NOTES AND CORRESPONDENCE

A Generalization of Bernoulli's Theorem

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1 January 1992 and 10 July 1992

ABSTRACT

The conservation of potential vorticity Q can be expressed as $\partial(\rho Q)/\partial t + \nabla \cdot \mathbf{J} = 0$, where $\mathbf{J}$ denotes the total flux of potential vorticity. It is shown that $\mathbf{J}$ is related under statistically steady conditions to the Bernoulli function B by

$$\mathbf{J} = \nabla\theta \times \nabla B,$$

where θ is the potential temperature. This relation is valid even in the nonhydrostatic limit and in the presence of arbitrary nonconservative forces (such as internal friction) and heating rates. In essence, it can be interpreted as a generalization of Bernoulli's theorem to the frictional and diabatic regime. The classical Bernoulli theorem—valid for inviscid adiabatic and steady flows—states that the intersections of surfaces of constant potential temperature and constant Bernoulli function yield streamlines. In the presence of frictional and diabatic effects, these intersections yield the flux lines along which potential vorticity is transported.

1. Introduction

It is now widely recognized that potential vorticity dynamics is a very illuminating and powerful tool for studying stratified flows on a range of scales (see the review of Hoskins et al. 1985). The justification for this approach has traditionally rested upon the inviscid and adiabatic framework, where the potential vorticity Q is conserved following the motion of the flow (Ertel 1942). As pointed out by Haynes and McIntyre (1987, 1990), however, there is a generalized conservation-theorem for the "potential vorticity substance" the quantity whose mixing ratio is Q, which remains valid in the presence of arbitrary diabatic and frictional processes. This theorem states that

$$\frac{\partial}{\partial t}(\rho Q) + \nabla \cdot \mathbf{J} = 0, \quad (1)$$

where $\mathbf{J}$ is the total flux of potential vorticity (or more precisely, the total flux of potential vorticity substance). Surfaces of constant potential temperature are impermeable to this flux, and it follows that the quantity ρQ cannot be created or destroyed in the interior of the flow but only where the isentropic surfaces intersect the ground. These general properties of stratified flows have proved useful in various applications, such as the study of the exchange of mass and potential vorticity between stratosphere and troposphere (Holton 1990) and across the thermocline in the oceanic context (Marshall and Nurser 1992).

In this note, a further general property of stratified dynamics is discussed. It will be shown that there is a generalized form of Bernoulli's theorem, which remains fully valid in the presence of diabatic heating and frictional forces.

Consider first a steady, inviscid, and adiabatic flow. This is the framework of the classical Bernoulli theorem,[1] which can be expressed as

$$\frac{DB}{Dt} = 0. \quad (2)$$

It states that the Bernoulli function is constant along streamlines. The Bernoulli function is given by

$$B = h + \frac{1}{2}\mathbf{u}^2 + \Phi \quad (3)$$

and can for most atmospheric applications be approximated as (see Gill 1982)

$$B = c_pT + \frac{1}{2}\mathbf{u}^2 + gz. \quad (4)$$

Here D/Dt is the derivative following the flow, Φ is the geopotential (or, more generally, the potential of

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[1] An early version, valid for homogeneous incompressible flow, took shape through the work of Daniel Bernoulli, Johannes Bernoulli, and Leonhard Euler in the years 1730–1760. Further historical background and an overview on the numerous forms of Bernoulli's theorem can be found in Truesdell (1954). A translation of some of the works of the Bernoullis is given in Bernoulli and Bernoulli (1968).

all conservative body forces), and $\mathbf{u}$ denotes the three-dimensional velocity vector. The quantity $h = e + p/\rho$ is the specific enthalpy with e, ρ, c_p, and p referring to the specific internal energy, the density, the specific heat at constant pressure, and the pressure, respectively. Together with the material conservation of the potential temperature θ, the Bernoulli law implies that both B- and θ-surfaces are impermeable to the flow. It follows that steady-state streamlines can be obtained by intersecting surfaces of constant potential temperature and constant Bernoulli function. The vector

$$\nabla\theta \times \nabla B \tag{5}$$

is hence aligned with the streamlines. This situation is indicated in Fig. 1a.

