The Optimal Balance in a Low-Order Atmospheric Model

SHUCAI GUAN

Division of Meteorology, Department of Geography, University of Alberta, Edmonton, Alberta, Canada

6 April 1992 and 3 November 1992

ABSTRACT

The Baer–Tribbia scheme leads to a divergent series in locating the slow manifold of Lorenz's model. The optimal asymptotic approximation is used to "sum" the divergent series. The method gives reasonable approximations to the full solutions of the model and provides the optimal balance relations. The "imbalance," which is the difference between the actual flow and the optimal balance state, is found to consist of nearly monochromatic inertial–gravity waves. However, the optimal asymptotic approximation fails to give a reasonable estimate of the level of inertial–gravity wave activity from the Rossby modes. The reason may be that the numerical experiments are undertaken at moderate Rossby numbers, whereas the notion of an optimal expansion strictly applies only in the limit of the small Rossby number.

1. Introduction

Leith (1980) and Lorenz (1980) introduce independently the notion of a slow manifold on which the flow is completely devoid of gravity wave activity. The concept of the slow manifold would reduce the initialization problem, whose purpose is to suppress rapid oscillations in numerical weather prediction models by adjusting input data, to a projection of the initial state of the atmosphere onto the manifold. A number of schemes (Baer and Tribbia 1977; Lorenz 1980) based on the assumption of small Rossby number and slow time scales have been proposed for constructing a sequence of manifolds $\mathbf{U}_n$. The assumption that the sequence $\mathbf{U}_n$ converges to an invariant manifold leads to the slave relation (Warn and Ménard 1986)

$$\mathbf{G} = \mathbf{U}(\mathbf{R}, \epsilon), \qquad (1)$$

where **G** and **R** are vectors whose components are the amplitudes of the fast (inertial–gravity) and slow (Rossby) modes used in the normal-mode representation of the dependent variables. Equation (1) defines the slow manifold. The existence of the relation (1) implies that long-time trajectories or states can be reconstructed from a knowledge of the Rossby modes alone.

Kopell (1985) showed that a slow manifold can exist in some particular cases and the results of Jacobs (1991) imply that balance schemes can sometimes converge. However, Warn and Ménard's (1986) numerical experiments with a simple nine-dimensional model (Lorenz 1980; called the L80 model herein) showed that for realistic atmospheric parameters, high-frequency gravity waves were almost always present, even in the limit of small Rossby numbers and long times. Numerical calculations by Lorenz (1986) and Vautard and Legras (1986) favored Warn and Ménard's results. If this indeed is the case, the balance schemes do not lead to a precise slave relation (1) implying that the notion of balance needs to be reexamined (Warn and Ménard 1986). Do all flows exhibit some degree of imbalance? If so, how much information can be obtained from balance relations?

In this note, we apply the balance scheme of Baer and Tribbia (1977, called the BT scheme herein) to the L80 model to construct a sequence of manifolds $\mathbf{U}_n$ and then employ the optimal asymptotic approximation to answer the above questions.

2. Results and discussion

The L80 model is the nine-dimensional system [Eqs. (33)–(35) in Lorenz (1980)]. The x_i, y_i, and z_i are scaled versions of the time coefficients of the velocity potential, the streamfunction, and the free surface height. For quasigeostrophic flows, the Rossby number ϵ is small, the vorticity and height are order ϵ, and the divergence is order ϵ^2, that is, $x_i \sim O(\epsilon^2)$, $y_i \sim z_i \sim O(\epsilon)$, where $i = 1, 2, 3$. In this case, $\mathbf{R} = (y_1, y_2, y_3)'$ is a three-dimensional variable and $\mathbf{G} = (x_1, x_2, x_3, y_1 - z_1, y_2 - z_2, y_3 - z_3)'$ is a six-dimensional variable. Note $y_i - z_i \sim O(\epsilon^2)$. All model parameters are identical to those used by Lorenz (1980), with the exception of $F_1 = 0.24$ to reduce the computing time. Model integrations were carried out for 220 days using a double precision, eighth-order version of the Taylor series time-stepping scheme used by Lorenz (1980)

Corresponding author address: Mr. Shucai Guan, Division of Meteorology, Department of Geography, University of Alberta, Edmonton, Alberta, T6G 2H4, Canada.

