Formation and Evolution of Frontal Rainbands and Geostrophic Potential Vorticity Anomalies

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(Manuscript received 1 November 1990, in final form 20 May 1991)

ABSTRACT

A viscous semigeostrophic model is developed and used to study the formation and evolution of frontal rainbands in association with the dry and moist geostrophic potential vorticity (GPV) anomalies. The numerical results show that when moist GPV (MGPV) becomes negative in the saturated region (but the flow is still stable to viscous symmetric perturbations), banded substructures can be generated internally by a positive feedback between the moist circulation bands and geostrophic forcing anomalies in association with the generation of mesoscale GPV anomalies. In addition to the previous diagnostic results for idealized forcings, the new aspect here is that the large-scale moist ascent evolves into much finer multiple moist bands as soon as the positive feedback begins to generate banded substructures in the forcing and GPV fields. When MGPV is positive, multiple rainbands can only be generated externally by preexisting GPV or MGPV anomalies. These rainbands can be self-maintained by a weak feedback between the vertical motion and warming anomalies that operates in a partially saturated layer between the maximum and minimum levels of the undulated cloud-base boundary in association with the preexisting GPV or MGPV anomalies. The bands are seen as weak cores of upward motion surrounded by the large-scale moist ascent, rather than separated by mesoscale dry subsidence as in the case of negative MGPV.

As the negative MGPV area diminishes (mainly due to the boundary MGPV flux) and the GPV anomalies are lifted into the saturated region, the later evolution of the bands is largely controlled by the Lagrangian advection and eddy dissipation. As the geostrophic confluence flow squeezes (stretches) the bands toward (along) the front, the fine structures of GPV anomalies are smoothed by eddy viscosity, and multibands gradually "merge" into a larger single band of moist ascent. Rainbands produce not only horizontal-mean positive (negative) GPV anomalies in the lower (upper) levels but also significant mesoscale GPV anomalies in the horizontal.

Boundary-layer processes can produce either positive or negative GPV flux, depending on the boundary conditions. In general, positive (negative) GPV flux is produced when warm (cold) air moves over a cold (warm) surface. The complex and yet somewhat subtle feature of GPV flux near the surface front is discussed in detail.

1. Introduction

The possible roles played by conditional symmetric instability (CSI) or small moist symmetric stability in the formation of frontal rainbands have been studied theoretically in the following four aspects.

(i) Pure CSI may produce either a single rainband or multiple rainbands, depending on the intensity of the instability and initial state. CSI occurs as the moist potential vorticity of the basic geostrophic flow or, say, the moist geostrophic potential vorticity (GPV) becomes negative (Bennetts and Hoskins 1979). Since the most unstable CSI modes tend to have infinitely narrow updrafts, the effect of eddy viscosity cannot be neglected (Xu 1986a). Pure viscous CSI can produce rainbands only if the moist GPV (MGPV) is negative enough, while multibands require even more negative MGPV. Because the observed MGPV is often near zero within frontal rainbands (Emanuel 1988) and the nonlinear CSI circulations may decay after their initial development (Xu 1986b,c), pure CSI is often hard-pressed to explain some observed long-lasting rainbands, though the CSI concept is useful for explaining and assessing some fast-growing bands.

(ii) A single rainband could evolve from a frontal circulation in the presence of small positive MGPV or, say, small moist symmetric stability (Emanuel 1985; Thorpe and Emanuel 1985). Since the band is stable and supported by large-scale forcing and moisture supply, it may last longer than pure CSI bands. This combination effect, however, cannot produce multiple bands and the semigeostrophic (SG) models used in these studies become invalid as the MGPV decreases to zero or negative

(iii) Mesoscale surface temperature anomalies may produce multifronts, but the associated banded structure tends to be very shallow (Hoskins et al. 1984; Chan and Cho 1989). Mesoscale GPV anomalies may produce multiple bands, but these bands decay with time
(Chan and Cho 1989). When moist processes are invoked, the above preexisting bands can turn into deep and intense rainbands, but the shape and intensity of the rainbands depend on the preexisting MGPV anomalies (Cho and Chan 1991). As in the above (ii), the SG models used in these studies do not allow zero or negative MGPV.

(iv) The combined effect of frontogenetical forcing and negative MGPV may produce either multiple rainbands or a strong single rainband, depending on the competition between the forcing, negative MGPV, and eddy viscosity (Xu 1989a,b). These bands are stable to viscous CSI and occur before the negative MGPV becomes low enough to initiate pure viscous CSI bands. Since zero or negative MGPV is often observed to coexist with frontogenetical forcing in regions of frontal rainbands (Sanders and Bosart 1985; Sanders 1986), rainbands are more likely to be produced by the combined effect, rather than by pure CSI. However, this combined effect was examined only for idealized smooth distributions of frontogenetical forcing, while the banded moist frontal circulations were computed only diagnostically from the extended viscous Sawyer–Eliassen (S–E) equation.

As a continued part of the author’s previous studies [cited in item (iv)], this paper develops a viscous SG model for two-dimensional frontogenesis of stretching deformation, which combines the previously derived viscous S–E equation with the tendency equations for momentum, potential temperature, and moist (wet-bubble or equivalent) potential temperature. This model is used to study the formation and evolution of frontal rainbands in association with the time evolution of GPV and MGPV anomalies. Scientific questions of particular interest to be addressed are as follows.

(i) How may stable multiple frontal rainbands be generated internally from smooth initial data in a prognostic model? Should these internally generated rainbands evolve with time differently from those generated externally by preexisting GPV and MGPV anomalies?

(ii) A feedback between latent heating and forcing anomalies was proposed by Cho and Chan (1991), but this feedback could not generate rainbands internally as MGPV is positive. Why should this be the case? Perhaps, we should consider more precisely a feedback between net warming and forcing anomalies and then examine how this feedback operates differently for externally generated rainbands compared with those generated internally due to negative MGPV.

(iii) According to Haynes and McIntyre (1987), the “potential vorticity (PV) substance” cannot be created or destroyed anywhere except at boundaries even for viscous diabatic flows, so it is interesting to examine how GPV and MGPV anomalies are generated at the boundaries and how the boundary GPV and MGPV fluxes may affect the development of frontal rainband.

(iv) How should eddy viscosity affect the evolution and final dissipation of the rainbands and MGPV anomalies?

The viscous SG model is introduced in the following section, but the detail derivations of well-posed boundary conditions are given in appendix A. The numerical method used for the model’s time integration is described in appendix B. In section 3, the dry and moist GPV equations are derived and their conservation properties are discussed. A point is also raised in this section concerning the use of negative moist PV as a measure of CSI. Section 4 presents four cases of simulation and examines the results in connection with the scientific problems raised above. Two types of feedback between net warming and forcing anomalies are proposed for stable rainbands generated externally and internally, respectively. Conclusions follow in section 5.

2. Viscous semigeostrophic model

Consider a unidirectional (along the y coordinate) moist geostrophic flow described by the following five parameters:

\[ F^2 = f(f + \eta), \]
\[ S^2 = (g/\Theta_0)\partial_\theta\partial_\theta - f\partial_\theta v_g, \]
\[ N_d^2 = (g/\Theta_0)\partial_\theta\partial_\theta, \]
\[ S_w^2 = (g/\Theta_0)\partial_\theta\partial_\theta, \]
\[ N_w^2 = (g/\Theta_0)\partial_\theta\partial_\theta, \]

(2.1)

where \( \eta = \partial_x v_g \); \( v_g \) and \( \Theta_g \) are the basic geostrophic wind and potential temperature satisfying the thermal wind relationship; \( \Theta_w \) is the equivalent potential temperature; \( \Theta_0 \) is the constant reference temperature; and the Coriolis parameter \( f \) and acceleration of gravity \( g \) are assumed to be constant. Here \( z = [1 - (p/p_0)]c_0/\Theta_0/g \) is the pseudohight (Hoskins and Bretherton 1972). Assume that the geostrophic flow (2.1) is superimposed on the following frontogenetical stretching deformation flow,

\[ u' = -\alpha x, \quad v' = \alpha y, \]
\[ \varphi' = -\alpha^2(x^2 + y^2)/2 + faxy, \]

(2.2)

where \( \varphi' \) is the balanced geopotential height associated with \( (u', v') \) and the balance includes the nonlinear inertial term. According to Xu (1989a), the viscous SG equations for a two-dimensional frontal geostrophic circulation \( (u, w) \) in the physical space \((x, z)\) are

\[ f_0 + D_\theta = 0, \]
\[ F^2 u + S^2 w - Df_0 = -f[\partial_t + D_\theta + \alpha - D]v_g, \]
\[ (g/\Theta_0)\theta + Dw = 0, \]
\[ S^2 u + N^2 w - (g/\Theta_0)D\theta = -(g/\Theta_0)[\partial_t + D_\theta - D]\theta_g, \]

(2.3a-d)
\[ \partial_x u + \partial_z w = 0, \quad (2.3e) \]

where \( D_u = u'_z \partial_x = -\alpha \partial_x \) is the geostrophic advection, \((g/\theta_0)\theta\) is the nonhydrostatic buoyancy, \(v\) is the alongfront geostrophic wind, \(D = v_1 \partial_x^2 + v_2 \partial_z^2\), the coefficients \((v_1, v_2)\) of eddy viscosity are assumed constant, and the Prandtl number is one. The conditional moist thermal stratification \(N^2\) is given by (Durrant and Klemp 1982)

\[ N^2 = \begin{cases} \gamma N^2_w = (\Gamma_w/\Gamma_f)(g/\theta_0)\partial_x \theta_w, & \text{when saturated} \\ N^2_d, & \text{otherwise} \end{cases}, \quad (2.4) \]

where \( \Gamma_w \) and \( \Gamma_f \) are the moist and dry adiabatic lapse rates, respectively, and \( N^2_w \) and \( N^2_d \) are defined in (2.1).

