Stochastic Radiative Transfer in a Partially Cloudy Atmosphere

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ABSTRACT

A radiation treatment of the broken-cloud problem is presented, based upon various stochastic models of the equation of radiative transfer that consider the clouds and clear sky as a two-component random mixture. These models, recently introduced in the kinetic theory literature, allow for non-Markovian statistics as well as both vertical and lateral variations in the cloudiness. Numerical results are given that compare different models of stochastic radiative transport and that point out the importance of treating the broken-cloud problem as a stochastic process. It is also shown that an integral Markovian model proposed within the atmospheric radiation community by Titov is entirely equivalent to a special case of a simple low-order differential model. The differential form of Titov's result should be easier than the integral form to implement in any general circulation model.

1. Introduction

It is generally accepted that the treatment of radiative transfer through a partially cloudy atmosphere is an important component of general circulation models (GCMs). We refer the reader to the papers of Stephens (1984), Ramanathan et al. (1989), Stephens et al. (1991), and the references therein. The need for a statistical description in which the clouds and clear sky are treated as a two-component stochastic mixture has also been recognized for some time (Stephens et al. 1991). In this paper, we wish to show that recent models introduced in the kinetic theory literature concerning particle and radiation transport in stochastic media can very easily and naturally be applied to this atmospheric radiative transfer problem. The kinetic theory applications of this stochastic transport formalism are many and include radiative transfer in Rayleigh–Taylor unstable inertially confined fusion pellets, neutron transport in boiling water reactors, gamma and neutron flow through concrete shields, and light transport through murky water and sooty air. For a list of published papers in the kinetic theory literature, we refer the reader to the recent paper by Malvagi and Pomraning (1992). We will not review all of this available literature but rather concentrate on a specific approach that we consider particularly promising for future implementation in GCMs, or more generally where the size of the spatial numerical cell does not allow resolution of individual clouds even if the description (size, shape, and location) of such clouds were known. This approach, based upon a modification of a Markovian model to account for arbitrary (non-Markovian) cloud size and spacing distributions, is applicable to an arbitrary binary mixture (and has been generalized to a mixture of more than two components). Further, it requires relatively few input parameters, possesses several exact limits, has proven to be robust and relatively accurate far from those limits, and has the form of integro-differential equations that are convenient for analysis and for which an extensive body of numerical solution methods is readily available. Finally, it allows for arbitrary variations of the cloudiness, both vertically and laterally.

We also show that in its most rudimentary and Markovian form, this approach is entirely equivalent to a Markovian model introduced in the atmospheric science community by Titov (1990) and coworkers (Zuev et al. 1987). The Titov formalism involves two coupled integral equations, and we show that these equations can be reduced to a more standard differential form. In this form, the Titov model should be much easier

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to implement in GCMs by introducing a two stream or any of a number of other well-known approximations for the classic equation of transfer. This differential form is, in fact, identical to the simplest kinetic theory model in the special case when the clear sky is treated as completely transparent (the case treated by Titov). Thus, the simplest kinetic theory model generalizes the Titov model by 1) allowing radiation interactions with both the clear sky and the clouds and 2) allowing the statistics to be non-Markovian (nonexponentially distributed cloud size and spacing).

The equation of radiative transfer we will be concerned with is written

$$\mathbf{\Omega}\cdot\nabla I(\mathbf{\Omega}) + \sigma I(\mathbf{\Omega}) = \sigma_s \int_{4\pi} f(\mathbf{\Omega}\cdot\mathbf{\Omega}')I(\mathbf{\Omega}')d\Omega' + S. \tag{1.1}$$

The dependent variable in (1.1) is the specific intensity of radiation $I(\mathbf{r}, \mathbf{\Omega})$, with $\mathbf{r}$ and $\mathbf{\Omega}$ denoting the spatial and angular (photon flight direction) variables, respectively. The quantity $\sigma(\mathbf{r})$ is the macroscopic total cross section (extinction coefficient), $\sigma_s(\mathbf{r})$ is the macroscopic scattering cross section, $f(\mathbf{\Omega}\cdot\mathbf{\Omega}')$ is the single-scatter angular redistribution function normalized according to

$$\int_{4\pi} f(\mathbf{\Omega}\cdot\mathbf{\Omega}')d\Omega = 2\pi\int_{-1}^{1} f(\xi)d\xi = 1, \tag{1.2}$$

and $S(\mathbf{r}, \mathbf{\Omega})$ denotes any emission source of photons. If one assumes local thermodynamic equilibrium for the matter, then $S = \sigma_a B$, where B is the Planck function, and σ_a is the macroscopic absorption cross section corrected for induced emission. We have assumed no time-dependence as well as coherent (no energy exchange) scattering in (1.1), but these simplifications are not necessary for the essentials of the models of stochastic transport considered. Thus, (1.1) is a time-independent, monochromatic (gray) equation of transfer, and there is no need to display the independent frequency variable that is simply a parameter.

To treat the case of a binary statistical mixture, the quantities σ, σ_s, f, and S in (1.1) are considered as discrete random variables, each of which assumes, at any $\mathbf{r}$, one of two sets of values characteristic of the two components constituting the mixture, namely, the clouds and the clear sky. These two sets are denoted by σ_i, σ_{si}, f_i, and S_i, where $i = 0$ for clear sky and $i = 1$ for clouds. That is, as a photon traverses the atmosphere along any path, it encounters alternating segments of clouds and clear sky, each of which has known deterministic values of σ, σ_s, f, and S. The stochastic nature of the problem enters through the statistics of the partially cloudy atmosphere, that is, through the statistical knowledge as to whether a cloud or clear sky is present at point $\mathbf{r}$. Since σ, σ_s, f, and S in (1.1) are (two-state, discrete) random variables, the solution of (1.1) for I is stochastic and we let $\langle I\rangle$ denote the ensemble-averaged intensity. The goal in any statistical model of cloud–radiation interaction is to obtain a relatively simple and accurate set of equations for $\langle I\rangle$. It may also be of interest to have a model for the higher moments of the stochastic radiation field, such as the variance.

In the next section, we give a brief description of the kinetic theory formulations of stochastic transport that have direct application to the problem of radiation transport through a partially cloudy sky. In our discussion, we give a prescription for incorporating any given cloud size and spacing distributions (non-Markovian behavior) into the models. We also make a specific suggestion as to how to model clouds that have different characteristic dimensions in the vertical and horizontal directions. In section 3, we give the analysis that demonstrates that the Titov integral model is entirely equivalent to a special case of the simplest kinetic theory model. Section 4 is devoted to representative numerical results comparing the kinetic theory models to a specific fractional cloud model, and to exact results when available, for a variety of test problems. The final section of our paper contains a few concluding remarks.

2. Stochastic models of radiative transfer

Equation (1.1) is considered in the case when σ, σ_s, f, and S are two-state discrete random variables obeying arbitrary statistics. In the atmospheric radiation context, arbitrary statistics means that the cloud size distribution and the cloud-spacing distribution are completely arbitrary and in general spatially and directionally dependent. An ensemble averaging of (1.1) leads to the exact set of two equations given by (Adams et al. 1989)

$$\mathbf{\Omega}\cdot\nabla(p_iI_i) + \sigma_i p_i I_i = \sigma_{si}\int_{4\pi} f_i(\mathbf{\Omega}\cdot\mathbf{\Omega}')p_iI_i(\mathbf{\Omega}')d\Omega' + p_iS_i + \frac{p_j\bar{I}_j}{\lambda_j} - \frac{p_i\bar{I}_i}{\lambda_i}, \quad i = 0, 1, \quad j \neq i. \tag{2.1}$$

Here $p_i(\mathbf{r})$ is the probability of component i of the mixture being at position $\mathbf{r}$, and the $I_i(\mathbf{r}, \mathbf{\Omega})$ and $\bar{I}_i(\mathbf{r}, \mathbf{\Omega})$ are conditional ensemble-averaged intensities. Specifically, I_i is an ensemble average conditioned upon position $\mathbf{r}$ being in component i, and $\bar{I}_i$ is an ensemble average conditioned upon position $\mathbf{r}$ being an interface between component i and component j, with component i to the left of the interface (the vector $\mathbf{\Omega}$ points from left to right). The desired quantity is the unconditional ensemble average of the intensity $\langle I(\mathbf{r}, \mathbf{\Omega})\rangle$, given by

$$\langle I\rangle = p_0I_0 + p_1I_1. \tag{2.2}$$

For inhomogeneous Markovian statistics, the $\lambda_i(\mathbf{r}, \mathbf{\Omega})$ in (2.1) are simply the Markov transition lengths defined by

$$\text{prob}(i \rightarrow j) = \frac{ds}{\lambda_i(s)}, \quad j \neq i, \tag{2.3}$$

where s is a spatial coordinate along the direction of sight $\mathbf{\Omega}$, and prob$(i \to j)$ is the (differential) probability that point $s + ds$ is in fluid j, given that point s is in fluid i. The $\lambda_i(\mathbf{r}, \mathbf{\Omega})$ are taken as prescribed by the cloud dynamics and completely define the stochastic cloud–clear-sky mixture. The probabilities $p_i(\mathbf{r})$ are related to the $\lambda_i(\mathbf{r}, \mathbf{\Omega})$ according to the Chapman–Kolmogorov equation given by

$$\mathbf{\Omega} \cdot \nabla p_i = \frac{p_j}{\lambda_j} - \frac{p_i}{\lambda_i}, \quad j \neq i. \tag{2.4}$$