In the following, we relax the restrictions of adiabatic and inviscid flow and allow for arbitrary diabatic and frictional processes. Under these conditions, there is a simple steady-state relation between the total flux of potential vorticity (referred to as $\mathbf{J}$ by Haynes and McIntyre) and the Bernoulli function. The shallow-water form of this theorem has been derived and utilized by Schär and Smith (1993), and the continuously stratified form is given by

$$\boxed{\mathbf{J} = \nabla\theta \times \nabla B.} \tag{6}$$

This equation is valid in the presence of arbitrary forces and heating functions including eddy flux divergences and only requires the presence of a statistical steady state. It can be interpreted as a generalized form of Bernoulli's theorem. In the presence of diabatic and frictional effects, surfaces of constant potential temperature and of constant Bernoulli function are no longer aligned with the streamlines, as indicated in Fig. 1b. The total flux of potential vorticity does, however, remain aligned with the intersections of θ- and B-surfaces.

The plan for this note is as follows: section 2 contains the derivation of the generalized Bernoulli theorem (6)

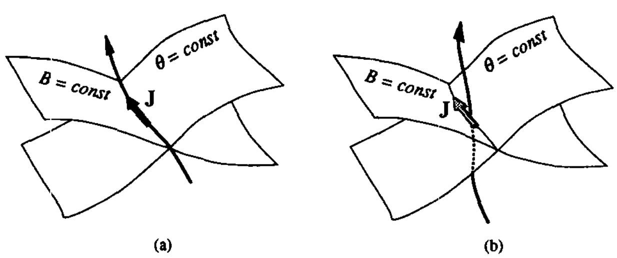

FIG. 1. (a) For steady, inviscid, and adiabatic flow, Bernoulli's theorem requires that the Bernoulli function B is constant along streamlines. It follows that streamlines can be obtained by intersecting B and θ surfaces. In the generalized form, valid for statistically steady, diabatic, and dissipative flows, the streamlines are no longer aligned with the θ surfaces. The total flux $\mathbf{J}$ of potential vorticity remains aligned, as indicated in panel (b). In panel (a), the flux of potential vorticity must be entirely advective, and this is indicated by a black $\mathbf{J}$-vector. On the other hand, the flux in (b) has a nonadvective (i.e., dissipative, turbulent, or diabatic) component, and this is indicated by the grey shading of the vector.

in z and θ coordinates, respectively. In these derivations the steady-state assumption is avoided. The results are applied in section 3 to the dynamics of gravity wave breaking, and some further remarks follow in section 4.

2. Bernoulli function and potential vorticity flux

a. z Coordinates

Consider a compressible, nonhydrostatic fluid in a rotating environment. The governing equations can then be expressed as

$$\frac{D\mathbf{u}}{Dt} + 2\mathbf{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi + \mathbf{F}, \tag{7}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{8}$$

$$\frac{D\theta}{Dt} = \dot{\theta}. \tag{9}$$

Here $\mathbf{\Omega}$ is the angular velocity of the earth. The effects of dissipation and diabatic heating enter through the nonconservative force $\mathbf{F}$ and the diabatic heating rate $\dot{\theta}$. The potential vorticity relation of this system is traditionally expressed as

$$\frac{DQ}{Dt} = \frac{1}{\rho}[\boldsymbol{\omega}\cdot\nabla\dot{\theta} + (\nabla\times\mathbf{F})\cdot\nabla\theta], \tag{10}$$

where $Q = \rho^{-1}\nabla\theta\cdot\boldsymbol{\omega}$ is the potential vorticity, and $\boldsymbol{\omega} = 2\mathbf{\Omega} + \nabla\times\mathbf{u}$ the absolute vorticity. As discussed by Haynes and McIntyre (1987, 1990), this equation can always be cast into the conservative flux form (1). The total flux of potential vorticity $\mathbf{J}$ is given as the sum of advective and nonadvective fluxes, that is,

$$\mathbf{J} = \mathbf{u}\rho Q + \mathbf{J}_N, \tag{11}$$

where the nonadvective contribution

$$\mathbf{J}_N = -\boldsymbol{\omega}\dot{\theta} - \mathbf{F}\times\nabla\theta \tag{12}$$

essentially replaces the source term in (10).