If the atmosphere is dry with a positive geostrophic potential vorticity (GPV, defined by \( q_d = N^2_d F^2 - S^2 \)) and the eddy viscous effect on the geostrophic wind is small \((D/f \ll 1)\), then (2.3a)–(2.3e) reduce to the conventional SG equations. However, if the atmosphere is saturated and the moist GPV (MGPV, defined by \( q_w = N^2_w F^2 - S^2 S^2 \)) is small, then the geostrophic wind deformation can be strong, especially along the internal boundary layer (IBL) between the dry and moist regions [see the IBL scale analysis in Xu (1989a)], and the effect of eddy viscosity on the geostrophic motion cannot be neglected from (2.3a)–(2.3e). The effect of eddy viscosity is also important in the planetary boundary layer (PBL). In this case, if \( w = 0 \) is assumed, then (2.3a)–(2.3e) reduce to the (baroclinic) Ekman PBL equations. Therefore, (2.3a)–(2.3e) are uniformly valid in both the dry and moist regions and for both IBL and PBL.

By using the thermal wind relationship, we can eliminate the tendency terms of \((v_x, \theta_x)\) in (2.3), and obtain the following extended viscous Sawyer–Elissen (S–E) equation for the geostrophic circulation:

\[ \partial_x (F^2 + D^2) \partial_x - \partial_z S^2 \partial_x - \partial_x S^2 \partial_x 
+ \partial_x (N^2 + D^2) \partial_1 \psi = 2Q, \quad (2.5) \]

where \( Q = f \partial_x (v_x + v_x')/\partial_x (x, z) = \alpha S^2 \) is the geostrophic forcing and \( \psi \) is the streamfunction defined by \( \psi = -\partial_x \psi \) and \( w = \partial_x \psi \). Again, when the basic flow (2.1) is dry and \( D \) is neglected, (2.5) reduces to the conventional S–E equation. It has been shown by Xu (1989a) that the problem of solving for the frontal circulation from the viscous S–E equation with an appropriate boundary condition is well posed. The existence, uniqueness, and stability of the solution are ensured until the basic geostrophic flow becomes unstable to the viscous conditional symmetric instability (CSI), even if the MGPV becomes zero or negative in the moist region. A variational formulation and a finite-element method with two-dimensional cubic-spline elements are developed in Xu (1989a) for solving the viscous S–E equation. One merit of this method is that the viscous CSI stability can be checked precisely and without any extra computational cost as the solution is computed. In this paper, the same finite-element method will be used to solve for the geostrophic circulation, at each time step, from (2.5) with the following boundary conditions:

\[ \psi = \partial_x \psi = (\partial_x D \partial_2 + \partial_z D \partial_2) \psi = 0 \quad \text{on} \quad z = 0, H \quad (2.6a) \]

\[ \psi = \partial_x \psi = (\partial_x D \partial_2 + \partial_z D \partial_2) \psi = 0 \quad \text{on} \quad x = \pm L, \quad (2.6b) \]

corresponding to \( w = \partial_x u = f \partial_x v - (g/\theta_0) \partial_\theta \psi = 0 \) on \( z = 0 \) and \( H \), and \( u = \partial_z w = f \partial_x v - (g/\theta_0) \partial_\theta \psi = 0 \) on \( x = \pm L \), respectively. It is shown in appendix A that the free-slip rigid boundary conditions used here in (2.6a,b) and the nonslip rigid boundary conditions used in Xu (1989a,b) are both well posed and compatible with the variational formulation of the viscous S–E equation.

The geostrophic field \((v_x, \theta_x)\) and moist field \(\theta_w\) are computed prognostically from (2.3b,d) and

\[ (\partial_t + U \cdot \nabla - D) \theta_w = 0, \quad (2.7) \]

where \( U = (u'_x + u, w) \) is the total cross-front wind and \( \nabla = (\partial_x, \partial_z) \). The boundary conditions are

\[ f \partial_x v_x = (g/\theta_0) \partial_x \theta_x, \quad (g/\theta_0) \partial_z \theta_x = N^2_0 \quad \text{at} \quad z = 0, H; \quad (2.8a) \]

no moisture flux at \( z = 0 \), \( RH = \text{const} \) at \( z = H; \quad (2.8b) \]

\[ (\partial_t + \alpha + D_g) v_x = 0 \quad \text{and} \quad (\partial_t + D_g) (\theta_x, \theta_w) = 0, \quad (2.8c) \]

where \( N^2_0 = \text{(const)} \) is the reference environmental static stability (which is also the initial static stability if the initial horizontal temperature gradient is negligibly small) and \( RH \) is the relative humidity. The first condition in (2.8a) assumes that the thermal wind balance holds at the free-slip boundaries \( z = 0 \) and \( H \), which is consistent with (2.6a). The second condition in (2.8a) assumes that the sensible heat flux remains constant at \( z = 0 \) and \( H \). The lateral boundary conditions in (2.8c) consider the inward geostrophic advection of the basic geostrophic field (specified analytically over \( |x| < \infty \) at the initial time) at \( x = \pm L \).

The model’s computational domain is chosen to be very wide \((L = 10^4 H = 10^4 \text{ km})\) so that the effect of the rigid lateral boundary condition (2.6b) on the geostrophic circulation is negligible. [A well-posed open lateral boundary condition for the geostrophic circulation is derived in appendix A, consistent with the open lateral boundary condition (2.8c) for the geostrophic flow. These open boundary conditions should be used if the computational domain is not large.
The domain is divided nonuniformly in the horizontal direction by 48 columns of subelements so that higher resolution is obtained in the central moist region (as shown in Figs. 1–14 and 17–18). In the vertical direction, there are 24 equally divided layers of subelements, so the total number of subelements are 48 × 24, as in Xu (1989a,b). Since bicubic-spline basis functions are used in our finite-element method, the resolution for \( \psi \) is actually three times higher than the subelement grids in each dimension. The grid meshes for the geostrophic fields \( (v_x, \theta_z, \omega_z) \) are 96 × 48, which doubles the subelement resolution in each dimension (see Figs. 5b and 6b). The numerical procedures used for the model’s time integration are described in appendix B.

3. Dry and moist GPV equations

Before deriving the dry and moist geostrophic potential vorticity (GPV) equations, I would like to make a point concerning the difference between potential vorticity (PV) and GPV in association with symmetric instability (SI). Negative PV has been widely used as a measure of SI. On the other hand, it is also well known that the SI basic state has to satisfy the thermal wind (or balanced wind) relationship. Thus, strictly speaking, it is the negative GPV, rather than negative PV, that should be used as a measure of linear inviscid SI. If the large-scale flow is nearly geostrophic and very stable to SI, then PV is very close to GPV and can be used approximately to measure the SI stability. In regions where the large-scale background flow is nearly neutral to inviscid SI, the GPV and PV may be both small but with opposite signs. The difference between PV and GPV becomes large as the SI circulation develops to a finite or large amplitude or, in general, as the geostrophic wind becomes strong. In this case, the linear SI theory becomes invalid and neither PV nor GPV can faithfully measure the evolution tendency of the SI circulation. The inviscid nonlinear SI theory (Xu 1986b,c) suggests that the initial growth of the SI circulation may be fairly well approximated by linear theory, but the longtime nonlinear evolution will be bounded energetically. After the circulation kinetic energy reaches the maximum, the circulation will decay and return at least a part of the perturbation energy to the basic flow and then evolve into the nonlinear oscillation stage. This may explain why the simulated nonlinear SI circulations decay while the PV remains negative (Thorpe and Rotunno 1989). The nonlinear inviscid SI circulation may itself become unstable to smaller perturbations, but this instability is neither SI nor measured by negative PV. Thus, as negative PV (or moist PV) is used as a measure of SI (or CSI), it is important to be aware of the above limitation. In this sense, the “paradox” of Thorpe and Rotunno (1989) that “the inviscid circulation can never remove the SI instability” was inferred from the conservation of negative PV, it should really mean that “the inviscid SI circulations cannot rearrange the flow to make the geostrophic state finally stationary and stable” (Thorpe and Rotunno, personal communication). This latter stated property exists not only for SI but also for convective instability (CI) in a rotating fluid, because SI and CI in a rotating fluid are similar regardless of whether the flow is linear or nonlinear, inviscid or viscous (Xu and Clark 1985).

When SI (or CSI) occurs in the real atmosphere, turbulent eddies should grow (Xu 1988) and “tear” the negative PV (or moist PV) structure into increasingly fine substructures. As these fine substructures of negative PV (or negative moist PV) go beyond the resolution of the observing system, the “observed” negative PV (or moist PV) can no longer be conserved. By the same reason, the negative PV (or negative moist PV) may not be precisely computed until the grid resolution reaches the molecular scale in a numerical model. In other words, the computed PV (or moist PV) depends on the resolution. On the other hand, since dry and moist GPVs are computed from the geostrophic geopotential height field (with the nongeostrophic geopotential height filtered out), the negative GPV field should be smoother and less dependent on resolution. Because of this and the reasons discussed above, it seems better to use MGPV as a measure of CSI.

In the SG model, GPV is conserved but not allowed to be negative, and the conservation of positive GPV ensures symmetric stability for all time. In our viscous SG model, though the dry and moist GPVs are no longer conserved following the Lagrangian advection, the GPV equations still have the “conservation forms” [see (3.1) and (3.4)], and the dry and moist GPVs are allowed to be negative, as long as the flow remains stable to viscous CSI. Since only the linear geostrophic terms are retained in (2.3), the inviscid CSI is linear and can be measured by the negative MGPV, while the viscous CSI stability can be determined numerically by the positiveness of the functional of (2.5)–(2.6) (see appendix A).