The dependence of the λ_i on $\mathbf{\Omega}$ must be just such that the p_i as determined by (2.4) are independent of $\mathbf{\Omega}$. For homogeneous Markovian statistics, the interpretation of these statistics is that, along any direction $\mathbf{\Omega}$, the alternating segments intersecting the clouds and clear sky populate two exponential distributions with mean chord lengths $\lambda_i(\mathbf{\Omega})$. In this case, the p_i and λ_i are both independent of $\mathbf{r}$ (but may depend upon $\mathbf{\Omega}$) and are related by

$$p_i = \frac{\lambda_i}{\lambda_0 + \lambda_1}. \tag{2.5}$$

From (2.5), it is seen that in order for the p_i to be independent of $\mathbf{\Omega}$, λ_0 and λ_1 must have the same dependence on $\mathbf{\Omega}$. The correlation length λ_c associated with a homogeneous Markovian mixture is given by (Pomraning 1989)

$$\frac{1}{\lambda_c} = \frac{1}{\lambda_0} + \frac{1}{\lambda_1}. \tag{2.6}$$

For homogeneous non-Markovian statistics, $\lambda_i(\mathbf{\Omega})$ has the physical interpretation of the mean chord length in component i. In the general case of inhomogeneous non-Markovian statistics, $\lambda_i(\mathbf{r}, \mathbf{\Omega})$ is well defined mathematically in terms of the statistics (Adams et al. 1989); and while a precise physical interpretation is not apparent, λ_i in this case can be qualitatively interpreted as a characteristic chord length in component i in direction $\mathbf{\Omega}$.

The $\mathbf{\Omega}$ dependence of $\lambda_i(\mathbf{r}, \mathbf{\Omega})$ can be used to introduce directionally dependent cloud sizes and spacings into the stochastic formalism. As an example, for an atmospheric layer with a volume fraction p_{cloud} occupied by clouds of characteristic vertical (along the z axis) dimension H and characteristic horizontal dimension D, we could write, recalling that the subscripts 0 and 1 refer to clear sky and clouds, respectively,

$$p_1 = p_{\text{cloud}}, \quad p_0 = 1 - p_{\text{cloud}}, \tag{2.7}$$

$$\frac{1}{\lambda_1} = \frac{1}{\lambda_{\text{cloud}}} = \left(\frac{\mu^2}{H^2} + \frac{1-\mu^2}{D^2}\right)^{1/2}, \tag{2.8}$$

$$\frac{1}{\lambda_0} = \left(\frac{p_{\text{cloud}}}{1 - p_{\text{cloud}}}\right)\frac{1}{\lambda_1}, \tag{2.9}$$

where μ is the cosine of the polar angle between the z axis and the direction $\mathbf{\Omega}$. Equation (2.8), which follows by characterizing the clouds as ellipses, accounts for clouds with an arbitrary aspect ratio $\gamma = D/H$. For clouds with aspect ratio $\gamma = 1$, the transition probabilities λ_i became independent of $\mathbf{\Omega}$ (isotropic statistics). Equation (2.9) follows from (2.5).

Returning to (2.1), it is clear that all quantities can in principle depend upon all three spatial coordinates x, y, and z. In the restricted problem of three-dimensional clouds embedded in a planar layer, the statistics and boundary conditions in the layer are assumed to be independent of x and y (in particular, this restricts the clouds to a common characteristic dimension along both x and y). If further assumptions are made that the physical properties of the clouds and of the clear sky do not change, on average, along the horizontal directions within a spatial cell, then the solution for the ensemble averages I_i and $\bar{I}_i$ will be independent of x and y, although they will still depend upon the azimuthal angle φ, which, along with μ, defines $\mathbf{\Omega}$. It is only these ensemble averages in (2.1) that depend upon φ, however. This allows integration of (2.1) over φ to obtain the planar equation

$$\mu \frac{\partial}{\partial z}[p_i(z)\psi_i(z, \mu)] + \sigma_i(z)p_i(z)\psi_i(z, \mu)$$
$$= \sigma_{si}(z)\int_{-1}^{1} g_i(z, \mu, \mu')p_i(z)\psi_i(z, \mu')d\mu'$$
$$+ p_i(z)S_i(z, \mu) + \frac{p_j(z)\bar{\psi}_j(z, \mu)}{\lambda_j(z, \mu)}$$
$$- \frac{p_i(z)\bar{\psi}_i(z, \mu)}{\lambda_i(z, \mu)}, \quad i = 0, 1, \quad j \neq i, \tag{2.10}$$

where

$$\psi_i(\mu) = \int_0^{2\pi} I_i(\mu, \varphi)d\varphi, \tag{2.11}$$

with a similar relationship between $\bar{\psi}_i(\mu)$ and $\bar{I}_i(\mu, \varphi)$. The redistribution function g_i is given by

$$g_i(\mu, \mu') = \int_0^{2\pi} f_i(\mathbf{\Omega} \cdot \mathbf{\Omega}')d\varphi, \tag{2.12}$$

and clearly has the normalization

$$\int_{-1}^{1} g_i(\mu, \mu')d\mu = 1. \tag{2.13}$$

Equation (2.10) still allows for altitude dependence of the cross sections and altitude and polar angle dependence of the cloud size and spacing. All horizontal effects of cloud–cloud and cloud–sky interactions are taken into account, however, by the coupling term on the right-hand side, and any explicit dependence on x and y has disappeared; it has been "averaged out" in a rigorous, nonapproximate way by the ensemble averaging. In other words, (2.10) is a one-dimensional model that rigorously accounts for the three-dimensional geometry of the clouds under the assumed translational invariance of the cross sections and the λ_i. If for some reason one is interested in the φ dependence of the ensemble-averaged intensities, one can expand the intensities in (2.1) in a Fourier series in φ, the result being an uncoupled set of equations for each Fourier component (Case and Zweifel 1967). Equation (2.10) is, in fact, the equation for the $n = 0$ cosine mode.

With regard to boundary conditions, it is assumed that the incoming intensity on the surface of the system is specified and nonstochastic. Then, for each physical realization of the statistical mixture, the boundary condition on (1.1) is taken as

$$I(\mathbf{r}_s, \Omega) = F(\mathbf{r}_s, \Omega), \quad \mathbf{n}\cdot\Omega < 0, \tag{2.14}$$

where $\mathbf{n}$ is a normal outward-pointing unit vector at a surface point $\mathbf{r}_s$, and F is the specified boundary data. Equation (2.14) implies that all conditional ensemble-averaged intensities satisfy the same boundary condition. In particular, the boundary conditions are

$$I_i(\mathbf{r}_s, \Omega) = \bar{I}_i(\mathbf{r}_s, \Omega) = F(\mathbf{r}_s, \Omega), \quad \mathbf{n}\cdot\Omega < 0. \tag{2.15}$$

The azimuthally integrated intensities contained in (2.10) then satisfy

$$\psi_i(z_s, \mu) = \bar{\psi}_i(z_s, \mu) = G(z_s, \mu), \quad \mathbf{n}\cdot\Omega < 0, \tag{2.16}$$

where z_s is a surface point of the planar system and

$$G(z_s, \mu) = \int_0^{2\pi} F(\mathbf{r}_s, \Omega) d\varphi. \tag{2.17}$$

We emphasize that (2.1), as well as (2.10) in restricted problems, are exact ensemble-balance equations for arbitrary (non-Markovian) statistics. As they stand, however, those equations do not represent calculational models in that they contain more unknowns than there are equations. For example, (2.1) represents two equations in the four unknowns I_0, I_1, $\bar{I}_0$, and $\bar{I}_1$. Either more equations are required, or a closure needs to be invoked. A very simple closure has been suggested (Adams et al. 1989), namely,

$$\bar{I}_i = I_i. \tag{2.18}$$

The closed set of equations resulting from using (2.18) in (2.1) is known to be exact for inhomogeneous Markovian statistics in the absence of photon scattering ($\sigma_{si} = 0$). With scattering in the underlying transport problem, (2.18) is an approximation, but the resulting model for stochastic transport has been shown to be robust and accurate for Markovian statistics (Adams et al. 1989). Additional derivations of the model resulting from using (2.18) in (2.1), using a variety of formalisms, have been given by Pomraning (1986), Levermore et al. (1986), Vanderhaegen (1986), and Sahni (1989a,b). A similar model for the higher moments of the stochastic radiation field, in particular the variance, has been given by Boffi et al. (1990).

