NOTES AND CORRESPONDENCE

A Scaling for the Three-Dimensional Semigeostrophic Approximation

GEORGE C. CRAIG

Department of Meteorology, University of Reading, Whiteknights, Reading, United Kingdom

4 September 1992 and 28 January 1993

ABSTRACT

The three-dimensional semigeostrophic equations are derived by a formal asymptotic analysis. The equations are scaled with typical values of the gradients across absolute momentum contours along isentropic surfaces and are accurate to first order in a small parameter that can be regarded as a modified Rossby number.

1. Introduction

The three-dimensional semigeostrophic equations were introduced by Hoskins (1975), and have been employed to study a number of atmospheric phenomena (e.g., Hoskins and West 1979; Schär and Davies 1990). The mathematical structure of the equations has also been widely investigated (e.g., Purser and Cullen 1987; Salmon 1985, 1988). However, recent studies comparing semigeostrophic and primitive equation solutions have emphasized inaccuracies in the approximate solutions (Moore and Peltier 1989; Snyder et al. 1991). The question of where semigeostrophic theory should be expected to be accurate is not easily answered. While the semigeostrophic momentum equations contain terms that are absent in the corresponding quasigeostrophic forms, asymptotic analysis shows that both contain errors at order Ro^2, where Ro is the Rossby number (McWilliams and Gent 1980; Synder et al. 1991). In the conventional scaling, semigeostrophic theory is formally no more accurate than quasigeostrophic.

That the semigeostrophic equations do not arise naturally from an expansion in Rossby number does not preclude the possibility that they can be derived from some other expansion. A clue is offered by Hoskins and Bretherton's (1972) derivation of the two-dimensional semigeostrophic approximation. Their method was to scale the two horizontal directions differently, and assume that one length scale (the "along-front" direction) is much larger than the other (the "cross-front" direction). The approximate equations were obtained as a truncated expansion in the ratio of the two horizontal scales. For the three-dimensional equations, the heuristic derivation of Hoskins (1975) suggests that an appropriate scaling would involve different length scales parallel and perpendicular to the local direction of the flow. There are two ways that one might write the equations to make such a scaling manifest. They could be written in a curvilinear coordinate system where the axes have the desired orientation, or in coordinates that are stretched so that the scaling in the transformed space is isotropic. It is the second approach that will be used here, and the appropriate transformed space turns out to be simply the well-known geostrophic and isentropic coordinate system. The Coriolis parameter will be allowed to vary, leading to Salmon's (1985) generalization of Hoskins' (1975) equations. A closely related generalization is discussed by Magnusdottir and Schubert (1990) for the β-plane case.

2. Derivation of the semigeostrophic equations

The starting point of the derivation is the Boussinesq primitive equations as given by Hoskins (1975) except that the Coriolis parameter f is assumed to be a given function of x and y, rather than constant. The primitive equations are transformed to geostrophic and isentropic coordinates,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} X \\ Y \\ \theta \end{pmatrix} = \begin{pmatrix} x + v_g/f(x, y) \\ y - u_g/f(x, y) \\ \theta \end{pmatrix}, \qquad (1)$$

where the components of the geostrophic wind are given by $u_g = -f(x, y)^{-1}\partial\phi/\partial y$ and $v_g = f(x, y)^{-1}\partial\phi/\partial x$. Note that z is the pseudoheight coordinate defined by Hoskins and Bretherton (1972)

$$z = \frac{\theta_0}{g}(c_p - \Pi(p)), \qquad (2)$$

with $\Pi(p) = c_p(p/p_0)^\kappa$ the Exner function. The transformed momentum, hydrostatic, and continuity equations are

Corresponding author address: Dr. George C. Craig, Department of Meteorology, University of Reading, 2 Early Gate, P.O. Box 239, Whiteknights, Reading, UK, RG6 2AU.