The GPV evolution equation is obtained from \( (F^2 \partial_x - S^2 \partial_t) (2.3d) + (N^2 \partial_x - S^2 \partial_t) (2.3b) \):

\[
\partial_t q = -\nabla \cdot (Uq + F_x + F_d) - \partial_x (v_x q),
\]

where \( F_x = (F_{x1}, F_{x2}) = [(g/\theta_0) S^2 D^2 - fN^2 D^2 V, fS^2 D^2 -(g/\theta_0) F^2 D^2] \theta \) is the viscous GPV flux, that is, the GPV flux caused by the eddy diffusions of total momentum and potential temperature; \( fS^2 D^2 S_h \) is the heating GPV flux along the absolute momentum surface; \( S_h = (N^2 - \gamma N^2) w \) is the rate of latent heating; \( \theta = \theta + \bar{\theta} \) is the total potential temperature; and \( V = v_x + v_y + v_z \) is the total alongfront wind. Equation (3.1) has the “conservation form,” which is similar to that of the PV equation in the more complete models of Haynes and McIntyre (1987). Clearly the dry GPV in our model can be transported...
from one location to another by advection and flux. The “GPV substance,” however, cannot be created or destroyed anywhere except at the boundaries. Following (4.2)-(4.4) of Haynes and McIntyre (1987), we can define a hypothetical velocity field \( U_0 \) so that \( \nabla \cdot \mathbf{n} = \nabla \cdot \mathbf{U}_0 \) is the velocity of the \( \theta \) surface in the direction normal to itself where \( \mathbf{n} = \nabla \theta / |\nabla \theta| \), and \( \nabla \cdot (\mathbf{U}_0 - \mathbf{U}) = 0 \). Then, (2.3d) and (3.1) can be rewritten as

\[
\nabla \cdot (\mathbf{U}_0 - \mathbf{U}) = \nabla \cdot (\mathbf{V} - \mathbf{U}) = 0,
\]

(3.2a)

\[
(\partial_t + \mathbf{U}_0 \cdot \nabla) q = -\nabla \cdot [(\mathbf{U} - \mathbf{U}_0) q] + (g/\theta_0)(\mathbf{D} \theta + S_h)(S_z^2, -S_z^2) - fD^2 (N_{a^2}^2, -S_z^2).
\]

(3.2b)

The left-hand side of (3.2b) is the GPV evolution following the \( \theta_g \) surfaces. The last term in (3.2b) is the GPV flux caused by eddy diffusion of the total momentum, which is perpendicular to \( \nabla \theta_g = (\nabla \theta / \theta)(S_z^2, N_{a^2}^2) \) or, equivalently, parallel to the \( \theta_g \) surface. The first and second terms on the right-hand side of (3.2b) are, respectively, the cross \( \theta \)-surface GPV advection and GPV flux caused by the diabatic processes (heating and eddy diffusion). Using (3.2a), it is easy to verify that the sum of these two terms is also parallel to the \( \theta_g \) surface. Thus, the following modified statement of Haynes and McIntyre (1987) are obtained for the model: the “GPV substance” can neither be created nor destroyed within a layer bounded by two \( \theta_g \) surfaces, except at the domain boundaries. It may be worthwhile to mention that the aforementioned layer is not a material one, rather a volume layer confined by two \( \theta_g \) surfaces. This volume layer changes its shape with time in physical space and moves “through” the air mass, just like the \( \theta_g \) surfaces do—not necessarily to follow the air mass (unless the process is adiabatic).

Since the reference environmental GPV is constant \( q_0 = f^2 N_{a^2}^2 \), the evolution equation for the GPV anomaly \( q' = q - q_0 \) is the same as (3.1) or (3.2b), except that \( q \) is replaced by \( q' \), and the aforementioned statement also applies to GPV anomalies. Integrating the GPV anomaly equation over the entire cross section \( \Omega \) yields the evolution equation of the integrated GPV anomaly:

\[
\partial_t \left\{ \{ q' \} \right\} = -\alpha \left\{ \{ q' \} \right\} - \Delta F_z, \quad (3.3a)
\]

where \( \{ \cdot \} = \int_{\Omega} \{ \cdot \} dxdz \) and \( \Delta F_z = F_z(H) - F_z(0) \) is the net outward dry GPV flux through the upper and lower boundaries. Clearly (3.3a) suggests that if there is no net flux of dry GPV at the boundary, then the GPV anomaly integrated over the cross section \( \Omega \) will decay exponentially as \( \exp(-\alpha t) \). More precisely, the solution of (3.3a) shows

\[
\left\{ \{ q' \} \right\} = \left\{ \{ q'(0) \} \right\} \exp(-\alpha t)
\]

- \int_0^t \Delta F_z \exp[\alpha(t - \tau)] d\tau \rightarrow -(1/\alpha)\Delta F_z,
\]

if \( \Delta F_z \rightarrow \text{const} \) as \( t \rightarrow \infty \). (3.3b)

The asymptotic behavior in (3.3b) can be understood as follows. As the GPV anomalies are stretched by the geostrophic deformation flow in the front-parallel direction, the cross section areas of the anomaly bands are compressed in the front-normal direction, and the integrated GPV anomaly becomes smaller and smaller. However, this does not necessarily mean that the locally individual GPV anomalies should become weak. Strong GPV anomalies can be produced by boundary GPV flux as well as internal up-gradient heating GPV flux.

The MGVP evolution equation can be obtained from

\[
(f^2 \partial_z S_z^2 \partial_x + N_{a^2}^2 \partial_x - S_z^2 \partial_x) (2.7) + (N_{a^2}^2 \partial_x - S_z^2 \partial_x) (2.3b):
\]

\[
\partial_t q_w = -\nabla \cdot [(\mathbf{U} q_w + F_w) - \rho \mathbf{v} q_w], \quad (3.4)
\]

where \( F_w = (F_{w1}, F_{w2}) = [(g/\theta_0)S_z^2 D\theta - fN_{a^2}^2 D\mathbf{v}, fS_z^2 D\mathbf{v} - (g/\theta_0)F_z^2 D\theta] \) is the viscous MGVP flux. The MGVP equation (3.4) also has conservation form and can be rewritten, together with (2.7), into the similar forms as (3.2a,b), so the statement that follows (3.2a,b) can similarly apply to the “MGVP substance” within a layer bounded by two \( \theta_g \) surfaces. Unlike the dry GPV flux, the MGVP flux is not affected by latent heating. Furthermore, since the background MGVP distribution is not uniform, the MGVP anomaly field is not integrable over the entire cross section. In this case, we define the MGVP perturbation \( q_w' = q_w - q_w(0) \) with respect to the initial MGVP field \( q_w(0) \), so \( q_w' \) is integrable and the evolution equation for \( \{ q_w' \} \) and its solution are similar to (3.3a,b).

The derivations of (3.1) and (3.4) do not depend on the assumed form of eddy viscosity. When the eddy viscosity coefficient \( \nu \) is constant and the Prandtl number is unity as assumed in (2.3), the second and third flux terms in (3.1) can be rewritten as

\[
-\nabla \cdot (F_w + F_h) = Dq_w - 2J[v_1J(F^2, S_z^2)
\]

- \( \nu_2 J(S_z^2, N_{a^2}^2) \) \( -\nabla \cdot (F_w + F_h), \quad (3.5) \)

where

\[
F_w = [(g/\theta_0)S_z^2 D\theta - fN_{a^2}^2 D\mathbf{v},
\]

\[
fS_z^2 D\mathbf{v} - (g/\theta_0)F_z^2 D\theta \]

is the GPV flux caused by the eddy viscous diffusion of the ageostrophic field (\( f\theta, \theta \)). The first term \( Dq_w \) in (3.5) gives the same second-order diffusion as those for the momentum and potential temperature in (2.1), while the second term highlights the difference between the GPV flux and the fluxes of momentum and potential temperature. Similarly, the second flux term in (3.4) can be rewritten as

\[
-\nabla \cdot F_w = Dq_w - 2J[v_1J(F^2, S_z^2)
\]

- \( \nu_2 J(S_z^2, N_{a^2}^2) \) \( -\nabla \cdot F_w, \quad (3.6) \)

where \( F_w = f[N_{a^2}^2 D\mathbf{v}, S_z^2 D\mathbf{v}] \) is the MGVP flux caused by the eddy viscous diffusion of the alongfront
ageostrophic wind $v$. Similarly to the viscous PV equation (6) of Thorpe and Rotunno (1989), (3.5)–(3.6) indicate that the dry/moist GPV flux does not correspond to a simple downgradient flux even when momentum and potential temperature move down their respective gradients. There is no consensus concerning how the inviscid conserved variables, such as dry and moist GPVs (or PVs in the full equation model), potential temperature, and two-dimensional absolute momentum, should be mixed downgradient in terms of turbulence parametrization. For simplicity, constant viscous coefficients $\nu_1$ and $\nu_2$ are used in this paper. In this case, (3.5)–(3.6) suggest that the viscous dry/moist GPV flux should be downgradient in regions where the dry/moist GPV anomalies are strong and, thus, the first term on the right-hand side of (3.5) and (3.6) is the largest. This feature is seen from the solutions in the following section.