For non-Markovian statistics, the closure given by (2.18) is an approximation in all cases, even in the absence of scattering. Such non-Markovian problems, in the absence of scattering, can be formulated exactly in terms of coupled integral equations using the theory of alternating renewal processes (Vanderhaegen 1986, Levermore et al. 1988). For homogeneous statistics, these integral equations are of the convolution type and are readily solved by Laplace transformation. It has been shown by Levermore et al. (1988) that certain characteristics, namely, the deep-in behavior and the mean distance to collision, of the non-Markovian solution can be captured by (2.1) and (2.18) by introducing an effective correlation length λ_{eff}, which plays the role of λ_c as given by (2.6). This effective correlation length distribution function. Now, (2.5) and (2.6) imply

$$\lambda_{\text{eff}} = q\lambda_c, \tag{2.19}$$

where

$$q = \frac{1}{\sigma_0}\left[\frac{1}{\tilde{Q}_0(\sigma_0)} - \frac{1}{\lambda_0}\right] + \frac{1}{\sigma_1}\left[\frac{1}{\tilde{Q}_1(\sigma_1)} - \frac{1}{\lambda_1}\right] - 1. \tag{2.20}$$

Here λ_i is the mean chord length in component i of the mixture, and $\tilde{Q}_i(\sigma_i)$ is the Laplace transform of $Q_i(s)$ evaluated at a transform variable σ_i, that is,

$$\tilde{Q}_i(\sigma_i) = \int_0^\infty e^{-\sigma_i s} Q_i(s) ds, \tag{2.21}$$

with $Q_i(s)$ representing the cumulative probability that the chord length in component i exceeds s. It has also been shown (Levermore et al. 1988) that $q \geq 0$ for all chord-length distributions. For Markovian statistics, $Q_i(s)$ is exponential and we have $q = 1$. In the particular case in which the clear sky is treated as a vacuum ($\sigma_0 = 0$), a simple limiting process gives

$$q = \frac{1}{\sigma_1}\left[\frac{1}{\tilde{Q}_1(\sigma_1)} - \frac{1}{\lambda_1}\right] + \frac{V_0}{2\lambda_0^2} - \frac{1}{2}, \tag{2.22}$$

where λ_0 is the mean spacing between clouds, and V_0 is the corresponding variance of the clear-sky chord-length distribution function. Now (2.5) and (2.6) imply

$$\lambda_i = \frac{\lambda_c}{p_j}, \quad j \neq i, \tag{2.23}$$

and thus replacing the correlation length λ_c by $q\lambda_c$ implies that the λ_i must be replaced by $q\lambda_i$ since the p_i are taken as given, independent of the statistics.

To summarize, one model of radiative transfer through a planar layer of partially cloudy sky is to use (2.10) with the closure given by

$$\bar{\psi}_i = \psi_i, \tag{2.24}$$

and the ensemble-averaged solution given by

$$\langle \psi \rangle = p_0\psi_0 + p_1\psi_1. \quad (2.25)$$

The $\lambda_i(z, \mu)$ in (2.10) account for the spatially and directionally dependent cloud size and spacing distributions in that λ_i, in any given direction $\mathbf{\Omega}$, is the product of the local mean chord length in component i in that direction and the modification factor q given by (2.20). This approximate method of treating non-Markovian mixing statistics has been tested numerically by Levermore et al. (1988) and was shown to predict exact results, found by solving the renewal equations, quite well. For example, if the clouds are modeled as ellipsoids of various sizes with an average vertical height H and horizontal extent D, then we would have [see (2.8)] for the effective value of λ_1,

$$\frac{1}{\lambda_{1,\mathrm{eff}}} = \frac{1}{\lambda_{\mathrm{cloud}}} = q\left(\frac{\mu^2}{H^2} + \frac{1-\mu^2}{D^2}\right)^{1/2} \quad (2.26)$$

with q given by (2.20). The quantity $\lambda_{0,\mathrm{eff}}$ then follows as [see (2.9)]

$$\frac{1}{\lambda_{0,\mathrm{eff}}} = \left(\frac{p_{\mathrm{cloud}}}{1-p_{\mathrm{cloud}}}\right)\frac{1}{\lambda_{1,\mathrm{eff}}}. \quad (2.27)$$

Recently, an improved model has been proposed (Pomraning 1991) that retains the exact balance result given by (2.10) but improves upon the closure given by (2.24). This model derives an exact ensemble balance-like equation for the $\bar{\psi}_i$, which involves still additional conditional ensemble averages, and then makes a closure to eliminate these additional averages. This improved closure is given by the following equations, with $i = 0, 1$ and $j \neq i$:

$$\mu\frac{\partial}{\partial z}(p_i\bar{\psi}_i) + \sigma_i p_i\bar{\psi}_i = \sigma_{si}\int_{-1}^{0} g_i(\mu,\mu')p_i\bar{\psi}_j(\mu')d\mu' + \sigma_{si}\int_0^1 g_i(\mu,\mu')p_i\bar{\psi}_i(\mu')d\mu' + \frac{p_j\bar{\psi}_j}{\lambda_j} - \frac{p_i\bar{\psi}_i}{\lambda_i} + p_iS_i, \quad \mu > 0, \quad (2.28)$$

$$\mu\frac{\partial}{\partial z}(p_i\bar{\psi}_i) + \sigma_i p_i\bar{\psi}_i = \sigma_{si}\int_{-1}^{0} g_i(\mu,\mu')p_i\bar{\psi}_i(\mu')d\mu' + \sigma_{si}\int_0^1 g_i(\mu,\mu')p_i\bar{\psi}_j(\mu')d\mu' + \frac{p_j\bar{\psi}_j}{\lambda_j} - \frac{p_i\bar{\psi}_i}{\lambda_i} + p_iS_i, \quad \mu < 0. \quad (2.29)$$

This more complex model has been shown to be more accurate than the simpler model given by invoking the closure given by (2.24) and possesses the proper results in all known limits. Non-Markovian behavior is accounted for in this model by, as before, replacing the λ_i by $\lambda_{i,\mathrm{eff}}$. In section 4, both the rudimentary model [(2.10) and (2.24)] and this more involved model [(2.10), (2.28), and (2.29)] are numerically tested.

The final model presented is that introduced by Titov (1990) within the atmospheric radiation context. He considered the special case of negligible emission ($S_i = 0$) and treated the clear sky as completely transparent ($\sigma_0 = \sigma_{s0} = 0$). To simplify the notation somewhat, the spatial variable s along the direction $\mathbf{\Omega}$ (so that $\mathbf{\Omega}\cdot\nabla = d/ds$) is introduced, and the collision operators C_i are defined as

$$C_i(p_iI_i) = \sigma_i p_i I_i - \sigma_{si}\int_{4\pi} f_i(\mathbf{\Omega}\cdot\mathbf{\Omega}')p_iI_i(\mathbf{\Omega}')d\Omega'. \quad (2.30)$$

Then the Titov model, derived under a Markovian assumption for both the mixing and the transport process, is given by the two coupled integral equations

$$\langle I(s)\rangle + \int_0^s C_1p_1(s')I_1(s')ds' = F, \quad (2.31)$$

$$p_1(s)I_1(s) + \int_0^s P_{11}(s',s)C_1p_1(s')I_1(s')ds' = p_1F. \quad (2.32)$$

Here the surface of the system is taken as $s = 0$, and F denotes the incoming intensity at that point. The remaining notation is the same as introduced earlier, with $P_{ij}(s', s)$ being defined as the (Markovian) conditional probability that position s is in mixture component j, given that position s' is in component i. These quantities satisfy the (forward form) Chapman–Kolmogorov equations given by

$$\frac{\partial P_{ii}}{\partial s} = \frac{P_{ij}}{\lambda_j} - \frac{P_{ii}}{\lambda_i}, \quad j \neq i, \quad (2.33)$$

$$\frac{\partial P_{ij}}{\partial s} = \frac{P_{ii}}{\lambda_i} - \frac{P_{ij}}{\lambda_j}, \quad j \neq i, \quad (2.34)$$

with boundary conditions

$$P_{ii}(s', s') = 1, \quad (2.35)$$

$$P_{ij}(s', s') = 0, \quad j \neq i. \quad (2.36)$$

It is clear from their definition that the $P_{ii}(s', s)$ satisfy the constraint

$$P_{ii} + P_{ij} = 1, \quad j \neq i. \quad (2.37)$$

In the next section, we prove in complete generality that the Titov integral model given by (2.31) and (2.32) is a special case of the rudimentary differential model given by (2.1) and (2.18).