$$\frac{Du}{Dt} - f(x, y)v + \frac{\partial M^*}{\partial X} = \frac{1}{f(x, y)}(u_g^2 + v_g^2)\frac{\partial f(x, y)}{\partial X} \quad (3)$$

$$\frac{Dv}{Dt} + f(x, y)u + \frac{\partial M^*}{\partial Y} = \frac{1}{f(x, y)}(u_g^2 + v_g^2)\frac{\partial f(x, y)}{\partial Y} \quad (4)$$

$$-\Pi + \frac{\partial M^*}{\partial \theta} = \frac{1}{f(x, y)}(u_g^2 + v_g^2)\frac{\partial f(x, y)}{\partial \theta} \quad (5)$$

$$\frac{\partial(u, y, \Pi)}{\partial(X, Y, \theta)} + \frac{\partial(x, v, \Pi)}{\partial(X, Y, \theta)} + \frac{\partial(x, y, D\Pi/Dt)}{\partial(X, Y, \theta)} = 0. \quad (6)$$

The Bernoulli function M^* is defined by

$$M^* = \phi + \theta\Pi + \frac{1}{2}(u_g^2 + v_g^2), \quad (7)$$

and the advective derivative is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{DX}{Dt}\frac{\partial}{\partial X} + \frac{DY}{Dt}\frac{\partial}{\partial Y} + Q\frac{\partial}{\partial \theta}, \quad (8)$$

where $Q = D\theta/Dt$ is the diabatic heating rate. The terms on the right-hand sides of (3)–(5) vanish if f is constant. In this coordinate system, u_g and v_g are defined implicitly by the relations

$$u_g = -\frac{1}{f(x, y)}\frac{\partial M^*}{\partial Y} + \frac{1}{f(x, y)^2}(u_g^2 + v_g^2)\frac{\partial f(x, y)}{\partial Y} \quad (9)$$

$$v_g = \frac{1}{f(x, y)}\frac{\partial M^*}{\partial X} + \frac{1}{f(x, y)^2}(u_g^2 + v_g^2)\frac{\partial f(x, y)}{\partial X}. \quad (10)$$

The equations will be nondimensionalized as follows,[1]

$$(X, Y) = (L\tilde{X}, L\tilde{Y}) \quad (11)$$

$$\theta = \theta_0 + \Theta\tilde{\theta} \quad (12)$$

$$M^* = f_0 UL\tilde{M}^* \quad (13)$$

$$f = f_0\tilde{f} \quad (14)$$

$$\Pi = \frac{f_0 UL}{\Theta}\tilde{\Pi} \quad (15)$$

$$\frac{D}{Dt} = \frac{U}{L}\frac{D}{D\tilde{t}} \quad (16)$$

$$Q = \frac{\Theta U}{L}\tilde{Q}. \quad (17)$$

The scaling for potential temperature in (12) does not require the vertical gradient to be large in comparison with the horizontal variations, as is assumed in quasi-geostrophic theory. Instead, a form is chosen following Hoskins and Bretherton (1972), which is appropriate for frontal zones where large horizontal gradients may exist. It should be emphasized that L is not a length scale since X and Y are actually components of the geostrophic absolute momentum, divided by f to give dimensions of length. The assumption of a characteristic scale in the transformed space implies a characteristic magnitude for variations of quantities across absolute momentum contours on isentropic surfaces. In regions such as frontal zones, where absolute momentum surfaces are close together, the gradients in physical space may be quite large. From a mathematical standpoint, it would perhaps be more natural to use the components of absolute momentum as horizontal coordinates (Gill 1981); geostrophic coordinates are used in this paper since it is in this system that the semigeostrophic equations take their most familiar form.