4. Results and discussions

In this section we present four cases of numerical simulation. Case I examines the effects of latent heating and boundary GPV flux on the evolution of a moist frontal circulation. Case II examines the role of negative MGPV in producing multiple mesoscale frontal rainbands from a smooth initial field. Case III and case IV examine the effects of preexisting mesoscale dry and moist GPV anomalies on the formation of multiple mesoscale frontal rainbands.

a. Case I—the effects of small MGPV and boundary GPV flux

The initial dry GPV field is assumed constant ($q = q_0 = f^2 N_0^2$) and the initial boundary value for $\theta_e$ is given by

$$\theta_e = \theta_s = \theta_0 + (\theta_0/g)(q_0/f^2)Z + \Delta \theta_s \tanh(X/L_s),$$

on $Z = 0$ and $Z = H$, (4.1a)

where $(X, Z) = (x + v_e/f, z)$ is the SG space. Based on the invertibility principle (Hoskins et al. 1985), the initial (synoptic scale) thermal wind field $(v_x, \theta_e) = (v_x, \theta_s)$ in the interior can be uniquely determined. It is convenient to first compute $(v_x, \theta_s)$ in $(X, Z)$ space and then map it to physical space $(x, z)$. The values of the parameters used are

$$f = 10^{-4} \text{ s}^{-1}, \quad g = 10 \text{ m s}^{-2}, \quad \nu_t = 10^5 \text{ m}^2 \text{ s}^{-1},$$
$$\nu_2 = 10^2 \text{ m}^2 \text{ s}^{-1}, \quad H = 10 \text{ km}, \quad L_s = 50H,$$
$$q_0 = f^2 N_0^2 = 10^{-12} \text{ s}^{-4}, \quad \alpha = 10^{-5} \text{ s}^{-1},$$
$$\Delta \theta_s = 12 \text{ K}, \quad \theta_0 = 273 \text{ K}.$$  (4.1b)

As shown in Fig. 1a, the initial geostrophic forcing field $Q = \alpha S^2$ is symmetric with respect to the middle-level coordinate center $(x, z) = (0, H/2)$, though the field is plotted in the subdomain (with higher computational resolution) centered at $(x, z) = (L/2, H/2)$ where $L = H N_0/f = 1000 \text{ km}$ is the Rossby radius of deformation. The initial value of the moist field is specified by

$$RH = \begin{cases} 0.5 + 0.5(1 - z/H), & 0.5 < z/H < 1 \\ 0.75, & 0 < z/H < 0.5 \end{cases}.$$  (4.1c)

Initially the air is unsaturated and there is a shallow region of negative MGPV on the warm side (see Fig. 1b). Note that $q_w \to f^2 N_0^2$ as $x \to \infty$ in Fig. 1b, so the (unsaturated) moist stratification is negative on the far warm side within the shallow region of negative MGPV. This initial field is conditionally stable in the conventional sense, that is, no positive convective available potential energy (CAPE). Here (4.1a) and (4.1c) are similar to that of Cho and Chan (1991), but their parameter settings ensure both the dry and moist GPVs to be positive.

Numerical simulation is carried up to 33.3 h ($=12.0/f$). Before the air becomes saturated (at 8.8 h), the ageostrophic circulation develops smoothly and symmetrically with respect to the maximum forcing. After the air becomes saturated, the moist ascent develops

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**Fig. 1.** (a) Initial geostrophic forcing $Q$ (drawn every $0.2 \times 10^{-12} \text{ s}^{-4}$) and (b) initial MGPV (drawn every $0.5 q_0 = 0.5 \times 10^{-12} \text{ s}^{-4}$) for case I. Three types of contour lines, that is, thick solid, thin solid, and dashed, are used for positive, zero, and negative values, respectively. The horizontal scale (from 0.0 to 1.0) is 1000 km.
rapidly on the warm side and the circulation evolves into a nonsymmetric pattern. By 16.7 h, the circulation has become highly nonsymmetric with a strong moist ascent on the warm side (Fig. 2a), and the moist (saturated) region has extended up to nearly the entire depth of the layer (Fig. 3a). At this time, the thermal wind field \( \mathbf{v}_g \) is also nonsymmetric with the maximum frontogenesis at the lower boundary (not shown), and the geostrophic forcing is strong in the lower levels and concentrated along the cold side of the moist region (Fig. 2b).

Figure 3a shows that at 16.7 h a strong positive GPV anomaly \( q' = q - q_0 \) is formed in the lower corner of the moist region. Clearly this GPV anomaly is produced mainly by the downward (along the absolute momentum surface) heating GPV flux \( F_h \) in the moist region. Along the lower surface boundary there is a GPV anomaly dipole with its positive anomaly on the warm side and negative anomaly on the cold side of the front, respectively. The enlarged details of these GPV anomalies are shown in Fig. 3b. It is also shown in Fig. 3b that the viscous GPV flux \( F_w \) is dominantly downgradient at least in regions where the GPV anomalies are strong, and this is consistent with the analysis in (3.5)–(3.6). Clearly the negative anomaly behind the surface front is produced by the downward bound-

The vertical component of \( F_w \) at the surface is

\[
F_{w2} \approx fS^2{\nu}_1\partial_x^2 V - (g/\Theta_0)F^2(\nu_2\partial_y^2 + {\nu}_1\partial_x^2)\Theta
\]

at \( z = 0 \), \( (4.2) \)

where \( \Theta \) and \( V \) are as in (3.1) and the condition \( \partial_z^2 V \approx 0 \) at \( z = 0 \) is used, which is implied by the boundary conditions \( f\partial_z V = (g/\Theta_0)\partial_z \Theta \) and \( (g/\Theta_0)\partial_z \Theta = N_0^2 = \text{const} \) in (2.6) and (2.8). Since \( \partial_z \Theta \approx \text{const} \) at \( z = 0 \) and increases with height (not shown), it follows that \( \partial_z^2 \Theta > 0 \) at \( z = 0 \). In the vicinity of the surface front, both \( S^2 \) and \( F^2 \) are positive, and \( \partial_z^2 V \) is also positive because the maximum negative \( V \) is immediately behind the front (not shown). Consequently, the first term on the right-hand side of (4.2) is positive and the second term \( -(g/\Theta_0)F^2{\nu}_2\partial_y^2 \Theta \) is negative. As these two terms nearly cancel each other, the sign of \( F_{w2} \) is largely determined by the last term. Since \( \partial_x \Theta \) reaches the maximum at the surface front, \( -{\nu}_1\partial_x \Theta \) must be negative behind the front and positive ahead of the front as is \( F_{w2} \). This explains the boundary GPV flux in Fig. 3b.

The total GPV anomaly flux \( Uq' + F_v + F_h \) and \( \theta_g \) contours are shown in Fig. 3c at 16.7 h. Since the total GPV anomaly flux is plotted in physical space, the cross \( \theta_g \)-surface flux should be in the direction that the \( \theta_g \) surface moves toward. If the hypothetical advection \( U_{\theta_0}' \) is subtracted from the total GPV anomaly flux, then, according to (3.2a,b), the residual flux should be parallel to the \( \theta_g \) surface. In this case, the upward component of \(-U_{\theta_0}' \) largely cancels the downward heating GPV flux, while the residual flux convergences/divergences along the \( \theta_g \) surface and produces positive/negative GPV anomalies within each (moving) \( \theta_g \) layer. The isentropic view of GPV flux in (3.2b) can be useful for checking the consistency of the model results. However, with our current numerical scheme, it seems difficult to compute \( U_{\theta} \) accurately from the motion of \( \theta_g \) surfaces (see appendix B). Nevertheless, as seen from Fig. 3c, the \( \theta_g \) surfaces have moved down slightly in the moist region, which is consistent with the downward (cross-\( \theta_g \) surface) GPV anomaly flux.

Figure 3d shows the MGPV field and viscous MGPV flux at 16.7 h. Comparing this figure with the initial MGPV field in Fig. 1b, it can be seen how the upward incoming MGPV flux \( F_{w2} > 0 \) from the lower boundary gradually destroys the lower-level negative MGPV and produces an area of positive MGPV ahead of the surface front, and how the downward outgoing MGPV flux \( F_{w2} < 0 \) enhances the negative MGPV behind the front. Similar to (4.2), the vertical component of \( F_w \) at the surface is

\[
F_{w2} = fS^2{\nu}_1\partial_x^2 V - (g/\Theta_0)F^2(\nu_2\partial_y^2 + {\nu}_1\partial_x^2)\Theta_w
\]

at \( z = 0 \). \( (4.3) \)
The first term on the right-hand side of (4.3) is positive, which is similar to that of (4.2). The boundary conditions (2.8a,b) imply that \( N_w^2 = N_\theta^2 = N_\phi^2 > 0 \) at \( z = 0 \), but \( N_w^2 \) decreases with height in the lower-level region ahead of the surface front. Consequently, the second term (proportional to \(- \partial_z^2 \theta_w\)) in (4.3) changes sign across the front, that is, from negative behind the front to positive ahead of the front. This is different from the second term in (4.2). Further note that \( \theta_w \) and \( N_w^2 \) have stronger cross-front variations than \( \Theta \) and \( N_\theta^2 \), as does the last term in (4.3) compared with the last term in (4.2). Thus, both the second and last terms in (4.3) have stronger cross-front variations with the same sign change. These explain why the boundary MGPV flux and the MGPV itself have stronger cross-front variations in Fig. 3d than the dry fields in Fig. 3b. During the entire simulation period (up to 33.3 h), the enhanced negative MGPV is confined to a shallow layer behind the surface front. As the negative MGPV is destroyed on the warm side of the front, the area of negative MGPV stays mostly below the saturated region. In the saturated region the MGPV remains positive almost all the time, so no multiple bands are produced in the simulation of case I. This agrees with the previous results of Xu (1989b).

b. Case II—the effect of negative MGPV

The initial thermal wind field \((v_x, \theta_y)\) and parameter settings are the same as in (4.1a)-(4.1c) for case I, except that \( \theta_0 = 278 \) K, that is, the background temperature is uniformly 5 K warmer than in case I. The initial geostrophic forcing is exactly the same as in Fig. 1a. With the same relative humidity distribution but higher temperature, the initial \( \theta_w \) field is less stable or more unstable than in case I and, thus, forms a deeper surface layer of negative MGPV on the warm side of the domain (Fig. 4). The air on the far warm side is slightly conditionally unstable in the conventional sense with a very small positive CAPE \([\max \int_0^z (\Delta \Theta / \theta_0) dz = 8.7 \, \text{m}]\), but this small CAPE still cannot support a viscous convection in our model [even if the surface air is lifted above the level of free (inviscid) convection (1.7 km) by a random external force].