3. The Titov model reformulated

The integral Titov model given by (2.31) and (2.32) will now be cast into an equivalent differential form.

First (2.32) is subtracted from (2.31) to find, recalling (2.2),

$$p_0(s)I_0(s) - \int_0^s P_{10}(s', s)C_1 p_1(s')I_1(s')ds' = p_0(s)F, \quad (3.1)$$

where $p_0 + p_1 = 1$ and [see (2.37)] $P_{11} + P_{10} = 1$ are used. If we differentiate Eqs. (2.31) and (2.32) we find

$$\frac{d\langle I(s)\rangle}{ds} + C_1 p_1(s)I_1(s) = 0, \quad (3.2)$$

$$\frac{d[p_1(s)I_1(s)]}{ds} + C_1 p_1(s)I_1(s) + \int_0^s \frac{\partial P_{11}(s', s)}{\partial s} \times C_1 p_1(s')I_1(s')ds' = \frac{dp_1(s)}{ds}F, \quad (3.3)$$

where use has been made of (2.35). We now use the Chapman–Kolmogorov equation given by (2.33) with $i = 1$ and $j = 0$, to rewrite (3.3) as

$$\frac{d[p_1(s)I_1(s)]}{ds} + C_1 p_1(s)I_1(s) - \frac{1}{\lambda_1(s)}\int_0^s P_{11}(s', s)C_1 p_1(s')I_1(s')ds' + \frac{1}{\lambda_0(s)}\int_0^s P_{10}(s', s)C_1 p_1(s')I_1(s')ds' = \frac{dp_1(s)}{ds}F. \quad (3.4)$$

Equations (2.32) and (3.1) can now be used to eliminate the integral terms in (3.4) and obtain, upon collecting terms,

$$\frac{d[p_1(s)I_1(s)]}{ds} + C_1 p_1(s)I_1(s) = \frac{p_0}{\lambda_0(s)}I_0(s) - \frac{p_1}{\lambda_1(s)}I_1(s) + \left[\frac{dp_1(s)}{ds} + \frac{p_1(s)}{\lambda_1(s)} - \frac{p_0(s)}{\lambda_0(s)}\right]F. \quad (3.5)$$

We recall the Chapman–Kolmogorov differential equation for the $p_i(s)$ as given by (2.4). We then see that the coefficient of F in (3.5) is zero and we have

$$\frac{d[p_1(s)I_1(s)]}{ds} + C_1 p_1(s)I_1(s) = \frac{p_0(s)}{\lambda_0(s)}I_0(s) - \frac{p_1(s)}{\lambda_1(s)}I_1(s). \quad (3.6)$$

The final algebraic manipulation is to subtract (3.6) from (3.2). The result is

$$\frac{dp_0(s)I_0(s)}{ds} = \frac{p_1(s)}{\lambda_1(s)}I_1(s) - \frac{p_0(s)}{\lambda_0(s)}I_0(s). \quad (3.7)$$

Equations (3.6) and (3.7) are the final results of our analysis and are entirely equivalent in content to the integral equations of Titov given by (2.31) and (2.32) once they are supplemented with the identity

$$\langle I(s)\rangle = p_0(s)I_0(s) + p_1(s)I_1(s), \quad (3.8)$$

and the initial conditions

$$I_0(0) = I_1(0) = F. \quad (3.9)$$

These two equations are written in a more explicit form by using $d/ds = \mathbf{\Omega}\cdot\nabla$ and the definition of the operator C_1 given by (2.30). Then

$$\mathbf{\Omega}\cdot\nabla(p_0 I_0) = \frac{p_1}{\lambda_1}I_1 - \frac{p_0}{\lambda_0}I_0, \quad (3.10)$$

$$\mathbf{\Omega}\cdot\nabla(p_1 I_1) + \sigma_1 p_1 I_1 = \sigma_{s1}\int_{4\pi} f_1(\mathbf{\Omega}\cdot\mathbf{\Omega}')p_1 I_1(\mathbf{\Omega}')d\mathbf{\Omega}' + \frac{p_0}{\lambda_0}I_0 - \frac{p_1}{\lambda_1}I_1 \quad (3.11)$$

is obtained. In this form, it is clear that the Titov integral model is a special case of the rudimentary differential model defined in the last section by (2.1) and (2.18). This special case is the case corresponding to no emission ($S_i = 0$), no interaction between the radiation and the clear sky ($\sigma_0 = \sigma_{s0} = 0$), and Markovian statistics. Thus, the rudimentary differential model effectively generalizes the Titov model to include emission and photon–sky interaction. It is clear that this generalization is also easily incorporated into the Titov integral equations. More importantly, as discussed in the last section, an algorithm is also available to account for non-Markovian statistics (nonexponential cloud size and spacing distributions) in this differential model. Lastly, a differential model should be much easier to deal with in applications since a wide variety of both analytical and numerical techniques are available in the literature to deal with differential equations of the equation of transfer type. It should be emphasized that this equivalence between the Titov integral model and the rudimentary differential model is completely general in the sense that all quantities are allowed complete three-dimensional spatial dependencies. In particular, there is no restriction to homogeneous statistics or one-dimensional symmetry.

4. Numerical results

This section presents some numerical results obtained by solving the stochastic radiative transfer in an atmosphere layer as described by (2.10), supplemented with the simple closure (2.24) (referred to as model 1) and the more complex closure given by (2.28) and (2.29) (referred to as model 2). For some test problems, our results can be compared to exact solutions. Those two models are also compared to a specific fractional cloud approximation.

The first problem considered (problem 1) is the case of a layer of thickness L populated with alternating layers of clouds and clear sky (which for our purposes is treated as a vacuum), with no internal sources of radiation. In the notation of section 2, this corresponds to considering clouds with an unbounded aspect ratio ($\gamma = \infty$ or $D = \infty$). While this problem has limited applicability to stratiform clouds, the main reason for considering it is that an explicit exact solution to this special stochastic problem is available for truly one-dimensional (or rod) geometry (Vanderhaegen and Deutsch 1989; Pomraning 1988; Stephens et al. 1991). This is mathematically equivalent to a two-stream approximation for the full planar problem, with quadrature angles chosen at $\mu = \pm 1$. Our purpose is then to use this problem as a test of the accuracy of models 1 and 2. In particular, it has been argued (Sahni 1989b) that the closure characterizing model 1 [see (2.24)] is the least accurate in this case of one-dimensional geometry. The presumption then is that this test case can provide a qualitative but reliable estimate of the accuracy of model 1, extendable to problems for which no exact solutions are available.

The two-stream approximation to (2.10) and (2.24) for a source-free medium and homogeneous statistics are written as

$$\pm \frac{d\psi_i^\pm}{dz} + \sigma_i \psi_i^\pm = \frac{\sigma_{si}}{2}(\psi_i^+ + \psi_i^-) + \frac{1}{\lambda_i}(\psi_j^\pm - \psi_i^\pm), \tag{4.1}$$

where $\psi_i^+(z)$ and $\psi_i^-(z)$ are the intensities in the $+z$ and $-z$ directions, respectively. Here $\lambda_1 = \lambda_{\rm cloud} = qH$ is the effective average vertical size of the clouds, and λ_0 is given by (2.27). Note that for any cloud size distribution that allows the possibility of infinite size clouds (such as an exponential, that is, Markovian, distribution that corresponds to $q = 1$), one would have to choose the atmospheric layer thickness L as infinite so that this layer would, for all realizations, completely contain all clouds. This points out the inappropriateness of using a Markovian model to describe an atmospheric layer in which all physical realizations completely contain the clouds in a layer of finite thickness L. Rather, one should use a cloud size distribution that involves some maximum allowable cloud size, with this maximum size being less than the layer thickness that contains all of the clouds. This in turn points out the importance of having a stochastic formalism available that allows for arbitrary cloud size distributions. In fact q, defined by (2.20), is the ingredient in the formalism described here that allows one to treat arbitrary cloud size distributions.

