivative of u), the shock is a special solution of the Kuramoto–Sivashinsky equation (Hooper and Grimshaw 1988). For $j = 2$ and 3, we have numerically computed unique antisymmetric solutions to (6.14). The numerical technique was to write $v(x) = \tanh(X) + V(X)$ and then expand $V(X)$ in a series of basis functions of the form $\varphi_n(X) = TB_{2n+1}(X) - TB_1(X)$, where the $TB_n(X)$ are the rational Chebyshev functions of Boyd (1989). A Newton–Kantorovich iteration is used to compute the series coefficients, as also described in Boyd (1989).

The shocks for higher-order viscosity are compared to the shock for $j = 1$, $-\tanh(X/2)$, in Fig. 7. Note that the shocks for higher-derivative dampings are nonmonotonic, asymptoting to ± 1 as damped oscillations. It is only within the narrow $O(1/\nu)$ frontal zone where X is $O(1)$ that the form of the dissipation

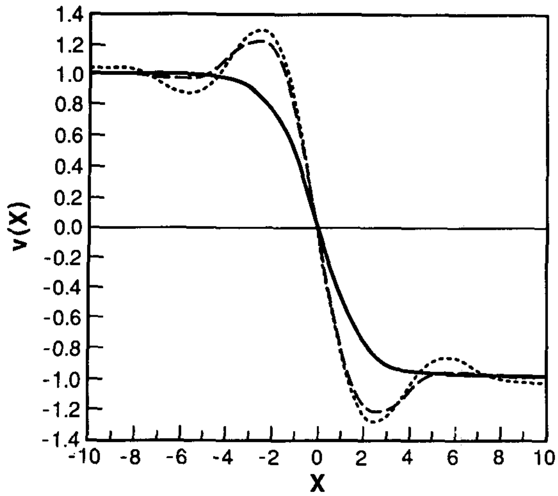

FIG. 7. A comparison of the shocks for damping proportional to the second derivative (solid), fourth derivative (long dashes), and sixth derivative (dotted). These shocks solve (6.14) for the cases j = 1, 2, and 3, respectively.

is important. Outside this zone, the damping is irrelevant save only that it must be capable of smoothing the jump discontinuity into a thin internal boundary layer. One could easily extend Theorem 3 to a huge variety of alternative damping laws (including nonlinear dissipation terms).

Figure 8 illustrates the accuracy of the matched asymptotics expansion for $j = 1$ (ordinary Burgers' equation). The uniform approximation gives a faithful representation of the solution both inside and outside the frontal zone.

7. Summary

In this work, we have extended the theory of frontogenesis in several ways. First, we have reexamined Burgers' equation. Meteorologists have known that the solution to the inviscid form of Burgers' equation, the one-dimensional advection equation, has a cube root singularity at the moment of breaking. For later times, however, the "geophysical view" has been limited to "the solution has no physical meaning" (Andrews and Hoskins 1978) and "the model is invalid because the [one-dimensional advection equation] solution is multivalued" (Blumen 1980). In contrast, the "aero view" of aeronautical engineering and applied mathematics has been that shocks and fronts are jump discontinuities (Roe 1986). Whitham (1974) is typical in making no special analysis of the time of breaking. Because of this, he quotes an erroneous number of roots to the characteristic equation (2.3), which we correct in sections 2 and 3. In the same spirit, Cai et al. (1989) numerically solve Burgers' equation by augmenting the usual Fourier series with a special basis function that is a discontinuity of time-varying magnitude and location.

Our first contribution has been to follow the evolution of Burgers' solution and of its time-dependent Fourier coefficients so as to merge these two incomplete viewpoints. We find that the shock is a discontinuity for all $t > t_B$, but because the discontinuity is initially very small, there is a brief interval when the slope of the Fourier coefficients more closely resembles the $1/k^{4/3}$ spectrum of the cube root branch point than the $1/k$ spectrum of the discontinuity. However, this interval is extremely short. For an initial sine wave, the cube root singularity persists only for about 3% of the time required for the wave to break. In applications to baroclinic instability, the slope S is within $\pm 1/6$ of its value at $t = t_B$, $-4/3$, for no more than *two hours.* The $-8/3$ energy spectrum, which is mentioned so often in the meteorological literature, is an order of magnitude more ephemeral than the mayfly, which lives and dies within a single day.