Before the air becomes saturated, the ageostrophic circulation develops smoothly and symmetrically, as
in case I. The air becomes saturated at a slightly earlier time (8.3 h) and the moist ascent develops more rapidly than in case I. By 11.1 h, the moist ascent (not shown) reaches the same intensity ($w = 3.0 \text{ cm s}^{-1}$) as in Fig. 2a for case I at the later time (16.7 h). At this time, the MGPV remains largely positive in the saturated region, except for the lowest saturated level.

As the MGPV becomes negative in the low corner portion (about one-quarter) of the saturated region, the meso-$\alpha$–scale moist ascent breaks into fine (meso-$\beta$–scale) multiple bands at 13.3 h (Figs. 5a–c). The maximum vertical velocity jumps from 5.6 cm s$^{-1}$ (in the smooth moist ascent) at 12.8 h to 43.5 cm s$^{-1}$ (within the fine bands) at 13.3 h, but the growth of the averaged upward motion over the entire (meso-$\alpha$–scale) moist area remains relatively slow and the flow is still stable (but close to marginally stable) to viscous CSI. Since the time derivative and advection terms of the ageostrophic components are neglected in the viscous S–E equation (2.5), the solution $\varphi$ is proportional to the forcing divided by the viscous CSI stability. Because of this and the fact that the viscous CSI stability is very small, the rainband circulations in Figs. 5a–c and the rapid growth of the vertical velocity from 12.8 h to 13.3 h could be spuriously exaggerated. Besides, the rapid growth is also partially boosted by the positive feedback between the banded moist circulations and forcing anomalies (see section 4e). Soon afterward, as the dry subsidences develop between the moist bands and the negative MGPV is diluted (mainly due to the downward transport and mixing of the positive MGPV from the middle levels) in the lower part of the saturated region, the vertical velocity within the bands decreases to 6.5 cm s$^{-1}$ at 14.4 h, and bands I and II (see Fig. 5b) merge into one larger relatively weaker band (i.e., band II as shown for the later time at 16.7 h in Fig. 6b). At this time, the bands become relatively weak and the streamlines can no longer form closed loops around the bands (compare Fig. 6c with Fig. 5c). After 14.4 h, the intensity of the moist bands increases again (due to the lifting of a new negative MGPV air mass to saturation), but the growth is very slow and slightly oscillatory. By 16.7 h, bands III and IV merge into one large band (i.e., band IV in Fig. 6b), while a new band (i.e., band V in Fig. 6b) is formed on the warm side of the moist region. At 20.0 h, the vertical velocity in band IV reaches 8.6 cm s$^{-1}$, which is nearly as same as in band II. Then the vertical velocity within the bands increases slowly to 9.2 cm s$^{-1}$ at 21.1 h and then falls gradually to 8.6 cm s$^{-1}$ at 22.2 h. By 22.2 h, band II and band IV further merge into one large relatively weak band and the triple bands evolve into double bands (not shown). The slow decay of the multiple bands is caused mainly by the intrusion of positive MGPV in association with the boundary MGPV flux.

![Fig. 4. Initial MGPV, as in Fig. 1b, but for case II.](image_url)

![Fig. 5. (a) Vertical velocity, (b) saturated region (dark meshes), (c) streamlines (drawn every $10^3 \text{ m}^2 \text{s}^{-1}$) at 13.3 h for case II. The contours in (a) are drawn with power increasing intervals ($\pm 2^a \times 0.5 \text{ cm s}^{-1}$) as in Fig. 2a.](image_url)
on the warm side of the front (see Fig. 7d and later discussion).

When the moist ascent breaks into bands, the heating GPV flux $F_h$ is maximally intensified along the bands and, soon afterward, strong banded GPV anomalies are produced. At 13.6 h, the positive GPV anomaly in the lower corner region of band II jumps about ten times and reaches a maximum. At the same time the downgradient viscous GPV flux also reaches a maximum. After this surging development, the heating GPV flux decays more rapidly than the viscous GPV flux, so the GPV anomaly decreases to $2.3 q_0$ at 16.7 h (Fig. 7a). The two positive GPV anomaly bands in Fig. 7a are associated with band II and band IV in Fig. 6b,
respectively. The GPV anomaly dipole along the lower boundary and the associated boundary GPV flux in Fig. 7b are similar to that in Fig. 3b and can be explained by the same analysis in the paragraph associated with (4.2). Figure 7c shows that at 16.7 h the \( \theta_s \) surfaces have moved down slightly along the moist bands, which is consistent with the downward (cross \( \theta_s \) surface) GPV anomaly flux along the bands. After 16.7 h, the saturated region is gradually replenished with the air of negative MGPV from the lower and warm side, the upgradient heating GPV flux becomes slightly dominant over the downgradient viscous GPV flux and the GPV anomalies start to increase again. By 22.2 h the GPV anomaly increases to 3.3\( \theta_0 \) (not shown).

Figure 7d shows the MGPV and viscous MGPV flux at 16.7 h. By this time, the upward incoming MGPV flux from the lower boundary has produced an area of positive MGPV ahead of the surface front. By 22.2 h (not shown) this positive MGPV area is further advected onto the front with two patches of positive MGPV intruded into the region of negative MGPV around the lower corner of the saturated region. Here the boundary MGPV flux and generation of positive MGPV in the boundary layer are similar to that in Fig. 3d. However, unlike the case in Fig. 3d, the area of negative MGPV in Fig. 7d is much larger and has raised into the moist region. Even for the later time, for example, by 22.2 h (not shown), the area of negative MGPV is still much larger than that in Fig. 3d, though the negative area is reduced and “eroded” by the intrusion of positive MGPV patches along and above the front. During the period from 13.3 h to 22.2 h, the area of negative MGPV overlaps the lower portion of the saturated region (Fig. 6b), but the air remains stable to the viscous CSI everywhere. Thus, the multiple rainbands are produced by the combined effects of geostrophic forcing and weak viscous CSI stability. This agrees with the theoretical results of Xu (1989b), except that the multiple bands are significantly narrower than the previously obtained. As mentioned in the introduction, the multiple bands in Xu (1989b) were obtained diagnostically from smooth forcings. The multibanded geostrophic circulation obtained here can give a feedback to the thermal wind \( (v_z, \theta_z) \) and produce multibanded substructures in the geostrophic forcing field (Fig. 8), though the forcing is initially smooth. Comparing Fig. 8 with Fig. 6b, we can see that the narrow (meso-\( \beta \)-scale) band of maximum forcing is along the cold side of the moist region and gives a positive feedback to band II, while another weaker forcing band is along the cold side of band IV and gives a positive feedback to band IV. These feedbacks, in addition to the negative MGPV, may explain why these bands are narrower than those in Xu (1989b) and why the individual bands (II and IV) can last for nearly 10 hours (from 13.3 h to 22.2 h). The detailed feedback processes are examined in section 4e.

c. Case III—the effect of preexisting GPV anomalies

The initial thermal field is given by \( \theta_s = \theta_i + \theta_m \). Here \( \theta_i \) is the same as in (4.1a–b) for case I, while \( \theta_m \) is the superimposed mesoscale anomaly

\[
\theta_m = \Delta \theta_m (x - x_0)/L_0 \exp[-(x - x_0)^2/L_0^2] h(z),
\]

(4.4)

where \( \Delta \theta_m = -2 K, x_0 = 70H, L_0 = 20H, \) and \( h(z) \) is unity from \( z = 0 \) to \( z = H/4 \), then decreases linearly to zero at \( z = H/2 \). This mesoscale anomaly field is similar to, but twice as strong as, that in (17) of Cho and Chan (1990). The initial \( v_z \) field is obtained via the thermal wind relationship in association with the above thermal field \( \theta_z = \theta_i + \theta_m \). The corresponding mesoscale anomalies in the initial forcing \( Q \) and GPV fields are shown in Figs. 9a,b. The initial \( \theta_z \) field is the same as in case I, with no mesoscale anomaly. Thus, as shown in Fig. 9c, the mesoscale anomaly in the initial MGPV field is relatively weak.

In response to the mesoscale forcing anomaly, the initial ageostrophic circulation also has a mesoscale anomaly (not shown). Before the air becomes saturated, the ageostrophic circulation develops slowly and smoothly, but the mesoscale anomaly does not grow proportionally because the viscous effect is stronger for smaller-scale anomalies. As the air becomes saturated, rainbands are formed within the subsynoptic-scale moist ascent and the mesoscale anomaly is enhanced in the vertical motion field (Fig. 10a). However, similar to the results of Cho and Chan (1991), these bands are seen as weak cores of upward motion surrounded by the synoptic-scale ascent rather than separated by mesoscale dry subsidence (compare with Figs. 5–6). As the geostrophic deformation flow stretches/squeezes the bands along/toward the front, the moist bands become more densely packed and subject to stronger eddy diffusion. At 16.7 h, the two bands merge into a larger single band (Fig. 10b) and the MGPV field becomes nearly the same as in Fig. 3d for case I, though the mesoscale anomaly remains identifiable in the geostrophic forcing field (Fig. 11). After this point, the subsequent evolution is very similar to that in case I.
Fig. 9. Initial (a) geostrophic forcing, (b) GPV anomalies, and (c) MGPV for case III. The contours are drawn as in Figs. 1a,b.