In writing (4.1), the scattering is treated as isotropic. Although the differential-scattering cross section is very forwardly peaked for clouds, for thick systems (such as clouds) a cross section of the Henyey–Greenstein or Mie scattering type can be adequately taken into account by using an effective cross section defined as

$$\sigma_{s,\rm eff} = \sigma_s(1 - \bar{\mu}), \tag{4.2}$$

where $\bar{\mu}$ is the average cosine of the scattering angle. Then σ_{si} is interpreted in (4.1) as meaning $\sigma_{si,\rm eff}$. For (4.1), the boundary condition

$$\psi_i^+(0) = 1, \quad \psi_i^-(L) = 0 \tag{4.3}$$

is used. In this two-stream approximation and in view of (4.3), the transmission probability T and reflection probability R are defined as

$$T = \frac{\langle \psi^+(L) \rangle}{\langle \psi^+(0) \rangle} = \langle \psi^+(L) \rangle,$$

$$R = \frac{\langle \psi^-(0) \rangle}{\langle \psi^+(0) \rangle} = \langle \psi^-(0) \rangle, \tag{4.4}$$

where

$$\langle \psi^\pm \rangle = p_0 \psi_0^\pm + p_1 \psi_1^\pm. \tag{4.5}$$

Analogously, the two-stream approximation to model 2 can be written as

$$\pm \frac{d\psi_i^\pm}{dz} + \sigma_i \psi_i^\pm = \frac{\sigma_{si}}{2}(\psi_i^+ + \psi_i^-) + \frac{1}{\lambda_i}(\bar{\psi}_j^\pm - \bar{\psi}_i^\pm), \tag{4.6}$$

$$\pm \frac{d\bar{\psi}_i^\pm}{dz} + \sigma_i \bar{\psi}_i^\pm = \frac{\sigma_{si}}{2}(\bar{\psi}_i^\pm + \bar{\psi}_j^\mp) + \frac{1}{\lambda_i}(\bar{\psi}_j^\pm - \bar{\psi}_i^\pm). \tag{4.7}$$

As boundary conditions on (4.6) and (4.7),

$$\psi_i^+(0) = \bar{\psi}_i^+(0) = 1, \quad \psi_i^-(L) = \bar{\psi}_i^-(L) = 0 \tag{4.8}$$

are used. The transmission and reflection are again given by (4.4).

For this layered geometry, two sets of cross sections representative of longwave and shortwave calculations in cumulus clouds are considered (Welch et al. 1980). The first set (referred to as case 1, corresponding to purely absorbing clouds) is given by

$$\sigma_1 = 30 \text{ km}^{-1}, \quad \sigma_0 = \sigma_{s0} = \sigma_{s1} = 0, \tag{4.9}$$

where all the cross sections are taken as constant along the vertical direction. The second set (referred to as case 2, corresponding to purely scattering clouds) is given by

$$\sigma_1 = \sigma_{s1} = 10 \text{ km}^{-1}, \quad \sigma_0 = \sigma_{s0} = 0. \tag{4.10}$$

In both case 1 and case 2, the total thickness of the layer is taken to be the unit length, that is,

$$L = 1 \text{ km}. \tag{4.11}$$

Figure 1 shows T as a function of the effective cloud thickness $\lambda_{\rm cloud}$ for case 1, as computed according to models 1 and 2. Since these curves extend to $\lambda_{\rm cloud} > L = 1$, this implies that certain realizations of the statistics correspond to the layer completely occupied by clouds. For a cloud size distribution with a maximum cloud size $M \leqslant 1$, the data in these curves beyond $\lambda_{\rm cloud}$

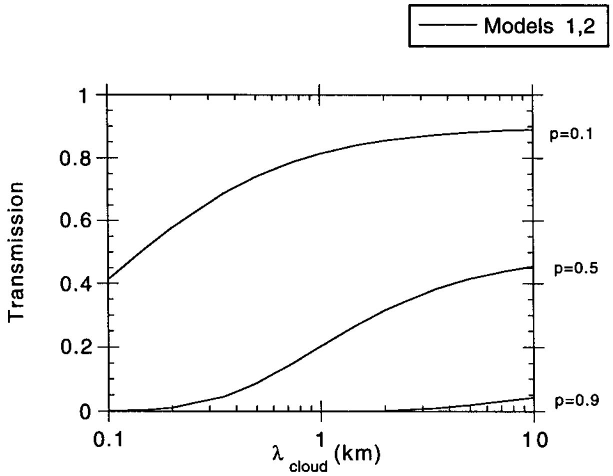

FIG. 1. Transmission versus effective cloud layer thickness for problem 1 (parallel layers and two stream approximation), values of the cross sections and total layer thickness corresponding to case 1 (pure absorber), and three values of the cloud volume fraction p.

$= q\lambda^*$ is physically not meaningful. The precise value for q and λ^* depends upon the details of the cloud size distribution. For example, if the distribution is flat (constant) between 0 and $M \leqslant 1$, one would have $\lambda^* = M/2$. Three values of the cloud volume fraction have been considered, $p_{\text{cloud}} = p = 0.1$, 0.5, and 0.9. Models 1 and 2 gives the same result, which is exact for a Markovian cloud field. For case 1 (purely absorbing clouds), $R = 0$ and $A = 1 - T$ are obviously obtained, where A is the absorption probability for the layer.

Figure 2 shows T versus λ_{cloud} for case 2. Here models 1 and 2 give different results. The thick line indicates the exact solution computed according to Pomraning (1988) and for the case of an exponential distribution. Model 2 gives a result that is very close to the exact solution. Model 1, while not as accurate as model 2, still provides a useful approximation, within about 10%, to the exact solution. This is in accordance with the results reported by Adams et al. (1989) and Titov (1990). Also note the interesting result that model 1 always overestimates the transmission. A physical explanation can be argued in view of the fact that in model 1 interface averages are approximated with volumetric averages. This has the effect of skewing the model 1 results towards the solution of a problem in which the clouds and clear sky are completely decoupled, thus overestimating the transmission. In other words, for any problem of clouds in vacuum, model 1 provides an upper bound for T. Model 2, while more accurate than model 1, does not share this property, and in fact it sometimes overestimates and sometimes underestimates T. This is also in accord with the findings of Pomraning (1991) and Malvagi and Pomraning (1992). For case 2 (purely scattering clouds), we have $R = 1 - T$ and $A = 0$.

The second problem considered (problem 2) is the more general problem of finite-size clouds ($D < \infty$) imbedded in a layer of constant thickness L. Model 1 is now written, for homogeneous statistics and after dividing (2.10) by p_i, as

$$\mu \frac{\partial \psi_i}{\partial z} + \sigma_i \psi_i = \frac{\sigma_{si}}{2} \int_{-1}^{1} \psi_i(\mu') d\mu' + \frac{1}{\lambda_i}(\psi_j - \psi_i), \tag{4.12}$$

where the $\lambda_i(\mu)$ are given by (2.26) and (2.27). Generic boundary conditions on (4.12) are given by

$$\psi_i(0) = G(\mu), \quad \mu > 0; \quad \psi_i(L) = 0, \quad \mu < 0. \tag{4.13}$$

The transmission and reflection probabilities of the layer are now defined as

$$T = \frac{\int_0^1 \mu \langle \psi_i(L, \mu) \rangle d\mu}{\int_0^1 \mu F(\mu) d\mu},$$

$$R = \frac{\int_0^1 \mu \langle \psi_i(0, -\mu) \rangle d\mu}{\int_0^1 \mu F(\mu) d\mu}. \tag{4.14}$$

Model 2 can now be written as

$$\mu \frac{\partial \psi_i}{\partial z} + \sigma_i \psi_i = \frac{\sigma_i}{2} \int_{-1}^{1} \psi_i(\mu') d\mu' + \frac{1}{\lambda_i}(\bar{\psi}_j - \bar{\psi}_i), \tag{4.15}$$

$$\mu \frac{\partial \bar{\psi}_i}{\partial z} + \sigma_i \bar{\psi}_i = \frac{\sigma_i}{2} \left[\int_{-1}^{0} \bar{\psi}_j(\mu') d\mu' + \int_0^1 \bar{\psi}_i(\mu') d\mu' \right] + \frac{1}{\lambda_i}(\bar{\psi}_j - \bar{\psi}_i), \quad \mu > 0 \tag{4.16}$$

