Flux of Potential Vorticity Substance: A Simple Derivation and a Uniqueness Property

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ABSTRACT

It is well known that even in the presence of diabatic effects a conservation law exists for potential vorticity Q in the form $\partial(\rho Q)/\partial t + \nabla\cdot\mathbf{J} = 0$, where $\mathbf{J}$ is a flux of potential vorticity substance. A new and extremely simple proof of this result is presented that uses only one fact: the vorticity vector is nondivergent. The flux vector derived by this method differs from that of Haynes and McIntyre by a divergence-free vector, calling attention to the nonuniqueness of $\mathbf{J}$. It is proved, however, that the Haynes–McIntyre flux vector is the unique choice that is the sum of a purely advective flux and a nonadvective flux that depends linearly on local heating rate and frictional forces.

1. Introduction

Potential vorticity (PV) has become a very illuminating and powerful theoretical tool for studying inviscid, adiabatic stratified flows on a range of scales (see the review of Hoskins et al. 1985). Seminal work by Haynes and McIntyre (1987, 1990) showed that even in the presence of arbitrary diabatic heating and frictional forces there are some important and somewhat inobvious constraints on the time evolution of PV. Their work rests on two remarkable facts about PV. The first is the existence of a conservation law for an imaginary "potential vorticity substance" (PVS) whose mixing ratio is the potential vorticity Q:

$$\partial(\rho Q)/\partial t + \nabla\cdot\mathbf{J} = 0. \qquad (1)$$

We call this a conservation law because (i) there is no net source of PVS in the interior of a fluid, and (ii) there exists a flux vector $\mathbf{J}$ of PVS that can be determined from *local instantaneous* values and spatial derivatives at time t of the prognostic fields.

The wording of the second condition is important. It corresponds to our intuitive notion of a flux as a net flow of a quantity driven by local spatial variations of that quantity or other fields. Without this condition, we could find a conservation law for an arbitrary time and space-varying field ϕ, constructing a "flux" vector $\mathbf{F}_\phi$ as follows. We find a field ψ whose Laplacian $\nabla^2\psi = \partial\phi/\partial t$ (for specificity, we can uniquely find such a ψ that also satisfies Dirichlet boundary conditions if the domain of interest is finite) and let $\mathbf{F}_\phi = -\nabla\psi$. In general, however, the ψ so constructed is a nonlocal function of $\partial\phi/\partial t$ rather than a local function of instantaneous values of ϕ and other prognostic fields, so the $\mathbf{F}_\phi$ does not satisfy condition (ii) and is not an admissible flux vector.

In analogy with the terms mass flux and momentum flux, $\mathbf{J}$ is most aptly called a PVS flux rather than the commonly used term PV flux. For adiabatic, frictionless fluid flow, Ertel's theorem (Ertel 1942) implies trivially that (1) is satisfied with a purely advective PVS flux $\mathbf{J}^A = \mathbf{u}\rho Q$. Equation (1), however, is by no means as obvious for nonadiabatic flows. Note that $\mathbf{J}$ is not mathematically unique but is arbitrary up to the curl of an arbitrary vector field. A best choice of $\mathbf{J}$, if it exists, must be made on physical grounds.

One such choice, the PVS flux that forms the basis of most of the Haynes and McIntyre and other analyses, is

$$\mathbf{J}_{\mathrm{HM}} = \mathbf{u}\rho Q + \mathbf{J}^N_{\mathrm{HM}}, \qquad (2a)$$

where ρ and $\mathbf{u}$ are the fluid density and velocity. The "nonadvective" PVS flux $\mathbf{J}^N_{\mathrm{HM}}$ can be written in terms of the diabatic tendency H of potential temperature θ and the acceleration $\mathbf{F}$ due to nonconservative forces (such as friction),

$$\mathbf{J}^N_{\mathrm{HM}} = -\boldsymbol{\omega}H - \mathbf{F}\times\nabla\theta, \qquad (2b)$$

where $\boldsymbol{\omega}$ is the absolute vorticity.

The second remarkable fact about PV is the "impermeability theorem," that isentropes are impermeable to PV substance. Again, this is trivial for adiabatic, frictionless flows for which isentropes are material surfaces, and the flux of PVS is due only to the rearrangement of parcels on this surface, but it is remarkable

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that the impermeability theorem holds quite generally. The impermeability theorem can be easily envisioned in terms of fluxes if the flux vector obeys

$$\partial\theta/\partial t + (\mathbf{J}/\rho Q)\cdot\nabla\theta = 0. \quad (3)$$

We interpret $\mathbf{J}/\rho Q$ as an effective velocity for PVS transport. Then (3) states that a particle of PVS that moves with the velocity $\mathbf{J}/\rho Q$ always remains on the same isentrope. While $\mathbf{J}_{\mathrm{HM}}$ obeys (3), one can easily find other PVS fluxes that satisfy (1) but not (3).