Fig. 10. Vertical velocity, as in Fig. 2a, but for case III at (a) 11.1 h and (b) 16.7 h.

d. Case IV—the effect of preexisting MGPV anomalies

The initial thermal wind \((v_x, \theta_e)\) and dry GPV fields are the same as in case III. The initial relative humidity (rather than \(\theta_n\)) is the same as in (4.1c) for case I, so there is a preexisting mesoscale anomaly in the \(\theta_e\) field. The preexisting MGPV anomaly is slightly stronger than in case III (compare Fig. 12 with Fig. 9c).

Before the air becomes saturated, the ageostrophic circulation develops slowly and smoothly, just as in case III. As the air becomes saturated, rainbands are formed more distinctly than in case III (compare Fig. 13a with Fig. 10a). At 16.7 h, double bands take shape, but these bands are still seen as weak cores of upward motion surrounded by the synoptic-scale ascent (Fig. 13b). At this time, not only is the mesoscale-forcing anomaly stronger than in case III (compare Fig. 14a with Fig. 11), but also the mesoscale dry and moist GPV anomalies are distinct (Figs. 14b–c). The total GPV anomaly flux and \(\theta_e\) contours at 16.7 h (not shown) are similar to Fig. 7c in that the \(\theta_e\) surfaces have moved down slightly along the moist bands, which again is consistent with the downward GPV anomaly flux along the bands. After 16.7 h, however, the bands dissipate gradually. Stretched/squeezed by the geostrophic deformation flow along/toward the front, the

Fig. 11. Geostrophic forcing, as in Fig. 8, but for case III at 16.7 h.
moist bands become more densely packed and subject to stronger eddy diffusion. By 22.2 h, the two moist ascents have merged into a large single band (not shown) and the geostrophic forcing and MGPV field have become nearly the same as in case I or case III. Clearly, it is the preexisting $\theta_w$ anomaly and stronger MGPV anomaly that leads to the stronger and longer-lasting development of the moist bands in this case than in case III. A further interpretation of the difference between case III and case IV involves the effect of moisture anomalies on the feedback between mesoscale rainbands and forcing anomalies, which is examined in the next subsection.

![Graphical representation](image)

**Fig. 12.** Initial MGPV, as in Fig. 1b, but for case IV.

![Graphical representation](image)

**Fig. 14.** (a) Geostrophic forcing, (b) GPV anomalies and heating GPV flux, and (c) MGPV and viscous MGPV flux at 16.7 h for case IV. The contours in (a)–(c) are drawn as in Figs. 1a, 3a, and 3d, respectively.

e. **Feedback between mesoscale rainbands and forcing anomalies**

A feedback mechanism between latent heating and forcing anomalies was proposed by Cho and Chan (1991). The arguments consisted of two parts: (i) a mesoscale warm core in a geostrophic confluenve flow produces a dipole forcing anomaly that tends to lift the warm core (Fig. 15) and (ii) the upward motion enhances the latent heating. However, the problem is that an upward-motion anomaly may not necessarily produce an anomaly of net warming, although it enhances the latent-heating anomaly. Note that $-w_d q_w$ is
Fig. 15. Schematic showing how a warm anomaly (shaded area) induces a dipole forcing anomaly (indicated by + and −) in the geostrophic confluent flow (\( a = -\delta \alpha \phi \)) and how this dipole forcing anomaly produces a pair of ageostrophic circulations that tends to lift the warm anomaly. Note that the circulations are indicated by streamlines and the streamfunction \( \psi \) is proportional to the forcing \( Q = \alpha \partial_\phi \partial_x \).

the net cooling (or warming if \( q_w < 0 \)) produced by a moist ascent with vertical velocity \( w_z \) (\( > 0 \)) in SG space (which is slantwise along the momentum surface in real space). When the MGPV is positive (\( q_w > 0 \)), the upward motion should produce cooling and give a negative feedback to the warm core. The situation is similar to the negative feedback between the vertical motion and middle-level mesoscale temperature anomalies in the dry cases of Chan and Cho (1989). Thus, a preexisting warm anomaly should also decay with time in the moist region of positive MGPV, though the decay may be much slower than in the dry region. This is seen in our cases III and IV after the warm anomalies have risen into the saturated region. This could also be seen in the inviscid model of Cho and Chan (1991) if the simulations were carried through long enough.

During the period when a preexisting warm anomaly is rising through the lower boundary of the saturated region or, say, cloud-base boundary (CBB), a positive feedback may occur between vertical motion and warming anomalies, even if the MGPV is positive. The situation is shown schematically in Fig. 16a. As shown, the upward motion induced by a preexisting warm anomaly is stronger than the surrounding large-scale ascent (imagine that the circulations in Fig. 15 are embedded within a large-scale ascent), so higher humidity air is brought up from lower levels, which lowers the CBB locally. In this way, the CBB is undulated in association with the preexisting warm anomalies. Between the maximum and minimum levels of the undulated CBB, the moist ascents are associated with the warm anomalies and the adiabatic cooling is weaker in the moist ascents than in the surrounding dry ascents. As this adiabatic cooling effect is superimposed on the dominant horizontal warm advection, the net result is that the warm anomalies are enhanced as are the forcing anomalies. This feedback may explain why a preexisting temperature anomaly grows when it is rising through the CBB into the moist region, as observed in cases III and IV and in the results of Cho and Chan (1991). As shown in Fig. 17a, the doublewave structure in the warming rate field is weak and confined in the lowest part of the moist region in association with the two lowest points of the CBB, so the positive feedback is weak and occurs only in a partially saturated shallow layer between the high and low CBBs (indicated by dashed lines). Above this shallow layer,

![Diagram](https://via.placeholder.com/150)

Fig. 16. Schematics for different feedback situations. Moist regions are shaded. The large and small arrows indicate strong and weak vertical motion, respectively. MGPV is positive in (a) and (b), but negative in (d), and below the upper dashed line in (c). Weak positive feedback occurs between the maximum and minimum levels (dashed lines) of the undulated cloud-base boundary (CBB) in (a) or undulated cloud-top boundary (CTB) in (b). Strong positive feedback occurs in the moist region of negative MGPV (between the two dashed lines) in (c). Negative feedback occurs in (d).
A strong positive feedback can occur in a layer of negative MGPV and this feedback can be further enhanced if the CBB is undulated in association with the warm anomalies (Fig. 16c). As shown in Fig. 18, the warming anomalies are strong and nearly in phase with the vertical motion anomalies within a relatively deep layer of negative MGPV (Figs. 6a–c and 7d), so that the feedback between the vertical motion and warming anomalies is strong. This may explain the growth of the stable rainbands at 16.7 h in case II. However, if the mesoscale circulations around each warm anomaly are very intense (as in Figs. 5a–c), then the dry descents may produce stronger warming than the moist ascents. In this case, as shown schematically in Fig. 16d, the dry descent warming tends to diminish the existing feedback and change the anomaly structure. This, in addition to the effect of eddy diffusion, may explain the rapid decay and merger of bands I and II (from Fig. 5 to Fig. 6) in case II.

In connection with the GPV anomalies, the above feedbacks may be viewed as the following four steps of interaction.

(i) As rainbands are formed, the heating GPV flux is intensified along the bands.
(ii) The banded GPV flux produces positive GPV anomalies (Fig. 7a or 14b).
(iii) The GPV anomalies “induce” warm anomalies through the GPV invertibility [see (15) of Chan and Cho (1989)].
(iv) Each warm anomaly produces a dipole forcing anomaly that reinforces the moist band.

Although this GPV view cannot clearly distinguish between the two types of feedback in Figs. 16a and 16c, it may help us visualize the interaction between the rainbands, GPV anomalies, and forcing anomalies in association with the feedback process. These four steps of interaction also may be considered as a nonlinear, self-maintaining mechanism that is consistent with an existing (whatever generated) band structure.
5. Conclusions

In this paper, a viscous semigeostrophic (SG) model is developed in combination with the extended Sawyer-Eliassen equation of Xu (1989a,b) and used to study the formation and evolution of frontal rainbands in association with the dry and moist geostrophic potential vorticity (GPV) anomalies. The analytical formulation of this model retains the “conservation forms” for the dry and moist GPVs [see (3.1)–(3.4)], although the semi-Lagrangian numerical scheme used here (see appendix B) does not exactly maintain the conservation properties. Different types of well-posed boundary conditions are derived for this model in appendix A, but only the free-slip and thermal conductive boundary condition is used, which is the simplest type of boundary condition, similar to that of Nakamura and Held (1989).

Four simulations (cases I–IV) are presented and discussed in association with the effects of negative MGPV, boundary GPV flux, and preexisting dry and moist GPV anomalies. Regarding the scientific problems raised in the introduction section, the basic findings are summarized as follows:

- Viscous stable multiple rainbands can be generated internally from a smooth moist frontal circulation if the MGPV becomes negative in a moderately deep (>1–2 km) saturated layer. In addition to the earlier proposed band-generating mechanism based on the diagnostic results of the extended viscous S–E equation that emphasized the competition between forcing, negative MGPV, and eddy diffusion (Xu 1989b), a positive feedback mechanism is suggested here by our prognostic results. The idea is similar to that of Cho and Chan (1991) but operates only when the MGPV is negative (Figs. 15 and 16c). Because this feedback requires negative MGPV, multiple rainbands cannot be generated internally until the MGPV becomes negative. The new aspect here is that the large-scale moist ascent evolves into much finer multiple moist bands as soon as the positive feedback begins to generate banded substructures in the forcing and GPV fields.