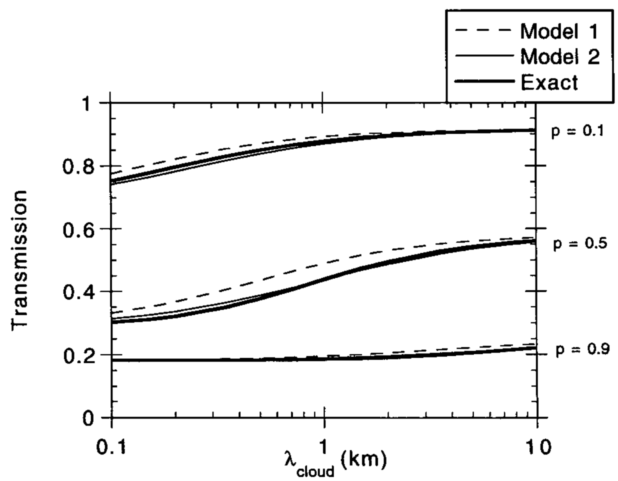

FIG. 2. Same as Fig. 1 but with values of the cross sections corresponding to case 2 (pure scatterer).

$$\mu \frac{\partial \bar{\psi}_i}{\partial z} + \sigma_i \bar{\psi}_i = \frac{\sigma_i}{2}\left[\int_{-1}^{0} \bar{\psi}_i(\mu')d\mu' + \int_{0}^{1} \bar{\psi}_j(\mu')d\mu'\right] + \frac{1}{\lambda_i}(\bar{\psi}_j - \bar{\psi}_i), \quad \mu < 0. \quad (4.17)$$

For (4.15), (4.16), and (4.17), we impose the boundary conditions

$$\psi_i(0) = \bar{\psi}_i(0) = G(\mu), \quad \mu > 0;$$
$$\psi_i(L) = \bar{\psi}_i(L) = 0, \quad \mu < 0. \quad (4.18)$$

Reflection and transmission are still defined by (4.14).

For the purpose of comparison, we also consider a specific fractional cloud model based on the solution of the two uncoupled equations

$$\mu \frac{\partial \tilde{\psi}_i}{\partial z} + \sigma_i \tilde{\psi}_i = \frac{\sigma_{si}}{2} \int_{-1}^{1} \tilde{\psi}_i d\mu', \quad (4.19)$$

with boundary conditions

$$\tilde{\psi}_i(0) = G(\mu), \quad \mu > 0; \quad \tilde{\psi}_i(L) = 0, \quad \mu < 0. \quad (4.20)$$

An approximation to the average intensity is then computed according to the weighting

$$\langle I(z, \mu) \rangle = (1 - N_c)\tilde{I}_0(z, \mu) + N_c \tilde{I}_1(z, \mu), \quad (4.21)$$

where N_c is the fractional cloud cover given by (Zuev et al. 1987)

$$N_c = 1 - (1 - p)e^{-pL/H}. \quad (4.22)$$

The N_c defined by (4.22) represents the probability of finding a cloud along the zenith line of sight under the assumption that the clouds are exponentially distributed. The model expressed by (4.19)–(4.22) is referred to as the fractional cloud model. Transmission and reflection are still defined by (4.14).

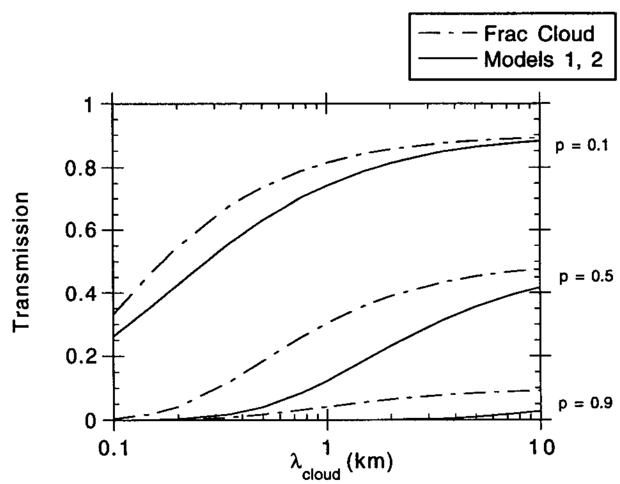

FIG. 3. Transmission versus effective cloud dimension for problem 2 (finite-size clouds imbedded in a layer of thickness L), values of the cross sections and total layer thickness corresponding to case 1, isotropic statistics (cloud aspect ratio $\gamma = D/H = 1$), isotropic incidence, and three values of the cloud volume fraction p.

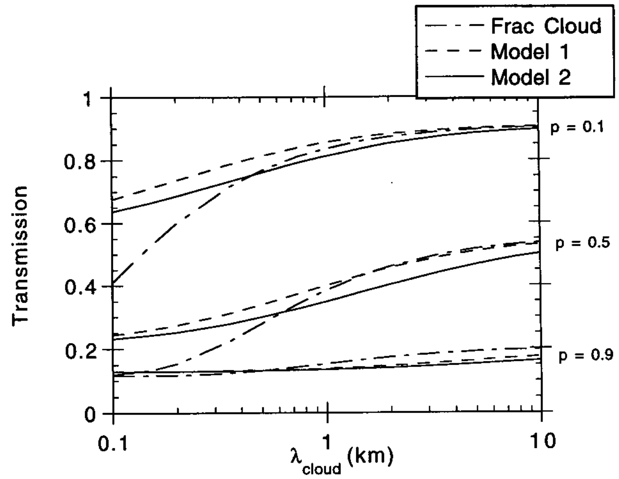

FIG. 4. Same as Fig. 3 but with values of the cross sections corresponding to case 2.

For the values of the cross sections and the slab thickness, the same two sets previously called case 1 and case 2 are used. The transfer equations are solved numerically using diamond differencing in space, the discrete ordinate method with 16 angles, and a power iteration on the scattering source (Duderstadt and Martin 1979; Bell and Glasstone 1970). First we consider the case of isotropic statistics [i.e., $\lambda_i \neq \lambda_i(\mu)$]. This corresponds to considering clouds with aspect ratio equal to unity, that is, $H = D$. For this case of isotropic statistics, two types of boundary conditions are considered. The first type considered is an isotropic incidence, that is,

$$G(\mu) = 1. \quad (4.23)$$

Figure 3 shows T versus λ_{cloud} for isotropic statistics ($\gamma = 1$), isotropic incidence, and values of the physical parameters corresponding to case 1, as computed according to models 1 and 2 and the fractional cloud model. For consistency with the assumption of the fractional cloud model (exponentially distributed clouds), a value of $q = 1$ is used in evaluating λ_{cloud} [see (2.26)]. As before, the three values $p = 0.1$, 0.5, and 0.9 are used. Both models 1 and 2 provide the same solution to the stochastic problem, which is exact for exponential distributions. The fractional cloud model overestimates the transmission, particularly for intermediate values of λ_{cloud}. Figure 4 shows T versus λ_{cloud} for the same problem, but with the values of the physical parameters corresponding to case 2. No exact solution is available here, and the solution of model 2 is taken as the best available approximation. Based on this assumption, model 1 and the fractional cloud model are roughly equivalent for values of λ_{cloud} from intermediate to large (λ_{cloud} between 1 and 10). For small values of λ_{cloud} (λ_{cloud} less than 1), the fractional cloud model significantly underestimates the transmission. Model 1 consistently provides an overestimate of the transmission by about 10% (or less).

Next isotropic statistics ($\gamma = 1$) with a beam incident upon the layer at some angle is considered. That is, the boundary condition $G(\mu)$ is taken to be a Dirac delta function

$$G(\mu) = \delta(\mu - \mu_0), \tag{4.24}$$

where μ_0 is the cosine of the angle θ_0 of incidence of the beam ($\theta_0 = 0$, or $\mu_0 = 1$, corresponds to a beam perpendicular to the layer). The azimuthal angle of the incident beam φ_0 is irrelevant since an integration has been made upon φ [see (2.17)]. Figures 5 and 6 show T versus λ_{cloud} for the case $\theta_0 = 0°$, and the values of the physical parameters corresponding to case 1 and case 2, respectively. Figures 7 and 8 show T versus λ_{cloud} for $\theta_0 = 30°$, and Figs. 9 and 10 show T versus λ_{cloud} for $\theta_0 = 60°$. We identify the following trends. The comparison between model 1 and model 2 follows the same pattern as before, with model 1 overestimating the transmission when scattering is present. The fractional cloud model provides a relatively good estimate of the transmission for $p = 0.1$, $\theta_0 = 0°$, and $\lambda_{\text{cloud}} \geqslant 1$. In all other cases, it overestimates the transmission except for small values of λ_{cloud}, for which it underestimates the transmission when scattering is present.