The nondimensionalized equations are

$$\epsilon\frac{D\tilde{u}}{D\tilde{t}} - \tilde{f}(x, y)\tilde{v} + \frac{\partial\tilde{M}^*}{\partial\tilde{X}} = \epsilon\frac{1}{\tilde{f}(x, y)}(\tilde{u}_g^2 + \tilde{v}_g^2)\frac{\partial\tilde{f}(x, y)}{\partial\tilde{X}} \quad (18)$$

$$\epsilon\frac{D\tilde{v}}{D\tilde{t}} + \tilde{f}(x, y)\tilde{u} + \frac{\partial\tilde{M}^*}{\partial\tilde{Y}} = \epsilon\frac{1}{\tilde{f}(x, y)}(\tilde{u}_g^2 + \tilde{v}_g^2)\frac{\partial\tilde{f}(x, y)}{\partial\tilde{Y}} \quad (19)$$

$$-\tilde{\Pi} + \frac{\partial\tilde{M}^*}{\partial\tilde{\theta}} = \epsilon\frac{1}{\tilde{f}(x, y)}(\tilde{u}_g^2 + \tilde{v}_g^2)\frac{\partial\tilde{f}(x, y)}{\partial\tilde{\theta}} \quad (20)$$

$$\frac{\partial(\tilde{u}, \tilde{Y} + \epsilon\tilde{u}_g/\tilde{f}(x, y), \tilde{\Pi})}{\partial(\tilde{X}, \tilde{Y}, \tilde{\theta})} + \frac{\partial(\tilde{X} - \epsilon\tilde{v}_g/\tilde{f}(x, y), \tilde{v}, \tilde{\Pi})}{\partial(\tilde{X}, \tilde{Y}, \tilde{\theta})} + \frac{\partial(\tilde{X} - \epsilon\tilde{v}_g/\tilde{f}(x, y), \tilde{Y} + \epsilon\tilde{u}_g/\tilde{f}(x, y), D\tilde{\Pi}/D\tilde{t})}{\partial(\tilde{X}, \tilde{Y}, \tilde{\theta})} = 0, \quad (21)$$

where the quantities $(\tilde{u}_g, \tilde{v}_g) = (u_g/U, v_g/U)$ have been defined for convenience. Since U represents a typical value for the total wind, no assumption is made about the magnitude of the geostrophic part.

The parameter

$$\epsilon = \frac{U}{f_0 L}, \quad (22)$$

[1] This scaling is also considered briefly by McWilliams and Gent (1980) in their appendix A.

which appears in these equations, resembles the usual definition of the Rossby number, differing only in the interpretation of L, which is no longer simply a length scale. It is reasonable to regard ϵ as a modified Rossby number, since it still represents the ratio of inertial over Coriolis forces. However, the inertial terms are now estimated by advection across absolute momentum contours on isentropic surfaces, rather than advection over distance on height or pressure surfaces.

To derive a set of equations that are asymptotically valid for small ϵ, the dependent variables are expanded in series,

$$\tilde{u} = \tilde{u}_0 + \epsilon\tilde{u}_1 + \cdots \quad (23)$$

$$\tilde{v} = \tilde{v}_0 + \epsilon\tilde{v}_1 + \cdots \quad (24)$$

$$\tilde{M}^* = \tilde{M}^*_0 + \epsilon\tilde{M}^*_1 + \cdots \quad (25)$$

$$\tilde{\Pi} = \tilde{\Pi}_0 + \epsilon\tilde{\Pi}_1 + \cdots. \quad (26)$$

An expansion for the Coriolis parameter can be found by taking, at each point (x, y), a Taylor series about the corresponding (X, Y), that is,

$$\tilde{f}(x, y) = \tilde{f}(X, Y) + \left.\frac{\partial\tilde{f}(x, y)}{\partial\tilde{x}}\right|_{X,Y}(\tilde{x} - \tilde{X}) + \left.\frac{\partial\tilde{f}(x, y)}{\partial\tilde{y}}\right|_{X,Y}(\tilde{y} - \tilde{Y}) + \cdots \quad (27)$$

$$= \tilde{f}(X, Y) + \epsilon\frac{\partial\tilde{f}(X, Y)}{\partial\tilde{X}}\left(\frac{-\tilde{v}_g}{\tilde{f}(X, Y)}\right) + \epsilon\frac{\partial\tilde{f}(X, Y)}{\partial\tilde{Y}}\left(\frac{\tilde{u}_g}{\tilde{f}(X, Y)}\right) + O(\epsilon^2). \quad (28)$$