- When MGPV is positive, multiple rainbands can only be generated externally by preexisting GPV anomalies or MGPV anomalies. These rainbands can be self-maintained by a weak feedback between the vertical motion and warming anomalies. This feedback operates in a layer between the maximum and minimum levels of an undulated cloud-base boundary (CBB) [or cloud-top boundary (CTB)] in association with the preexisting warm anomalies (Figs. 16a–b) and it will be enhanced/reduced if the relative humidity anomalies are positively/negatively correlated to the warm anomalies. The bands maintained by this feedback mechanism evolve slowly and their life periods depend very much on the preexisting GPV and MGPV anomalies. The bands are seen as weak cores of upward motion surrounded by the large-scale moist ascent rather than separated by mesoscale dry subsidences.

- Boundary-layer processes can produce either positive or negative GPV flux. Even with the simplest free-slip and thermal conductive boundary conditions, the boundary GPV flux and MGPV flux can be very strong in the vicinity of a surface front. In our case, positive GPV flux is produced mainly in association with the cooling process of the warm air ahead of the front, while negative GPV flux is produced mainly in association with the warming process of the cold air behind the front. Although these warming and cooling processes are caused by the assumed strong horizontal diffusion in the model, they may occur in a real atmospheric frontal region due to the vertical heat exchange as the cold (or warm) air moves over a warm (or cold) surface behind (or ahead of) the front. However, the free-slip boundary condition used in the model underestimates the boundary momentum flux and its contribution to boundary GPV flux in comparison with a real atmospheric situation. Surface moisture flux is completely suppressed in the model, so the boundary MGPV flux is more positive than the dry GPV flux. The strong positive MGPV flux ahead of the front causes the low-level negative MGPV region to gradually diminish. Surface moisture flux may have a strong impact on the boundary MGPV flux, but is not studied in this paper.

- Eddy viscosity plays an important role in the later evolution and final dissipation of the rainbands and GPV anomalies. For a strong GPV anomaly, the viscous GPV flux is generally downgradient [see (3.5)–(3.6)] and tends to dissipate the anomaly. As the negative MGPV area diminishes and the GPV anomalies are lifted into the saturated region, the later evolution of the bands is largely controlled by the Lagrangian advection and eddy dissipation, no matter how the bands were previously generated (either internally or externally). As the geostrophic confluence flow squeezes (stretches) the bands toward (along) the front, the fine structures of GPV anomalies are smoothed by eddy viscosity and multiband gradually “merge” into a larger single band of moist ascent. During this process, the bands may also be partially suppressed by a negative feedback between the bands and forcing anomalies within the moist layer, as the MGPV is positive and the mesoscale CBB-undulation is largely leveled off.

A point is made concerning the use of PV (or MPV) as a measure of SI (or CSI). It is the negative GPV (or MGPV), instead of negative PV (or MPV), that should be used as a measure of inviscid linear SI (or CSI), because the SI (or CSI) basic state has to satisfy the thermal wind relationship. Even though the real atmospheric flow is rarely in a balanced state, it still has
some advantages to use negative MGPV as a measure of CSI (see section 3). In addition to the use of negative GPV, here we would like to make a related point concerning the PV invertibility, which in combination with the PV conservation has made the isentropic PV thinking very useful in gaining direct insights into many important large-scale dynamic processes (Hoskins et al. 1985). [The invertibility is also helpful for understanding some mesoscale processes as shown in section 4e and by Chan and Cho (1989).] Again, it is only the geostrophic wind (or balance wind, such as gradient wind) that can be inverted from GPV (or balanced PV), so the PV invertibility really means GPV (or balanced-PV) invertibility and the PV thinking is really a GPV (or balanced-PV) thinking. GPV can be directly computed from the geostrophic geopotential height field, but it is not a simple task to filter the nongeostrophic geopotential height (Temperton 1988). Thus, it is much easier to examine the GPV dynamics in a SG model than in a primitive equation (PE) model.

Since only the linear terms of the ageostrophic front circulation are retained in our viscous SG model [see (2.3)], the inviscid CSI can be measured exactly by the negative MGPV. Besides, the viscous CSI stability can be checked precisely by the positiveness of the functional of the viscous S–E equation [see (A.3)], as the solution is computed by the variational method (Xu 1989a). These are the advantages of our model in comparison with either an inviscid SG model or PE model. However, like the inviscid SG model, our viscous SG model filters the inertial-gravity waves, so it is less complete than a PE model. In addition, our model becomes inaccurate when the viscous CSI stability is very small. Although one could argue how well this viscous SG model represents a full PE system, the inconsistency between the vertical and horizontal resolutions in many PE models for properly resolving internal gravity waves could also nullify the assumed advantage of PE models (Lindzen and Fox-Rabinovitz 1989). In this sense, a balanced model may still offer some advantages. Whether or not our viscous SG model represents a "slow manifold" of the viscous PE system for a moist atmosphere would be an interesting subject.

Acknowledgments. I am thankful to Dr. Alan Shapiro for reading the manuscript and to anonymous reviewers for their comments. The financial support was provided by the NOAA contract NA85-RAH05046 and NSF Grant ATM-8822782 at CIMMS, University of Oklahoma.

APPENDIX A

Variational Derivation of the Boundary Conditions

It was shown in Xu (1989a) that the boundary value problem of (2.5) with the homogeneous nonslip boundary conditions, that is,

\[
\psi = \partial_z \psi = D \partial_z \psi = 0 \quad \text{at} \quad z = 0, H \quad (A.1a-c)
\]

\[
\psi = \partial_x \psi = D \partial_x \psi = 0 \quad \text{at} \quad x = \pm L, \quad (A.2a-c)
\]

is equivalent to the variational principle \( \delta \Phi(\psi) = 0 \) for \( \psi \in W^0(\Omega) \), where \( \Omega \) is the domain in (A.1)–(A.2),

\[
\Phi(\psi) = \Phi_0(\psi) + 2 \int \int Q \psi dx dz, \quad (A.3)
\]

\[
\Phi_0(\psi) = \| \psi \|^2 + \frac{1}{2} \int \int \left\{ F^2(\partial_z \psi)^2 + N^2(\partial_x \psi)^2 \right\} dx dz,
\]

\[
= -2S^2 \partial_x \psi \partial_z \psi \right\} dx dz, \quad (A.4)
\]

and \( W^0(\Omega) \) is the space of functions satisfying (A.1)–(A.2) in association with the norm defined in (A.5). The nonslip boundary conditions in (A.1)–(A.2) are well posed as the existence, uniqueness, and (viscous CSI) stability of the solution of (2.5) and (A.1)–(A.2) are ensured by the positiveness of the functional \( \Phi_0(\psi) \) in (A.4) (see Theorems 1–3 of Xu 1989a).

Now we can derive different types of well-posed boundary conditions for Eq. (2.5) from the variational principle. First, homogeneous boundary conditions are considered. By using integration by parts, we can show that

\[
\delta \Phi(\psi) = -\int \int \left\{ L_0 \psi - 2Q \right\} \delta \psi dx dz
\]

+ boundary variation terms, \quad (A.6)

where

\[
L_0 = \partial_x (D^2 + F^2) \partial_x - \partial_x S^2 \partial_x - \partial_x ^2 \partial_x S^2 + \partial_x (D^2 + N^2) \partial_x
\]

is the differential operator in (2.5); the boundary variation terms are

\[
\delta B_{12} = \pm \int \int \nu_2 (\partial_x \psi)(-L_2 \psi) dx \quad \text{at} \quad z = 0, H,
\]

\[
\delta B_{13} = \pm \int \int \nu_2 (\partial_x \psi)(D \partial_x \psi) dx \quad \text{at} \quad z = 0, H,
\]

\[
\delta B_{21} = \pm \int \int \delta \psi (L_1 \psi) dz \quad \text{at} \quad x = \pm L,
\]

\[
\delta B_{22} = \pm \int \int \nu_1 (\partial_x \psi)(-L_2 \psi) dz \quad \text{at} \quad x = \pm L,
\]

\[
\delta B_{23} = \pm \int \int \nu_2 (\partial_x \psi)(D \partial_x \psi) dz \quad \text{at} \quad x = \pm L;
\]

and \( L_1 = (D^2 + N^2) \partial_x - S^2 \partial_x, L_2 = \partial_x D \partial_x + \partial_x D \partial_x \). In the derivation of (A.6), the boundary conditions are undetermined except for the nonpenetrative condition \( \psi = 0 \) at \( z = 0 \) and \( H \). The variational principle
\( \delta \Phi(\psi) = 0 \) indicates that the well-posed homogeneous boundary conditions are
\begin{align*}
\psi &= 0 \quad \text{at } z = 0, H \\
\partial_z \psi &= 0 \quad \text{or } L_2 \psi = 0 \quad \text{at } z = 0, H, \quad (A.8a,b) \\
\partial_z^2 \psi &= 0 \quad \text{or } D \partial_z \psi = 0 \quad \text{at } z = 0, H, \quad (A.9a,b) \\
\psi &= 0 \quad \text{or } L_1 \psi = 0 \quad \text{at } x = \pm L, \quad (A.10a,b) \\
\partial_x \psi &= 0, \quad \text{or } L_2 \psi = 0 \quad \text{at } x = \pm L, \quad (A.11a,b) \\
\partial_x^2 \psi &= 0 \quad \text{or } D \partial_x \psi = 0 \quad \text{at } x = \pm L. \quad (A.12a,b)
\end{align*}

The six boundary conditions can be chosen from (A.8)–(A.12) in different combinations, but each of (A.8)–(A.12) has to give one and only one condition. All (a) equations in (A.8)–(A.10) are boundary conditions of the first type (similar to the Dirichlet boundary condition for a Poisson equation). These boundary conditions can be specified explicitly as constraints on the functional space \( W^0(\Omega) \). All (b) equations in (A.8)–(A.10) are boundary conditions of the second type (similar to the Neumann boundary condition for a Poisson equation). They contain higher order derivatives and cannot be specified explicitly as constraints on the functional space \( W^0(\Omega) \) because the generalized solutions in \( W^0(\Omega) \) are not smooth enough. These (second-type) conditions need to be imposed on the solution implicitly through the boundary variation terms in (A.6). For a homogeneous second-type boundary condition, the corresponding boundary variation term is zero. Thus, only inhomogeneous second-type boundary conditions enter the variational formulation as boundary-forcing terms [see (A15) and (A17)].