We begin the proof of (6) by casting the momentum equation (7) into the form

$$\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla_p - \frac{1}{2}\nabla(\mathbf{u}^2) - \nabla\Phi + \mathbf{F}. \tag{13}$$

Next, the pressure term is split using a version of Gibbs' equation,

$$\nabla p = \rho\nabla h - \rho T\nabla s, \tag{14}$$

where s and h denote the entropy and enthalpy per unit mass, respectively. Substituting (14) into (13) gives an alternative form of the momentum equation, that is,

$$\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u} = T\nabla s - \nabla B + \mathbf{F}, \tag{15}$$

where the Bernoulli function is as defined in (3). The foregoing derivation is standard (see, e.g., Batchelor 1967, section 3.5). We continue by taking $\nabla\theta \times (15)$. This yields

$$\nabla\theta \times \frac{\partial \mathbf{u}}{\partial t} + \nabla\theta \times (\boldsymbol{\omega} \times \mathbf{u}) = \nabla\theta \times T\nabla s - \nabla\theta \times \nabla B + \nabla\theta \times \mathbf{F}. \quad (16)$$

The first term on the right-hand side disappears since $\nabla\theta \| \nabla s$. The second term to the left is further rearranged using the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ to obtain

$$\nabla\theta \times \frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega}(\nabla\theta \cdot \mathbf{u}) - \mathbf{u}(\nabla\theta \cdot \boldsymbol{\omega}) = -\nabla\theta \times \nabla B + \nabla\theta \times \mathbf{F}. \quad (17)$$

From the thermodynamic equation (9) we have $\nabla\theta \cdot \mathbf{u} = \dot{\theta} - \partial\theta/\partial t$ and this is substituted into the second term of (17). After minor rearrangements, one obtains

$$\nabla\theta \times \left(\nabla B + \frac{\partial \mathbf{u}}{\partial t}\right) - \boldsymbol{\omega}\frac{\partial \theta}{\partial t} = \mathbf{u}(\nabla\theta \cdot \boldsymbol{\omega}) - \boldsymbol{\omega}\dot{\theta} - \mathbf{F} \times \nabla\theta. \quad (18)$$

Next, it is noted that the first term on the right-hand side is the advective flux of potential vorticity, whereas the other terms to the right combine to the nonadvective flux [cf., (11), (12)]. This is expressed as

$$\mathbf{J} = \nabla\theta \times \left(\nabla B + \frac{\partial \mathbf{u}}{\partial t}\right) - \boldsymbol{\omega}\frac{\partial \theta}{\partial t}. \quad (19)$$

Equation (19) establishes a surprisingly simple relation between the total flux of potential vorticity and the other dynamical fields. At steady state, it reduces to (6) and states that *the total flux of potential vorticity is aligned with both Bernoulli and isentropic surfaces.* The transport of potential vorticity is generally not along the flow (cf., Fig. 1b), and the **J**-vectors are hence associated with a nonadvective contribution.

b. Isentropic coordinate version

A particularly transparent derivation is possible within the hydrostatic isentropic coordinate framework. Following Haynes and McIntyre (1987), the momentum equation, the hydrostatic equation, and the continuity equation can be expressed as

$$\frac{D\mathbf{v}}{Dt} + \dot{\theta}\frac{\partial \mathbf{v}}{\partial \theta} + f\mathbf{k} \times \mathbf{v} = -\nabla M + \mathbf{F}, \quad (20)$$

$$g\sigma = -\frac{\partial p}{\partial \theta}, \quad (21)$$

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot (\mathbf{v}\sigma) + \frac{\partial(\dot{\theta}\sigma)}{\partial \theta} = 0. \quad (22)$$

Here $M = \Phi + h$ is the Montgomery potential, $\mathbf{v} = (u, v)$ denotes the horizontal wind vector, f is the Coriolis parameter, $\mathbf{k}$ the vertical unit vector, and σ the mass density in (x, y, θ)-space. All derivatives are taken on isentropic surfaces with $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ and $\nabla = (\partial/\partial x, \partial/\partial y)$.