Using the definitions of $\tilde{X}$ and $\tilde{Y}$, the advective derivative can be expanded as

$$\frac{D}{D\tilde{t}} = \frac{\partial}{\partial\tilde{t}} + \frac{D\tilde{X}}{D\tilde{t}}\frac{\partial}{\partial\tilde{X}} + \frac{D\tilde{Y}}{D\tilde{t}}\frac{\partial}{\partial\tilde{Y}} + \tilde{Q}\frac{\partial}{\partial\tilde{\theta}} \quad (29)$$

$$= \frac{\partial}{\partial\tilde{t}} + \left(\tilde{u} + \epsilon\frac{D}{D\tilde{t}}\left[\frac{\tilde{v}_g}{\tilde{f}(x, y)}\right]\right)\frac{\partial}{\partial\tilde{X}} + \left(\tilde{v} - \epsilon\frac{D}{D\tilde{t}}\left[\frac{\tilde{u}_g}{\tilde{f}(x, y)}\right]\right)\frac{\partial}{\partial\tilde{Y}} + \tilde{Q}\frac{\partial}{\partial\tilde{\theta}} \quad (30)$$

$$= \frac{D_0}{D\tilde{t}} + O(\epsilon), \quad (31)$$

where

$$\frac{D_0}{D\tilde{t}} = \frac{\partial}{\partial\tilde{t}} + \tilde{u}_0\frac{\partial}{\partial\tilde{X}} + \tilde{v}_0\frac{\partial}{\partial\tilde{Y}} + \tilde{Q}\frac{\partial}{\partial\tilde{\theta}}. \quad (32)$$

These expressions, as well as the definitions of u_g, v_g, x, and y, are substituted into the equations, and terms of like order in ϵ are equated.

At order ϵ^0, the equations reduce to

$$-\tilde{f}(X, Y)\tilde{v}_0 + \frac{\partial\tilde{M}^*_0}{\partial\tilde{X}} = 0 \quad (33)$$

$$\tilde{f}(X, Y)\tilde{u}_0 + \frac{\partial\tilde{M}^*_0}{\partial\tilde{Y}} = 0 \quad (34)$$

$$-\tilde{\Pi}_0 + \frac{\partial\tilde{M}^*_0}{\partial\tilde{\theta}} = 0 \quad (35)$$

$$\frac{\partial(\tilde{u}_0, \tilde{\Pi}_0)}{\partial(\tilde{X}, \tilde{\theta})} + \frac{\partial(\tilde{v}_0, \tilde{\Pi}_0)}{\partial(\tilde{Y}, \tilde{\theta})} + \frac{\partial}{\partial\tilde{\theta}}\left[\frac{D_0\tilde{\Pi}_0}{D\tilde{t}}\right] = 0. \quad (36)$$

If f is constant, the first three equations transform exactly to the geostrophic and hydrostatic balance relations in physical coordinates. If f varies, u_0 and v_0 differ from the exact geostrophic wind by terms of $O(\epsilon)$, while Π_0 gives the hydrostatic pressure distribution to $O(\epsilon^2)$. The continuity equation can be further simplified using (33)–(35). It can be shown that the first two terms of (36) vanish, which implies that $\partial w_0/\partial\theta = 0$, where $w_0 = (-\theta_0/g)D_0\Pi_0/Dt$ is the lowest-order approximation to Dz/Dt. Requiring w_0 to be zero at the surface and finite at infinity then implies that $w_0 = 0$, that is, the balanced part of the flow is along pressure surfaces, consistent with the hydrostatic approximation.