Finally, we consider the effects of the cloud aspect ratio on the predictions of our models. Figure 11 shows T versus H for an isotropic incidence and for values of the physical parameters corresponding to case 2, as computed according to model 2 with values of the aspect ratio $\gamma = 0.1$, 1, and 10. The values $p = 0.1$, 0.5, and 0.9 for the cloud volume fraction are used. It is clear that the transmission of the atmospheric layer decreases with decreasing values of γ. This is due to the fact that H is kept constant so that small values of γ correspond to small clouds. Figures 12 and 13 are the same as Fig. 11 but with an incident beam at an angle $\theta_0 = 0°$ and $\theta_0 = 30°$, respectively. The effect of the aspect ratio increases dramatically with the angle of incidence. Results for model 1, not reported here, show the same dependence of the transmission on the cloud aspect ratio.

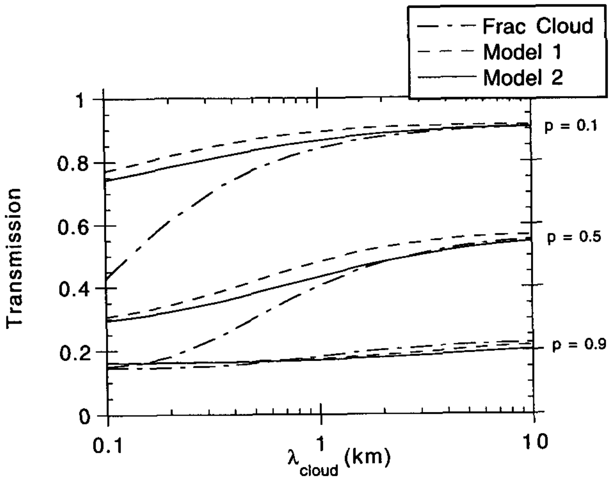

FIG. 6. Same as Fig. 4 but with a beam incident upon the layer at an angle $\theta_0 = 0°$.

5. Summary and conclusions

In this paper, we have presented what we feel is a promising approach for the treatment of radiative transfer through an atmospheric layer populated with randomly distributed clouds. This approach treats the radiative transfer problem as a stochastic process and provides predictions for the average radiative intensity once the statistical description of the cloud field is given. This approach is entirely based on the properties of the equation of transfer itself, and there is no need to use approximate empirical relations.

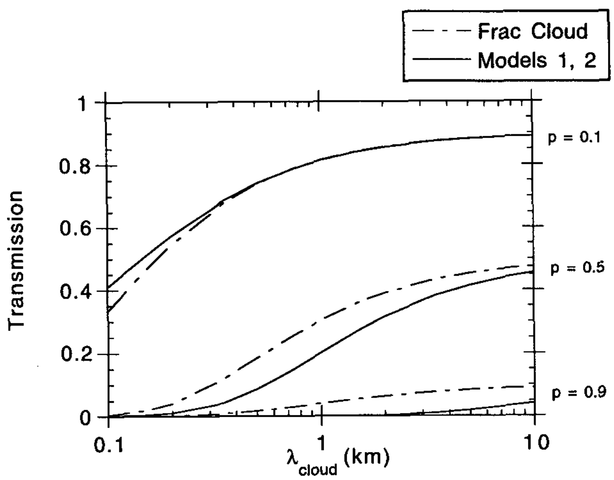

FIG. 5. Same as Fig. 3 but with a beam incident upon the layer at an angle $\theta_0 = 0°$.

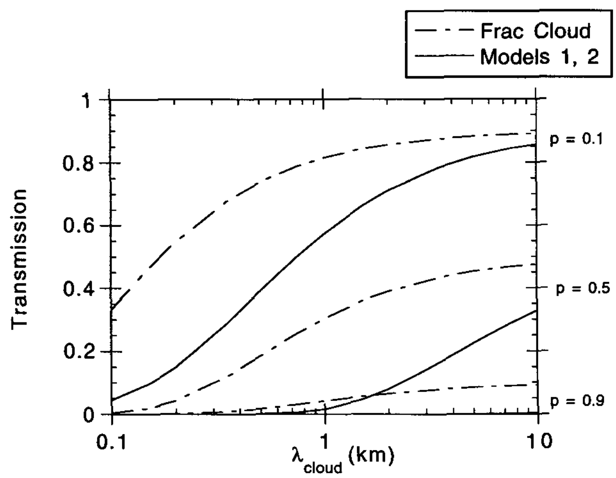

FIG. 7. Same as Fig. 5 but with $\theta_0 = 30°$.

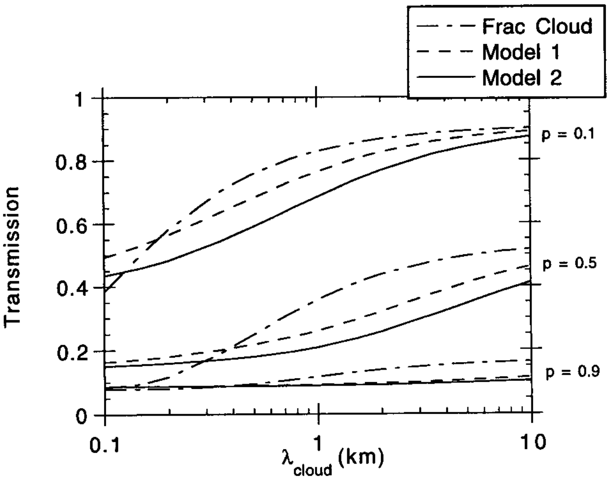

FIG. 8. Same as Fig. 6 but with $\theta_0 = 30°$.

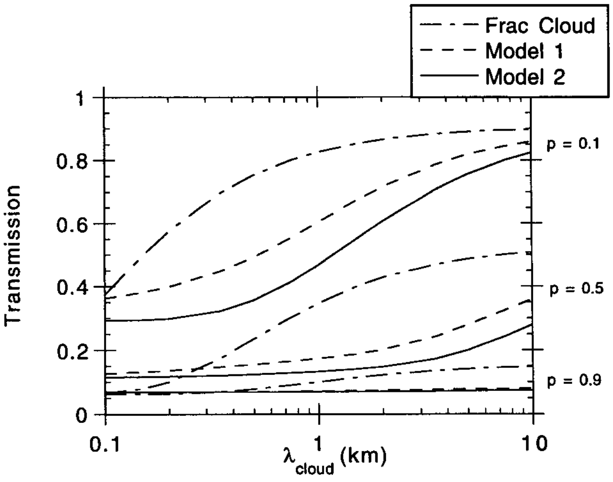

FIG. 10. Same as Fig. 6 but with $\theta_0 = 60°$.

In dealing with the stochastic equation of transfer, one is confronted with a problem of closure since the balance equation for the volumetric averages involves interface averages. Two possible closures are discussed. The simplest closure produces a set of two coupled integro–differential equations [(2.1) and (2.18)], that are referred to as model 1. This model is exact for purely absorbing media with Markovian statistics, but it is approximate when scattering is present and/or if the statistics are non-Markovian. This model is also shown to be equivalent to the earlier Markovian model of Titov and coauthors involving integral equations in the particular case considered by those authors. A more sophisticated closure, [(2.1), (2.28), and (2.29)] produces a set of four coupled integro–differential equations, which are referred to as model 2. This second model, while still approximate when scattering is present, is shown to give very accurate predictions in the cases when exact solutions are available. The models of stochastic radiative transfer discussed here apply equally well to all wavelengths. In particular, they treat the incoming short wavelengths and the reemitted long wavelengths by the same formalism. It should be emphasized that both of these models are formulated for non-Markovian statistics (nonexponential distributions). This is an important issue since experimental characterizations of cloud fields often employ power-law distributions.

Both models account for nonhomogeneous statistics and in particular allow for vertical variations of the statistical characteristics of the cloud field. Also, these models, while presented here for a mixture of two components (clouds and clear sky), can be easily gen-

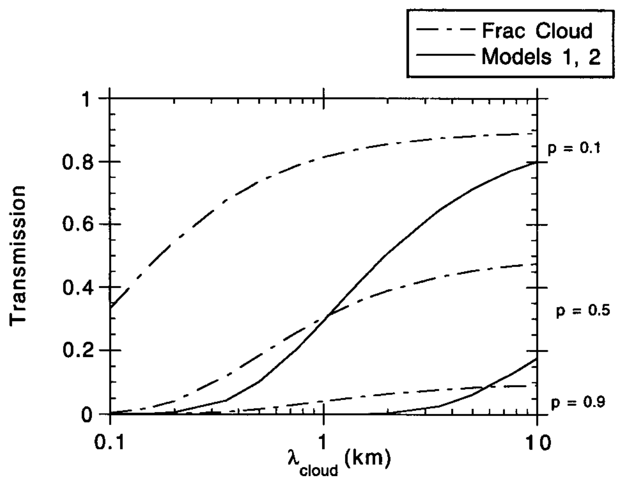

FIG. 9. Same as Fig. 5 but with $\theta_0 = 60°$.