The horizontal momentum equation is arranged as

$$\left(\frac{\partial}{\partial t} + \dot{\theta}\frac{\partial}{\partial \theta}\right)\mathbf{v} + \mathbf{k} \times \mathbf{v}\xi_{a\theta} = -\nabla\left(\Phi + h + \frac{1}{2}\mathbf{v}^2\right) + \mathbf{F}, \quad (23)$$

where

$$\xi_{a\theta} = f + \mathbf{k} \cdot (\nabla \times \mathbf{v}) \quad (24)$$

is the absolute isentropic vorticity. Inferring from (23), the isentropic Bernoulli function B_θ is defined as

$$B_\theta = \Phi + h + \frac{1}{2}\mathbf{v}^2 = M + \frac{1}{2}\mathbf{v}^2. \quad (25)$$

It differs from the full Bernoulli function (3) by the contribution $1/2 w^2$ arising from the vertical motion. It can next be verified that the momentum equation can be expressed as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{k} \times \mathbf{J}_\theta = -\nabla B_\theta, \quad (26)$$

where

$$\mathbf{J}_\theta = \xi_{a\theta}\mathbf{v} - \mathbf{k} \times \dot{\theta}\frac{\partial \mathbf{v}}{\partial \theta} + \mathbf{k} \times \mathbf{F} \quad (27)$$

is the isentropic version of the total flux of potential vorticity (cf., Haynes and McIntyre 1987). Equation (26) is of significant conceptual interest since the potential vorticity flux now appears in the momentum equation. After taking $\nabla \times (26)$ and using $\sigma Q = \xi_{a\theta}$, one obtains

$$\frac{\partial \sigma Q}{\partial t} + \nabla \cdot \mathbf{J}_\theta = 0, \quad (28)$$

that is, the isentropic form of the generalized PV equation in flux form (Haynes and McIntyre 1987). Taking $\mathbf{k} \times (26)$ yields

$$\mathbf{J}_\theta = \mathbf{k} \times \left(\nabla B_\theta + \frac{\partial \mathbf{v}}{\partial t}\right), \quad (29)$$

that is, the isentropic form of the generalized Bernoulli theorem (19). The simplification gained through the isentropic framework is obvious at steady state, where

$$\mathbf{J}_\theta = \mathbf{k} \times \nabla B_\theta \quad (30)$$

states that the *isentropic Bernoulli function simply becomes the streamfunction of the total flux of potential vorticity on isentropic surfaces.*

c. Discussion

Relation (19), (29) and their steady state versions (6), (30) appear well suited for the diagnostic computation of **J** from numerical and observational data. It is of particular importance that the validity of (6), (30) does not require a perfect steady state; a statistical steady state is sufficient. This follows provided the interaction of the unresolved components with the mean flow can formally be described through the terms **F** and $\dot{\theta}$ in (7) and (9) and (20) and (22), respectively. In such a situation, the governing equations describe the time mean, and the computation of θ and B for application with (6) and (30) is based on the mean quantities. The **J** fluxes diagnosed in this way then include contributions that arise from unresolved (e.g., turbulent wave–mean interaction) components of the flow, but the diagnosis does not require knowledge of these processes. In essence, (6) and (30) yield the **J** fluxes that are required to maintain a steady state.

The isentropic version (30) appears particularly attractive for the analysis of synoptic- and large-scale atmospheric processes. The display of B_θ and M on isentropic surfaces for a statistically steady flow will immediately reveal information on the advective and nonadvective contributions to the potential vorticity budget. Transport of potential vorticity across surfaces of constant Montgomery potential can only be important where the variation of the kinetic energy term $\frac{1}{2}\mathbf{v}^2$ significantly contributes to that of the Bernoulli function (25).