Collecting terms of $O(\epsilon^1)$ results in

$$\frac{D_0\tilde{u}_0}{D\tilde{t}} - \tilde{f}(X, Y)\tilde{v}_1 + \frac{\partial\tilde{M}^*_1}{\partial\tilde{X}} = \frac{\tilde{u}_0}{\tilde{f}(X, Y)}\left(\tilde{u}_0\frac{\partial\tilde{f}(X, Y)}{\partial\tilde{X}} + \tilde{v}_0\frac{\partial\tilde{f}(X, Y)}{\partial\tilde{Y}}\right) \quad (37)$$

$$\frac{D_0\tilde{v}_0}{D\tilde{t}} + \tilde{f}(X, Y)\tilde{u}_1 + \frac{\partial\tilde{M}^*_1}{\partial\tilde{Y}} = \frac{\tilde{v}_0}{\tilde{f}(X, Y)}\left(\tilde{u}_0\frac{\partial\tilde{f}(X, Y)}{\partial\tilde{X}} + \tilde{v}_0\frac{\partial\tilde{f}(X, Y)}{\partial\tilde{Y}}\right) \quad (38)$$

$$-\tilde{\Pi}_1 + \frac{\partial\tilde{M}^*_1}{\partial\tilde{\theta}} = 0 \quad (39)$$

$$\frac{\partial(\tilde{u}_1, \tilde{\Pi}_0)}{\partial(\tilde{X}, \tilde{\theta})} + \frac{\partial(\tilde{u}_0, \tilde{u}_0/\tilde{f}(X, Y), \tilde{\Pi}_0)}{\partial(\tilde{X}, \tilde{Y}, \tilde{\theta})} + \frac{\partial(\tilde{u}_0, \tilde{\Pi}_1)}{\partial(\tilde{X}, \tilde{\theta})} + \frac{\partial(-\tilde{v}_0/\tilde{f}(X, Y), \tilde{v}_0, \tilde{\Pi}_0)}{\partial(\tilde{X}, \tilde{Y}, \tilde{\theta})} + \frac{\partial(\tilde{v}_1, \tilde{\Pi}_0)}{\partial(\tilde{Y}, \tilde{\theta})} + \frac{\partial(\tilde{v}_0, \tilde{\Pi}_1)}{\partial(\tilde{Y}, \tilde{\theta})}$$
$$+ \frac{\partial(-\tilde{v}_0/\tilde{f}(X, Y), D_0\tilde{\Pi}_0/D\tilde{t})}{\partial(\tilde{X}, \tilde{\theta})} + \frac{\partial(\tilde{u}_0/\tilde{f}(X, Y), D_0\tilde{\Pi}_0/D\tilde{t})}{\partial(\tilde{Y}, \tilde{\theta})}$$
$$+ \frac{\partial}{\partial\tilde{\theta}}\left[\frac{D_0\tilde{\Pi}_1}{D\tilde{t}} + \left(\tilde{u}_1 + \frac{D_0}{D\tilde{t}}\left(\frac{\tilde{v}_0}{\tilde{f}(X, Y)}\right)\right)\frac{\partial\tilde{\Pi}_0}{\partial\tilde{X}} + \left(\tilde{v}_1 - \frac{D_0}{D\tilde{t}}\left(\frac{\tilde{u}_0}{\tilde{f}(X, Y)}\right)\right)\frac{\partial\tilde{\Pi}_0}{\partial\tilde{Y}}\right] = 0. \quad (40)$$