Our second contribution is a simple topological demonstration that waves that are continuous and single valued at the instant of breaking must have a cube root singularity. This argument is independent of the wave equation or equations since it depends only on simple assumptions about continuity and smoothness.

This in turn has important implications for the numerical solution of many nonlinear, shock-forming wave equations other than Burgers'. For example, the discontinuity-fitting Fourier algorithm of Cai et al. (1989) would be improved near the time of breaking if their basis function were generalized so that it could match the cube root singularity that occurs at $t = t_B$. Fitting a singularity by a discontinuity is a good idea if and only if a discontinuity is the only type of singularity that occurs.

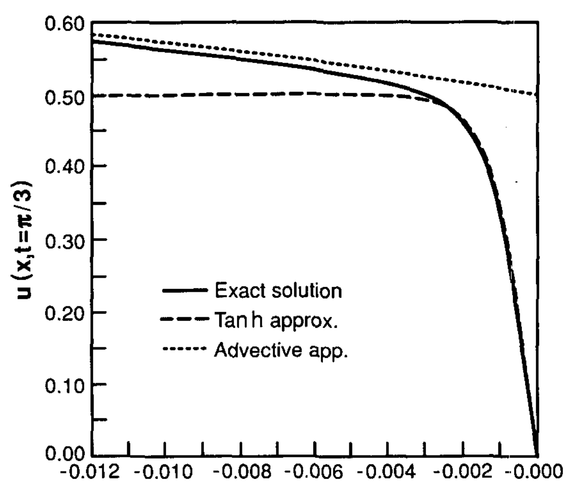

FIG. 8. A comparison of approximations to the solution of Burgers' equation for a sinusoidal initial condition at $t = \pi/3$ with $\nu = 0.0001$. Solid: exact solution. Long dashes: inner solution [$= -(1/2) \times \tanh\{x/(2\nu)\}$]. Not shown: uniformized approximation [$= -w(x, t) \tanh\{x/(2\nu)\}$]. Dotted: outer solution $w(x, t)$ (solves one-dimensional advection equation).

Our third contribution is to generalize Burgers' equation so as to incorporate instability. Blumen (1980, 1981), who showed that unstable, semigeostrophic Eady waves could be modeled by the one-dimensional advection equation with an "antifriction," also solved his model via an exact transformation that reduced his equation to the ordinary, frictionless advection equation. Similarly, we generalize his model still further by adding a time-dependent viscosity and then mapping the generalized equation to Burgers' equation via a similar exact transformation. (See also Blumen 1990b.)

Our fourth contribution is to generalize Burgers' equation still further by employing the method of matched asymptotic expansions to show that for small viscosity, the dynamics *outside* the frontal zone (where the discontinuity is smoothed into a thin layer of high gradient) is *independent* of the damping or mixing processes *within* the frontal zone. To illustrate this point, we give explicit solutions for three different forms of damping: ordinary viscosity, a hyperviscosity proportional to the fourth derivative of the solution, and a sixth-derivative hyperviscosity. Independent of the form of damping, the frontal zone shrinks to zero width in the limit of zero dissipation and the front becomes a jump discontinuity.

We received a preprint of Blumen (1990b) from its author after he was a referee for the first submitted version of this article. He, too, applies matched asymptotics to a generalized Burgers' equation. However, he does not consider higher-order viscosities, nor does he offer the numerical comparison between matched asymptotics and the exact solution given in our Fig. 8.

Charney (1971) showed that the observed energy spectrum of the atmosphere, approximately proportional to $1/k^3$, could be explained by quasigeostrophic turbulence theory. However, in the words of Andrews and Hoskins (1978): "Charney suggests that the formation of fronts near the ground may produce a weak $1/k^2$ energy spectrum superposed on the basic $1/k^3$ spectrum." Charney's suggestion is supported by (7.1), which implies a frontal energy spectrum proportional to $1/k^2$.

Our extensions to frontogenesis theory fill in some important gaps, but many questions remain. At present, the modeling of baroclinic instability by a generalized Burgers' equation has been carefully justified only for simple Eady waves. It would be very useful to calculate energy spectra and apply matched asymptotics to three-dimensional, primitive equation models. The generalized Burgers' visualization of Eady waves is very simple: the shock-forming advection and the time-amplifying instability mechanisms operate almost independently of one another and of boundaries, cumulus convection, small-scale mixing, and many other complications. We have a reasonable approximation to atmospheric frontogenesis, but it is likely that many subtle details still elude us.

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