The physical meanings of the conditions (A.7)–(A.12) can be easily seen from
\begin{align*}
\partial_z \psi &= -u, \quad \partial_x \psi = w, \\
D \partial_z \psi &= f v, \quad D \partial_x \psi = -(g/\Theta_0) \theta, \\
L_2 \psi &= f \partial_z v - (g/\Theta_0) \partial_x \theta, \\
L_1 \psi &= S^2 u + N^2 w - (g/\Theta_0) D \theta, \quad (A.13)
\end{align*}

where (2.3a–d) are used. The homogeneous nonslip boundary conditions in (A.1) correspond to (A.7), (A.8a), (A.9b), (A.10a), (A.11a), and (A.12b), while the free-slip boundary conditions (2.6a–b) correspond to (A.7), (A.8b), (A.9a), (A.10a), (A.11b), and (A.12a).

The homogeneous nonslip boundary conditions (A.1a–c) assume that the ageostrophic wind \((u, w, v)\) vanishes on \( z = 0 \) and \( H \), or, equivalently, the lower and upper boundaries move with the geostrophic wind. If the lower boundary is fixed with the earth surface, then the following inhomogeneous condition [see (A.13) for their physical meanings],
\[ \partial_z \psi = u_\psi \quad \text{and} \quad D \partial_z \psi = -f (v_\psi + v'_\psi) \quad \text{at} \quad z = 0, \quad (A.14a-b) \]
should be used to replace \( \partial_z \psi = D \partial_z \psi = 0 \) at \( z = 0 \) in (A.1b–c). In this case, the following boundary-forcing term,
\[ BF_{13} = \int v_2 (\partial_z^2 \psi) f (v_\psi + v'_\psi) dx \quad \text{at} \quad z = 0, \quad (A.15) \]
should be added to the functional \( \Phi(\psi) \) in (A.3). The variation of this term, \( \delta BF_{13} \), cancels the boundary variation term \( \delta B_{13} \) in (A.6), so the variational principle \( \delta \Phi(\psi) = 0 \) is ensured. With the inhomogeneous boundary condition (A.14a–b), the solution can have (i) a classic Ekman boundary layer (EBL) in a region where the geostrophic flow is uniform and unidirectional, (ii) a complex EBL flow in the surface frontal region, or (iii) a very deep EBL in the region far from the front where \( |u_\psi| = |\alpha x| \) becomes very large. Because our computational domain is very large (\( L = 10^3 H = 10^4 \) km), the last property (iii) with (A.14a–b) rendered the far-field flow unrealistic (not shown).

It is reasonable to expect that (A.14a–b) can produce more realistic results than both (A.1b–c) and (2.6a) if the domain is not too large and/or the constant eddy viscous coefficients \( v_1, v_2 \) are replaced by a more realistic eddy parametrization scheme (e.g., Xu 1988).

Note that \( L_1 \psi \) is the same as the left-hand side of (2.3d), so (A.10b) cannot be satisfied unless the right-hand side of (2.3d) vanishes at \( x = \pm L \). In general, (A.10b) should be replaced by the following inhomogeneous condition:
\[ L_1 \psi = -(g/\Theta_0) [\partial_t + D_\theta - D] \theta \quad \text{at} \quad x = \pm L. \quad (A.16) \]

In this case, the following boundary-forcing term,
\[ BF_{21} = \pm \int \psi (g/\Theta_0) [(\partial_t + D_\theta - D) \theta] dz \]
\[ \text{at} \quad x = \pm L, \quad (A.17) \]
should be added to the functional \( \Phi(\psi) \) in (A.3). The variation of this term, that is, \( \delta BF_{21} \), cancels the boundary variation term \( \delta B_{21} \) in (A.6), so, again, the variational principle \( \delta \Phi(\psi) = 0 \) is ensured. Note that (A.16) is an open lateral boundary condition. For an inviscid flow, \( L_1 \) reduces to \( N^2 \partial_z - S^2 \partial_z \), and (A.16) reduces to an open lateral boundary condition for the conventional S–E equation.

**APPENDIX B**

Semi-Lagrangian Numerical Scheme

The computational procedures for the model’s time integration are as follows:
(i) Compute $\psi$ diagnostically from (2.5)–(2.6) at the time level of, for example, $t = n \Delta t$.
(ii) By using the bicubic-spline basis functions, construct the cross-front ageostrophic wind fields $(u, w) = (-\partial_\psi, \partial_\psi)$ at $n \Delta t$.
(iii) Compute $(v, \theta_g) = D(u/v, w/\theta_0/g)$ from (2.3a,c) at $n \Delta t$ (the central finite-difference scheme is used for those higher order spatial derivatives of $\psi$ that are beyond the description of the bicubic-spline basis functions).
(iv) By using the central finite-difference scheme, compute the diffusion terms $D(v_g + v), D(\theta_g + \theta)$, and $D\theta_w$ at $n \Delta t$.
(v) By using the semi-Lagrangian method with bilinear upstream interpolation scheme (Bates and McDonald 1982), compute $(v_g, \theta_g, \theta_w)$ for the next time level $(n + 1) \Delta t$ from (2.3b,d) and (2.7). Here the diffusion terms obtained in (iv) are treated explicitly, that is, fixed from time level $n \Delta t$ to $(n + 1) \Delta t$. During the Lagrangian advection computation, the conditional moist thermal stratification $N^2$ in (2.3d) is set to the dry value $N_d^2$. Then, after each step of advection, the parcel supersaturation, if any, is adjusted to saturation by using the method of Soong and Ogura (1973). Since all condensations are removed as rain from the model, the computation of latent heating is consistent with the idealized formulation, $(N_d^2 - \gamma N^2)w$, implied by (2.3d) and (2.4).
(vi) Using the finite difference form of the thermal wind relationship, compute the thermal wind $v_g$ from $\theta_g$, check the difference (error) between this thermal wind and the $v_g$ obtained in (v), and use the thermal wind to replace the $v_g$ obtained in (v). Here, as the thermal wind is computed from the vertical integration of $\partial_\theta \theta_g$, the middle level $v_g$ at $(z = H/2)$ obtained in (v) is used. The differences (errors) between the two $v_g$ fields obtained here and in (v) are found generally smaller than 5% of the maximum value of $v_g$ in most cases, except for the development period of the fine multiple bands in case II.
(vii) Compute the moist geostrophic flow parameters in (2.1) at $(n + 1) \Delta t$ and go to (i).

The semi-Lagrangian scheme is absolutely stable for the advection terms (Bates and McDonald 1982), but the explicit scheme for the diffusion terms is stable only if

$$\Delta t \leq \min \left[ 0.5/(v_1/\Delta x^2 + v_2/\Delta z^2) \right].$$

(B.1)

Here (B.1) is the two-dimensional version of (5.133) of Haltiner and Williams (1979). With the parameter values $\min \Delta x = 22$ km, $\min \Delta z = 0.21$ km, $v_1 = 10^3$ $\text{m}^2\text{s}^{-1}$, and $v_2 = 10^1$ $\text{m}^2\text{s}^{-1}$, (B.1) estimates $\Delta t \leq 1155$ s. As a verification, it is found that the numerical runs can (or cannot) remain computationally stable as $\Delta t = 1000$ s (or $\Delta t = 2000$ s).

According to McCalpin (1988), the numerical dissipation inherent in the semi-Lagrangian advection can be estimated by the following equivalent "eddy" coefficients:

$$\kappa_1 = 0.25(\Delta x^2/\Delta t) \quad \text{and} \quad \kappa_2 = 0.25(\Delta z^2/\Delta t)$$

(B.2)
in the horizontal and vertical, respectively. Obviously, the smaller the $\Delta t$, the larger the numerical diffusion. To reduce the numerical diffusion, $\Delta t = 1000$ s (0.28 h) is chosen, which is close to the maximum value allowed by (B.1). In this case, (B.2) estimates $(\kappa_1, \kappa_2) \approx 0.5(v_1, v_2)$. Clearly the numerical diffusion is not very small, though smaller than the "physical diffusion." This numerical diffusion, in addition to the numerical truncation errors, could also have increased the differences (errors) between the two $v_g$ fields, respectively, computed in the above (v) and (vi). When a smaller $\Delta t$ is used, the time truncation error is reduced, but the numerical diffusion is increased and the net result is that the differences (errors) between the two $v_g$ fields computed in (v) and (vi) are increased. Thus, with the above numerical scheme and parameter settings, choosing $\Delta t = 1000$ s seems to give a nearly optimum error-control and high computational efficiency. In this case, the time truncation error is also more consistent with the spatial truncation error than in the cases of smaller $\Delta t$.

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