As Haynes and McIntyre (1987, section 5) recognized, these general results about PV are kinematic consequences of the definition $Q = \rho^{-1}\boldsymbol{\omega}\cdot\nabla\theta$. By this we mean that an analogous conservation law could be found if θ were replaced by *any* scalar field and if $\boldsymbol{\omega}$ were replaced by the curl of *any* vector field. Similarly, they showed that one could formulate an impermeability theorem for a PV analog formed using any scalar field. The mathematical derivations they offered for PVS flux, however, are not entirely satisfying. While they expose the kinematic nature of these results in their Eq. (5.5), the proof of this equation, while fairly straightforward, is not transparent. Similarly, while they recognize that their choice of PVS flux is nonunique, they do not address whether other choices might be better for some purposes.

Truesdell (1951, 1954) and Palmer (1988) also noted some kinematic properties of PV. Equations (2) and (3) of Truesdell (1951) (derived for an arbitrary fluid flow and an arbitrary scalar field θ) and the nondivergence of $\boldsymbol{\omega}$ can be combined (though Truesdell did not do this) to give the formula $\rho DQ/Dt = -\nabla\cdot\mathbf{J}^N_{\mathrm{HM}}$, which is equivalent to (1) using the mass conservation equation. Palmer used the elegant formalism of Lie derivatives to look at magnetohydrodynamic analogs of PV and their conservation properties in the presence of arbitrary dissipation.

In this note, we present a one-line "kinematic" derivation of PVS conservation simpler than any that we have seen in the literature and find a sense in which $\mathbf{J}_{\mathrm{HM}}$ is the unique physically simplest choice of PVS flux.

2. A simple kinematic derivation of a conservation law for PVS

By PVS conservation, we mean that the tendency of PVS per unit volume ρQ can be written as the divergence of some vector field, which can then be interpreted as a PVS flux. This can be shown as follows, using only the definition of PV and that $\nabla\cdot\boldsymbol{\omega} = 0$,

$$\frac{\partial}{\partial t}(\rho Q) = \frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\nabla\theta) = \frac{\partial}{\partial t}\nabla\cdot(\boldsymbol{\omega}\theta) = -\nabla\cdot\mathbf{J}, \quad (4a)$$

$$\mathbf{J} = -\frac{\partial}{\partial t}(\boldsymbol{\omega}\theta). \quad (4b)$$

This flux is quite different than $\mathbf{J}_{\mathrm{HM}}$. It is zero for steady flow. It can be shown to be a true flux in the sense defined after (1), removing time derivatives by substituting for $\partial\boldsymbol{\omega}/\partial t$ and $\partial\theta/\partial t$ from the vorticity and thermodynamic equations.

The relationship between $\mathbf{J}$ and $\mathbf{J}_{\mathrm{HM}}$ is illuminated by writing $\mathbf{J}_{\mathrm{HM}}$ in an alternative form from (2). Following Schär (1993), we write the momentum equation in terms of the Bernoulli function $B = \mathbf{u}^2/2 + e + p/\rho + \Phi$, where e is the internal energy per unit mass and Φ is a potential energy associated with conservative body forces; the vector $\mathbf{F}$ includes all other nonconservative forces. If s is specific entropy, then without assumption (Batchelor 1967, section 3.5)

$$\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u} = T\nabla s - \nabla B + \mathbf{F}. \quad (5)$$

Taking the cross product of $\nabla\theta$ with (5), Schär (1993) showed that

$$\mathbf{J}_{\mathrm{HM}} = \nabla\theta\times\left(\nabla B + \frac{\partial\mathbf{u}}{\partial t}\right) - \boldsymbol{\omega}\frac{\partial\theta}{\partial t}. \quad (6)$$

Vector identities applied to (6) show that $\mathbf{J}_{\mathrm{HM}}$ differs from $\mathbf{J}$ only by a nondivergent vector, specifically,

$$\mathbf{J}_{\mathrm{HM}} = \nabla\times\left\{\theta\left[\nabla B + \frac{\partial\mathbf{u}}{\partial t}\right]\right\} - \theta\frac{\partial\boldsymbol{\omega}}{\partial t} - \boldsymbol{\omega}\frac{\partial\theta}{\partial t},$$

$$= \mathbf{J} + \nabla\times\left\{\theta\left[\nabla B + \frac{\partial\mathbf{u}}{\partial t}\right]\right\}. \quad (7)$$

As pointed out by a reviewer, the derivation (4) can equally well be applied to show that the hierarchy of PV extensions $Q^{(n)} = \rho^{-1}\boldsymbol{\omega}\cdot\nabla\theta^n = n\theta^{n-1}Q$ all obey conservation laws. Once again, the associated fluxes $\mathbf{J}^{(n)}$ are not unique, but if $D\theta^n/Dt \equiv H^{(n)} = n\theta^{n-1}H$, we can extend the same arguments leading to the HM PVS flux to obtain an extended HM flux $\mathbf{J}^{(n)}_{\mathrm{HM}} = \rho\mathbf{u}Q^{(n)} + \boldsymbol{\omega}H^{(n)} - \mathbf{F}\times\nabla\theta^n = n\theta^{n-1}\mathbf{J}_{\mathrm{HM}}$. The impermeability theorem (3) then follows from (1) and the analogous conservation equation for $Q^{(2)}$ using the flux $\mathbf{J}^{(2)}_{\mathrm{HM}}$. A similar derivation would *not* work for the alternate flux vector in (4b), for which $\mathbf{J}^{(2)}$ cannot be taken equal to $2\theta\mathbf{J}$; this $\mathbf{J}$ does not satisfy the impermeability theorem.