Correct through O(ϵ), the coordinate transformation can be approximated as

$$x = X - v_0/f(X, Y) \quad (41)$$

$$y = Y + u_0/f(X, Y). \quad (42)$$

The semigeostrophic equations are obtained by adding together the O(ϵ^0) and O(ϵ^1) terms. In this approximation, $u = u_0 + \epsilon u_1$, with similar expressions for v, M^*, and Π. The result is

$$f(X, Y)\frac{D_0}{Dt}\left(\frac{u_0}{f(X, Y)}\right) - f(X, Y)v + \frac{\partial M^*}{\partial X} = 0 \quad (43)$$

$$f(X, Y)\frac{D_0}{Dt}\left(\frac{v_0}{f(X, Y)}\right) + f(X, Y)u + \frac{\partial M^*}{\partial Y} = 0 \quad (44)$$

$$-\Pi + \frac{\partial M^*}{\partial \theta} = 0 \quad (45)$$

$$\frac{\partial(u, y, \Pi)}{\partial(X, Y, \theta)} + \frac{\partial(x, v, \Pi)}{\partial(X, Y, \theta)} + \frac{\partial(x, y, D_*\Pi/Dt)}{\partial(X, Y, \theta)} = 0, \quad (46)$$

where

$$\frac{D_*}{Dt} = \frac{\partial}{\partial t} - \frac{1}{f(X, Y)}\frac{\partial M^*}{\partial Y}\frac{\partial}{\partial X} + \frac{1}{f(X, Y)}\frac{\partial M^*}{\partial X}\frac{\partial}{\partial Y} + Q\frac{\partial}{\partial \theta}. \quad (47)$$

To obtain a compact form for the continuity equation (46), some terms of O(ϵ^2) and O(ϵ^3) have been retained [although not in the derivative defined in Eq. (47)]. Following Salmon (1985), u_0 and v_0 will be redefined in terms of $M^* = M_0^* + \epsilon M_1^*$ rather than M_0^* alone, namely,

$$u_0 = \frac{-1}{f(X, Y)}\frac{\partial M^*}{\partial Y} \quad (48)$$

$$v_0 = \frac{1}{f(X, Y)}\frac{\partial M^*}{\partial X}. \quad (49)$$

The errors introduced by this substitution are of O(ϵ^2) and higher, so no accuracy is lost. An analogous substitution must be made in an asymptotic derivation of the quasigeostrophic momentum equations. Magnusdottir and Schubert (1990) employ a slightly different definition of u_0 and v_0:

$$u_0 = \frac{-1}{f(X, Y)}\frac{\partial \phi}{\partial y} = \frac{-1}{f(X, Y)}\frac{\partial M^*}{\partial Y} + \frac{1}{f(X, Y)^2}(u_0^2 + v_0^2)\frac{\partial f(X, Y)}{\partial Y} \quad (50)$$

$$v_0 = \frac{1}{f(X, Y)}\frac{\partial \phi}{\partial x} = \frac{1}{f(X, Y)}\frac{\partial M^*}{\partial X} + \frac{1}{f(X, Y)^2}(u_0^2 + v_0^2)\frac{\partial f(X, Y)}{\partial X}. \quad (51)$$

As with (48) and (49), the equations obtained using this definition remain accurate to O(ϵ). This version has a simple form in physical coordinates, while the previous definition is simplest in the transformed space. If f is constant, both reduce to the statement that u_0 and v_0 are precisely equal to the geostrophic wind components.

Using the coordinate transformation (41) and (42), the approximate equations can be written in physical coordinates as

$$\frac{Du_0}{Dt} - f(X, Y)v + \frac{\partial \phi}{\partial x} = \frac{v_0}{f(X, Y)}\left(v_0\frac{\partial f(X, Y)}{\partial X} + u_0\frac{\partial f(X, Y)}{\partial Y}\right) \quad (52)$$

$$\frac{Dv_0}{Dt} + f(X, Y)u + \frac{\partial \phi}{\partial y} = \frac{u_0}{f(X, Y)}\left(v_0\frac{\partial f(X, Y)}{\partial X} + u_0\frac{\partial f(X, Y)}{\partial Y}\right) \quad (53)$$

$$-\frac{g}{\theta_0}\theta + \frac{\partial \phi}{\partial z} = 0 \quad (54)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (55)$$

$$\frac{D\theta}{Dt} = Q, \quad (56)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \quad (57)$$

If f is constant, the terms on the right-hand sides of (52) and (53) vanish, and since u_0 and v_0 are equal to the geostrophic wind in that case, the equations represent the familiar geostrophic momentum approximation.