d. The J flux across an area

Before proceeding, we derive a useful expression for the steady-state flux of potential vorticity through an arbitrary area A, which is enclosed by the contour ∂A. The integrated flux through A is

$$J^{(A)} = \iint_A \mathbf{J} \cdot d\mathbf{a} \qquad (31)$$

and can—using (6), the identity $\nabla\theta \times \nabla B = \nabla \times (\theta\nabla B) = -\nabla \times (B\nabla\theta)$, and the Stokes theorem—be expressed as

$$J^{(A)} = \oint_{\partial A} \theta\nabla B \cdot d\mathbf{s} = -\oint_{\partial A} B\nabla\theta \cdot d\mathbf{s},$$

where $d\mathbf{a}$ and $d\mathbf{s}$ denote area and line elements, respectively. The second of these expressions transforms into θ-coordinates as

$$J^{(A)} = -\oint_{\partial A} B d\theta. \qquad (32)$$

The isentropic framework discussed earlier gives virtually the same result. This indicates that it can in principle account for PV-fluxes associated with dissipative regions (in which the assumptions implied by isentropic coordinates can locally become invalid), provided that the effects of dissipation on the Bernoulli function are properly represented.

3. Application to gravity wave breaking

Internal gravity wave breaking in airflow past topography is an important aspect of to the mesoscale flow (Clark and Peltier 1977), and it can also influence the planetary-scale circulation (e.g., Palmer et al. 1986). It has been stated (Haynes and McIntyre 1987; Smith 1989; Staquet and Riley 1989) that three-dimensional gravity wave breaking will be associated in general with the turbulent production of potential vorticity anomalies, even if the upstream flow is fully irrotational. Such a process is only possible if the dissipative or turbulent flux of potential vorticity is upgradient (McIntyre and Norton 1990), and this issue has been debated in the literature (e.g., Danielson 1990; Keyser and Rotunno 1990). Using the previously developed Bernoulli arguments, it is possible to gain some insight into the dynamics of dissipative gravity wave breaking without resorting to detailed computations.

Here we consider statistically steady airflow past two- and three-dimensional topography. We will assume the absence of background rotation and a uniform incident flow, yielding an upstream profile of zero potential vorticity, that is, $Q = 0$. Under these assumptions, unperturbed B and θ surfaces are aligned horizontally and parallel to one another. The relative alignment of these surfaces is intrinsically linked through (6), with the total flux of potential vorticity, and the question whether dissipation will be associated with a nonvanishing flux of potential vorticity can hence be addressed by asking whether the initially parallel alignment of B and θ surfaces can be maintained in dissipative regions of the flow.

a. Two-dimensional geometry

Consider first a two-dimensional flow example. The flow is in the direction of the positive x axis, and $\partial/\partial y = 0$. This geometry yields $Q \equiv 0$ in the whole domain. As shown below, however, there can nevertheless be a flux of potential vorticity along the wave-breaking region in the y direction. For further examination, we study the flux of potential vorticity across an area A, which is defined by

$$A = [(x, \theta) \mid x_1 \leqslant x \leqslant x_2, \theta_l \leqslant \theta \leqslant \theta_u].$$

The total flux of potential vorticity across this area (directed into the positive y direction) follows from (32) as

$$J^{(y)} = \int_{\theta_l}^{\theta_u} [B(x_2) - B(x_1)] d\theta. \qquad (33)$$

Here we have required that the stratification is stable at $x = x_{1,2}$. Below it is furthermore assumed that the profiles $x_{1,2}$ are sufficiently far upstream and downstream of the wave-breaking region, respectively, such that the flow is inviscid, adiabatic, and hydrostatic at these locations. For simplicity it is assumed that $\mathbf{u}$ $(\mathbf{x}=\mathbf{x}_{2,2}) \geqslant 0$. In the interior of A, however, the flow can be arbitrarily complex except for two assumptions. First, we neglect the dissipative heating and assume that the corresponding amount of energy is lost. It follows that the entropy principle is replaced by the requirement that there is a net loss of total energy. Second, we assume that the fluid is nonconductive and neglect other diabatic processes. This approximation, which represents the limit of infinite Prandtl number, is certainly not always appropriate in gravity wave breaking events and is only used as an idealized thought experiment.