3. A special property of $\mathbf{J}_{\mathrm{HM}}$

Given that different choices for the PVS flux can emerge quite naturally from different mathematical derivations of the PVS conservation equation, is $\mathbf{J}_{\mathrm{HM}}$ in some sense still the best choice? To have a simple physical interpretation as a flux of PVS, a flux $\mathbf{J}$ should

(i) reduce to the advective flux $\mathbf{u}\rho Q$ where diabatic and frictional forcings are absent and

(ii) have as simple as possible a dependence on H and $\mathbf{F}$, so that the interpretations are not made unnecessarily complicated by "do-nothing" nonadvective fluxes circulating around diabatic or frictionally forced regions.

Criterion (i) does not uniquely determine $\mathbf{J}$, even if we impose the impermeability property (3) as an additional requirement; for instance (i) and the impermeability property are satisfied by any flux differing from $\mathbf{J}_{\mathrm{HM}}$ by the nondivergent vector $\nabla\theta \times \nabla g$, where g is an arbitrary function.

However, (i) and (ii) together *are* enough to uniquely determine $\mathbf{J}$ (and guarantee the impermeability property) provided that we interpret (ii) in the following way. We require that the nonadvective part of $\mathbf{J}$ be a *linear* function of the local *values* of H and $\mathbf{F}$ (and not involve their first or higher derivatives). Hence, $\mathbf{J}$ has the form

$$\mathbf{J} = \mathbf{u}\rho Q + \mathbf{a}H + \mathbf{M}\mathbf{F}, \qquad (8)$$

where $\mathbf{a}$ is an arbitrary vector and matrix independent of H and $\mathbf{F}$, and $\mathbf{M}$ is a similarly arbitrary matrix. The components of $\mathbf{a}$ and $\mathbf{M}$ may depend in any way on the instantaneous position, velocity field, or thermodynamic variables. Then $\mathbf{J}_{\mathrm{HM}}$ is the unique PVS flux of this form.

The remainder of this paper is devoted to a simple proof of this fact. Suppose there is a PVS flux vector $\mathbf{J} \neq \mathbf{J}_{\mathrm{HM}}$ of the form (8). Let $\tilde{\mathbf{J}} = \mathbf{J} - \mathbf{J}_{\mathrm{HM}}$. Then $\tilde{\mathbf{J}}$ must be nondivergent for all H and $\mathbf{F}$ and have the form $\tilde{\mathbf{a}}H + \tilde{\mathbf{M}}F$, where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{M}}$ do not depend on H and $\mathbf{F}$. We now show $\tilde{\mathbf{J}} = 0$ by looking at some special choices of H and $\mathbf{F}$. First we consider a heating field $H = 1$ and a forcing $\mathbf{F} = 0$. Since this $\tilde{\mathbf{J}}$ is nondivergent, $\nabla \cdot \tilde{\mathbf{a}} = 0$. The $\tilde{\mathbf{J}}$ corresponding to any other heating field is also nondivergent. Together, these facts imply that

$$0 = \nabla \cdot (\tilde{\mathbf{a}}H) = \tilde{\mathbf{a}} \cdot \nabla H \quad \text{for all functions } H(x). \qquad (9)$$

Now suppose there was some position x_0 at which $\tilde{\mathbf{a}}(x_0)$ was nonzero. Consider the heating field $H(x) = \tilde{\mathbf{a}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$, for which $\nabla H = \tilde{\mathbf{a}}(\mathbf{x}_0)$. For this heating field, $\tilde{\mathbf{a}} \cdot \nabla H > 0$ at $\mathbf{x}_0$, in contradiction to (9). Hence, the above supposition is false, and $\tilde{\mathbf{a}}(\mathbf{x}_0) = 0$ for all $\mathbf{x}_0$. If $\tilde{\mathbf{M}}$ is split up into its columns (the vectors $\tilde{\mathbf{m}}_1, \tilde{\mathbf{m}}_2, \tilde{\mathbf{m}}_3$) and $\mathbf{F}$ is written in terms of its components (F_1, F_2, and F_3), then $\tilde{\mathbf{M}}\mathbf{F} = \tilde{\mathbf{m}}_1 F_1 + \tilde{\mathbf{m}}_2 F_2 + \tilde{\mathbf{m}}_3 F_3$. By considering each term separately, the same argument as above shows that $\mathbf{M} = 0$. Hence, $\tilde{\mathbf{J}} = 0$ and $\mathbf{J} = \mathbf{J}_{\mathrm{HM}}$.

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