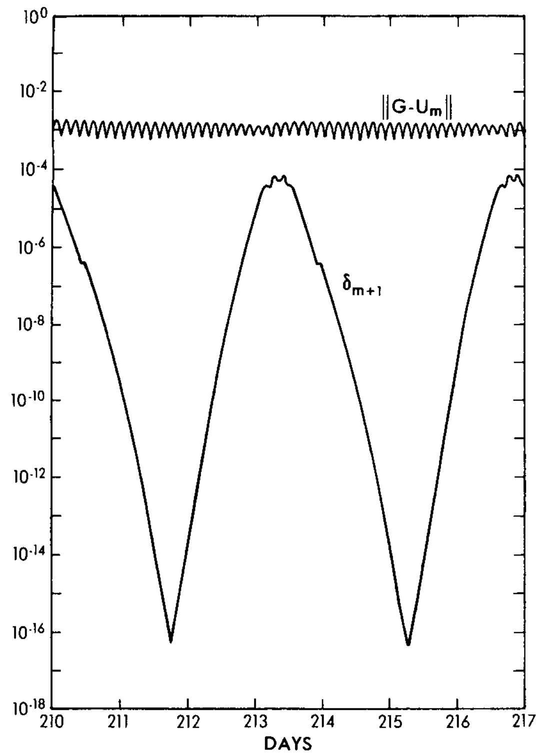

FIG. 1. Time variations of actual error $\|\mathbf{G} - \mathbf{U}_m\|$ and estimated error δ_{m+1} by the optimal asymptotic approximation for $F_1 = 0.24$.

with a time step of 7.5 min and initial conditions $x_i = y_i = z_i = 0.1$. The long-time (200 days) trajectory involved intermittent gravity-wave activity (Vautard and Legras 1986). The solution became periodic with a period of about 7 days (Guan 1991). In this case, it is clear that there can be no slow manifold that is completely devoid of gravity waves. There would evidently be a limitation to accuracy of balance relations. It is, therefore, of interest to find the accuracy to which the three variables **R** determine the other six variables **G**.

In our case, all manifolds $\mathbf{U}_n$ constructed by the BT scheme diverge because the sequence $\delta_n = \|\mathbf{U}_n - \mathbf{U}_{n-1}\|$, where the Euclidean norm is implied, first decays rather rapidly but soon decreases less rapidly and finally increases (Guan 1991). To examine the divergent series, we consider the optimal asymptotic approximation in an attempt to approximate the "sum" of the divergent series. The optimal estimate is given by $\mathbf{U}_m$, where m is chosen so that δ_{m+1} is the smallest. The unbalanced residual by subtracting the optimal approximation from the full numerical solution, that is, $\|\mathbf{G} - \mathbf{U}_m\|$, is found to be of the order 10^{-3}, while $\|\mathbf{G}\|$ is of order 10^{-1}. This indicates that despite the fact that the balance sequence $\mathbf{U}_n$ diverges, the optimal estimate $\mathbf{U}_m$ gives reasonable approximations to the full solution **G**, that is, **R** determines **G** accurately, especially in the more realistic case of $F_1 = 0.1$, $\|\mathbf{G} - \mathbf{U}_m\|$ is about order 10^{-8} (Guan 1991).

It would be of interest to give an a priori estimate of the error associated with the optimal balance condition, that is, of $\|\mathbf{G} - \mathbf{U}_m\|$. Warn and Ménard (1986) speculated that this might be of the order of δ_{m+1}, since the latter should be an estimate of the first neglected term in an optimal Rossby number expansion. Figure 1 shows $\|\mathbf{G} - \mathbf{U}_m\|$ and δ_{m+1} calculated every 0.375 h beginning at day 210. Imbalance consists of nearly monochromatic gravity waves and δ_{m+1} is from 2 to 13 orders of magnitude smaller than $\|\mathbf{G} - \mathbf{U}_m\|$. Thus, the optimal approximation fails to give a reasonable estimate of the associated error. This result seems to be at variance with the discussion of Warn and Ménard (1986). The reason may be that ϵ is relatively large or that the interpretation of balance schemes as asymptotic representations to full solutions is not entirely correct.

Acknowledgments. T. Warn's guidance and G. Reuter's support are greatly appreciated. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

Baer, F., and J. Tribbia, 1977: On complete filtering of gravity waves through nonlinear initialization. *Mon. Wea. Rev.*, **105**, 1536–1539.

Guan, S., 1991: The optimal balance in a low-order atmospheric model. M.Sc. thesis, Department of Meteorology, McGill University, Montreal, 81 pp.

Jacobs, S. J., 1991: Existence of a slow manifold in a model system of equations. *J. Atmos. Sci.*, **48**, 893–901.

Kopell, N., 1985: Invariant manifolds and the initialization problem for some atmospheric equations. *Physica D*, **14**, 203–215.

Leith, C., 1980: Nonlinear normal-mode initialization and quasi-geostrophic theory. *J. Atmos. Sci.*, **37**, 958–968.

Lorenz, E. N., 1980: Attractor sets and quasigeostrophic equilibrium. *J. Atmos. Sci.*, **37**, 1685–1699.

——, 1986: On the existence of a slow manifold. *J. Atmos. Sci.*, **43**, 1547–1557.

Vautard, R., and B. Legras, 1986: Invariant manifolds, quasigeostrophy and initialization. *J. Atmos. Sci.*, **43**, 565–584.

Warn, T., and R. Ménard, 1986: Nonlinear balance and gravity–inertial wave saturation in a simple atmospheric model. *Tellus*, **38A**, 285–294.