The governing equations (7)–(9) describe a statistically steady flow, and we allow for $\dot{\theta} \neq 0$ in the wave-breaking region, which is required in order to represent wave-mean interactions resulting from the folding of isentropes and transient excursions of fluid elements away from their initial isentropic layer. A nonconductive flow nevertheless implies that all airparcels recover their original potential temperature and return to their static equilibrium level at some location downstream, say x_2, provided the dissipative region is finite. It follows that the conservation of mass can be expressed for each layer separately as

$$\sigma u|_{\theta,x=x_1} = \sigma u|_{\theta,x=x_2}, \qquad (34)$$

where σ corresponds to the density in isentropic coordinates as defined in (21).

The energy relation for the governing set of equations (7)–(9) is (see Gill 1982)

$$\frac{\partial}{\partial t}(\rho E) + \nabla\cdot(\mathbf{u}\rho E) + \nabla\cdot(\mathbf{u}p) = \rho\mathcal{D}, \qquad (35)$$

where

$$E = e + \frac{1}{2}\mathbf{u}^2 + \Phi \qquad (36)$$

is the total energy per unit mass, and where

$$\mathcal{D} = \mathbf{u}\cdot\mathbf{F} + \frac{Tc_p}{\theta}\dot{\theta} \qquad (37)$$

denotes the energy dissipation per unit mass. Using the relation $B = E + p\rho^{-1}$, Eq. (35) can be converted into a viscous and diabatic flux form of the Bernoulli law (2), that is,

$$\frac{\partial(\rho B)}{\partial t} - \frac{\partial p}{\partial t} + \nabla\cdot(\mathbf{u}\rho B) = \rho\mathcal{D}. \qquad (38)$$

This equation is now integrated under the steady-state assumption over the area A', which extends in the vertical throughout the whole domain and which is confined in the x direction by $x = x_{1,2}$. The result is

$$\int_0^\infty (u\rho B)|_{x_1}^{x_2} dz = \iint_{A'} \rho\mathcal{D}\,dxdz.$$

Next we substitute for dz from (21) and use (34) to obtain

$$\int_{\theta_{z=0}}^\infty u\sigma|_{x=x_1}[B(x_2) - B(x_1)]d\theta = \iint_{A'} \rho\mathcal{D}\,dxdz < 0, \qquad (39)$$

where $\theta_{z=0}$ is the surface potential temperature (or the lowest potential temperature that occurs on the surface). The inequality on the right stems from the simplified form of the entropy principle noted earlier. In order to satisfy this inequality, there must be some θ surfaces on which the Bernoulli function is reduced within the dissipative region, that is, $B(x_2) - B(x_1) < 0$. It then follows that one can always find some values for θ_l and θ_u in (33), such that the resulting area A is associated with a net flux of potential vorticity in the negative y direction.

Within the present framework, two-dimensional wave-breaking regions are hence inevitably linked with net fluxes of potential vorticity on certain θ surfaces. Even in this simplified geometry and in abscence of potential vorticity itself, fluxes of PV cannot be avoided.

b. Three-dimensional geometry

A perspective view of one particular θ surface in a three-dimensional situation is shown in Fig. 2. The shaded area on the θ surface represents the isolated region of dissipation, which is induced by gravity wave breaking. Following the arguments of the previous subsection, we assume that the passage of air parcels across this region produces a wake, which is characterized by reduced values of the Bernoulli function. These "Bernoulli anomalies" are advected downstream, where the dynamics is again close to inviscid and B is materially conserved.

The intersections of the initially parallel B and θ surfaces with one another are indicated in Fig. 2 by the bold contours of B. There is an associated flux of potential vorticity $\mathbf{J}$. It can be diagnosed from (39), which states that the Bernoulli function is the streamfunction of the potential vorticity flux on isentropic surfaces. For a left–right symmetric flow (looking downstream), the central streamline on the θ surface cuts perpendicular across the Bernoulli contours. The resulting flux of potential vorticity hence has a vanishing component in the alongflow direction and must be purely nonadvective (i.e., dissipative), as is indicated by the white arrow. Further downstream, on the other hand, we may assume absence of dissipation, and the flow is aligned with the flux of potential vorticity (black arrows). These fluxes are entirely advective and directly relate to the presence of positive (negative) potential