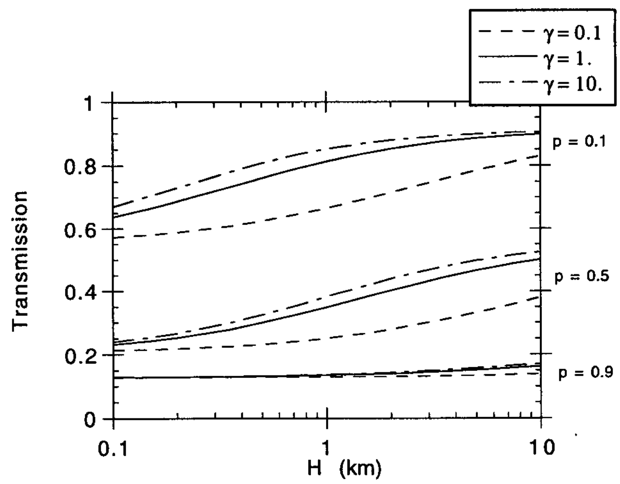

FIG. 11. Transmission versus effective cloud vertical dimension for problem 2, values of the cross sections and total layer thickness corresponding to case 1, three values of the cloud volume fraction p, three values of the aspect ratio γ, and isotropic incidence upon the layer. All curves are computed according to model 2.

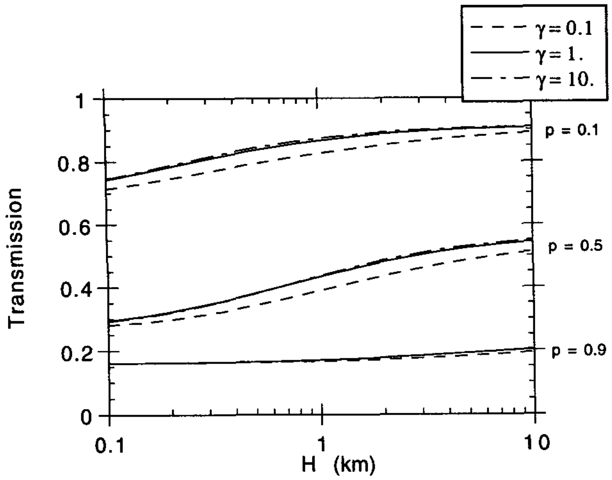

FIG. 12. Same as Fig. 11 but with a beam incident upon the layer at an angle $\theta_0 = 0°$.

eralized to a mixture of an arbitrary number of components (Malvagi and Pomraning 1990). For instance, clear sky and several kinds of clouds with different physical properties can be treated simultaneously in a single set of coupled equations. Combining these two features, inhomogeneous statistics and an arbitrary number of components, one could in principle solve for the radiative intensity in the entire vertical extension of the single-cell atmospheric column at once, without the need to subdivide it into relatively thin layers (except for numerical discretization purposes).

Numerical results for a number of test problems clearly indicate that the discrepancy between the predictions of the stochastic transfer model and a specific fractional cloud model can be quite significant. These results thus underscore the need for an approach to the problem of transfer in partially cloudy atmospheres that is more realistic than the ones currently in use. Any existing numerical algorithms for the solution of the equation of transfer, such as the popular two-stream (diffusive) model, apply directly to our stochastic method. Thus, it would be relatively straightforward to incorporate our method into existing GCM radiative transfer treatments. Also note in this regard that the variation of cloud characteristics and fractional cloud cover are easily incorporated. The various parameters that enter are simply allowed to be spatially (both vertically and laterally) dependent.

Let us now touch upon what these parameters are. That is, what information is required from the cloud model to be able to implement this stochastic radiative transfer technique? In all instances, one requires the radiative properties, namely, the total and scattering cross sections and the scattering phase function for pure clouds and clear atmosphere. In the simplest model of Markovian statistics, one additionally requires the mean cloud size and the cloud volume fraction (or the mean cloud spacing). The theory allows these lengths to be direction dependent. For non-Markovian statistics, one requires the distributions of both the cloud size and the cloud spacing, although numerical tests seem to indicate that to a good approximation, only the mean and the variance of these distributions affect the radiative transfer (Levermore et al. 1988).

The advantage of the stochastic approach is that it can accurately calculate the radiative heating rates through a broken cloud layer without requiring an exact description of the cloud geometry. The methods discussed here provide the average solution for this radiative transfer problem that depends on macroscopic properties of each component in the broken cloud layer and the statistical description (sizes and distribution) of the clouds within the layer. Some cloud data of this type is already available and more will become available in the future. This technique is especially relevant to the future predictive capability of GCMs because, as they become more sophisticated, it is expected that they will be able to provide the type of cloud data required by any stochastic technique. In addition, the stochastic treatment is sufficiently general to allow the inclusion of higher moments of the cloud distribution functions should such detail prove necessary.

We hasten to add, however, that the stochastic treatment presented here is far from a complete description of radiative transfer through a partially cloudy atmosphere. It may be useful for the reader if we end these concluding remarks with an explicit discussion of the limitations of the models presented. The first point to be emphasized is that these models are just that; they are models that are only exact in the no scattering limit with both the cloud and the cloud separation chord lengths described by Markovian (exponential) statistics. While the no scattering limit is applicable at certain wavelengths, the Markovian assumption is not realistic for the real-world situation. The second point to be made is that two basic assumptions are built into these

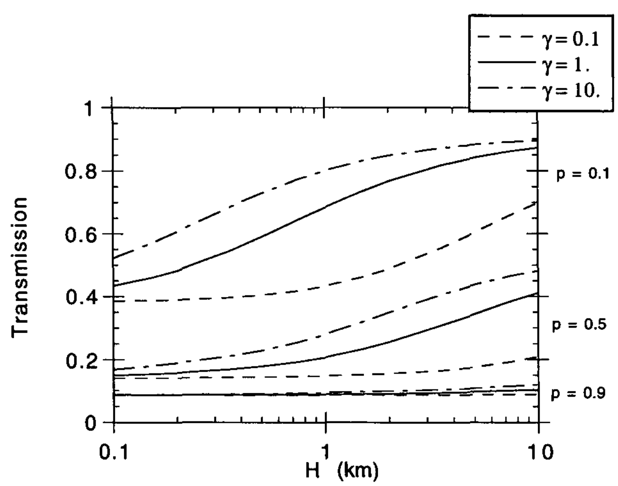

FIG. 13. Same as Fig. 12 but with $\theta_0 = 30°$.

models. The first of these is that while the models are capable of treating non-Markovian statistics, this non-Markovian treatment is restricted to renewal statistics (Levermore et al. 1988). These statistics are characterized by prescribing the chord-length distributions separately for the clouds and cloud separation. This is a very narrow subset of all possible statistics and ignores any complex correlations between the clouds and their spacing. Such correlations would be described by a nonseparable joint distribution function, rather than separate chord-length distributions for the clouds and their spacing.

The second limitation of the models presented here has to do with the internal constitution of the clouds. Although these models allow both vertical and lateral average property variation of the clouds, they do not take into account the relatively small-scale statistical nature of the internal structure of the clouds. Specifically, the absorption and scattering coefficients have been taken to be deterministic functions of space; the only stochasticity allowed in the models are the location and geometry of the clouds and their separation. This aspect of the problem, however, can probably be treated, albeit in an approximate way, by applying statistical techniques to the clouds themselves in order to homogenize a statistically heterogeneous cloud (Malvagi and Pomraning 1990). This procedure averages out the statistical nature of a cloud and leads to effective deterministic absorption and scattering coefficients. This analysis, which we have not discussed in this paper, is an asymptotic limit of our model 1, corresponding to a small amount of large extinction coefficient material admixed with a large amount of small extinction coefficient material. If this strategy is added to the models discussed in this paper, the result would be a two-step statistical treatment of radiative transfer through a broken cloud field. One would first concentrate on the clouds themselves, using stochastic methods to replace the statistically heterogeneous clouds with (approximately) equivalent homogeneous clouds. This would be followed by the application of the models given in this paper to study the radiative transfer through a stochastic mixture of homogeneous clouds and clear sky. This calculational strategy has not, to our knowledge, been tried or tested in any way. It represents a research activity for the future.

In summary, we feel that the kinetic theory methods outlined in this paper represents a reasonable starting point for a stochastic treatment of radiative transfer through a partially cloudy atmosphere. This problem is clearly not a simple one, and we regard the methods discussed here as a step along the way in dealing with the complex real-world situation.

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