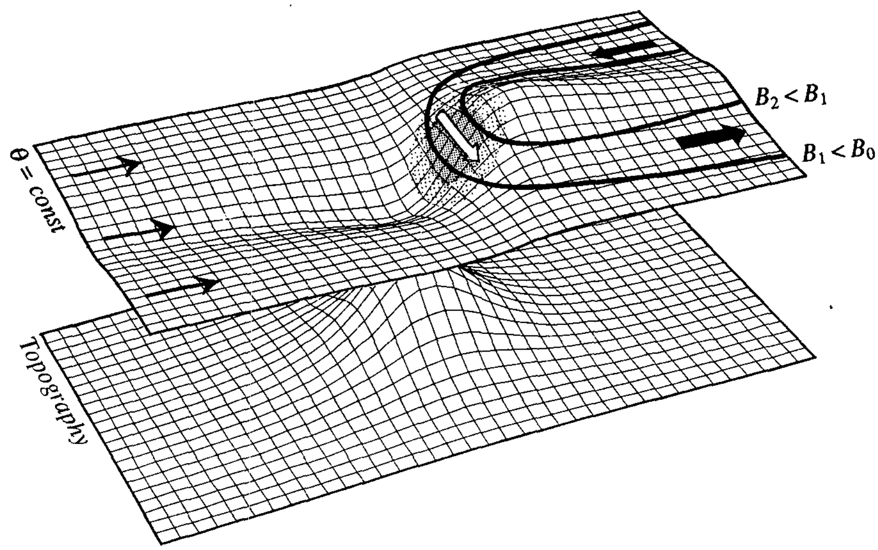

FIG. 2. Schematic view of a θ surface in steady stratified flow past an isolated obstacle. The flow is from left to right and has zero potential vorticity initially. Breaking gravity waves induce dissipation and result in a decrease of the Bernoulli function from its upstream value B_0 on the streamlines that pass across the dissipative region. The bold lines on the θ surface are contours of B. The resulting flux of potential vorticity is diagnosed from (6) or (30) and indicated by the perspective arrows. White and black vectors refer, respectively, to purely nonadvective (i.e., dissipative) and advective fluxes. The black vectors to the right (left) are associated with the downstream advection of positive (negative) potential vorticity.

vorticity to the right (left) of the obstacle. The presence of such anomalies can intrinsically alter the nature of the flow since it introduces the potential for barotropic instabilities.

4. Further remarks

The relationship between the Bernoulli function and the flux of potential vorticity can be expressed by a generalized version of the Bernoulli law. The classical version states that streamlines of steady, adiabatic, and inviscid flows can be obtained by intersecting surfaces of constant potential temperature and constant Bernoulli function. The generalized form, expressed by (6), is valid in the presence of diabatic heating and frictional forces and only requires a statistical steady state. It states that the intersections of B and θ surfaces yield the flux lines, along which the potential vorticity is transported. This transport can have advective and nonadvective components. The latter arise from wave-mean interaction, turbulence, dissipation, diabatic heating, and other nonconservative effects (cf., Haynes and McIntyre 1987, 1990). The generalized version of Bernoulli's theorem, and its time-dependent version (23), appear well suited for the diagnosis of potential vorticity flux vectors both from observed flows and from numerical model output.

The application of this theorem to dissipative gravity wave breaking has confirmed that dissipation can result in the creation of PV-anomalies, even if the flow is initially characterized by zero potential vorticity. The associated fluxes of PV can occur in an environment where there is no PV at all (even in the unresolved components of the flow). It follows that, in general, it is not possible to think of the total flux of potential vorticity as arising from time or space averages of the form $\overline{\rho Q'\mathbf{v}'}$. This distincts the flux of potential vorticity on a fundamental level from other fluxes, as, for example, those of momentum and chemical substances.

Acknowledgments. The author wishes to thank Dale R. Durran for his careful comments on earlier versions of this paper. Discussions and correspondence with Christof Appenzeller, Christopher S. Bretherton, Huw C. Davies, James R. Holton, Daniel Keyser, Michael E. McIntyre, Andy Rossa, Ronald B. Smith, George Veronis, and Heini Wernli, as well as the help of Candace Gudmundson with the English, are also appreciated. An anonymous reviewer has suggested section 2b and has generously provided the incisive relation (26). Support for this study has been provided by NSF under Grant ATM-9106494.

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