3. Other results

a. Surface boundary condition

The preceding scale analysis can also be applied to the surface boundary condition,

$$\frac{D\phi}{Dt} = g\mathbf{v}\cdot\nabla h \quad \text{at} \quad \phi = gh, \quad (58)$$

where $h(x, y)$ is the physical height of the surface topography. This condition is sometimes approximated by $w = 0$ at $z = 0$, in the absence of topography (e.g., Hoskins and West 1979). It is consistent with the hydrostatic approximation to apply the boundary condition on a pressure surface ($z = 0$), but it is harder to justify setting $w = 0$, which implies that the surface pressure tendency vanishes [Haynes and Shepherd (1989) discuss this problem in the quasigeostrophic approximation]. Transforming (58) to isentropic and geostrophic coordinates and nondimensionalizing results in

$$\frac{D\Pi}{D\tilde{t}} - \left(\frac{\Theta}{\theta_0}\right)\frac{D}{D\tilde{t}}\left[\tilde{M}^* - \left(\frac{\theta_0}{\Theta} - \tilde{\theta}\right)\tilde{\Pi} - \frac{\epsilon}{2}(\tilde{u}_g^2 + \tilde{v}_g^2)\right]$$
$$= -\left(\frac{\Theta}{\theta_0}\right)\frac{g}{f_0UL}\left[\frac{D\tilde{X}}{D\tilde{t}}\frac{\partial h}{\partial X} + \frac{D\tilde{Y}}{D\tilde{t}}\frac{\partial h}{\partial Y}\right] \text{ at } z = h. \quad (59)$$

If terms of $O(\Theta/\theta_0)$ are ignored, this expression reduces to $D\Pi/Dt = 0$, that is, $w = 0$. However, the anelastic approximation, implicit in the Boussinesq primitive equations used as a starting point for the present analysis, requires that terms up to first order in this ratio be retained (Ogura and Phillips 1962). At the same accuracy as the semigeostrophic equations, the boundary condition is

$$\frac{D}{D\tilde{t}}\left[M^* - \theta\Pi - \frac{1}{2}(u_0^2 + u_0^2)\right]$$
$$= g\left[u_0\frac{\partial h}{\partial X} + v_0\frac{\partial h}{\partial Y}\right] \quad \text{at} \quad z = h, \quad (60)$$

provided that $h \leqslant O(f_0UL/g)$. This inequality implies that the topography must have a reasonably shallow slope in the transformed space. Alternatively, the condition can be written $\mathrm{Fr} \geqslant \epsilon^{1/2}$, where $\mathrm{Fr} = U/\sqrt{gh_0}$ is a Froude number based on a typical topographic height h_0.

b. Potential vorticity

The semigeostrophic equations are most frequently integrated using conservation of potential vorticity. The form of potential vorticity conserved by (52)–(56) is

$$q_{\mathrm{SG}} = f(X, Y)\frac{\partial(X, Y, \theta)}{\partial(x, y, z)} \quad (61)$$
$$= \zeta_{\mathrm{SG}} \cdot \nabla\theta, \quad (62)$$

where

$$\zeta_{\mathrm{SG}} = \begin{pmatrix} -\dfrac{\partial v_0}{\partial z} \\ \dfrac{\partial u_0}{\partial z} \\ f - f\dfrac{\partial u_0}{\partial y} + f\dfrac{\partial v_0}{\partial x} + \dfrac{1}{f}\left(u_0\dfrac{\partial f}{\partial y} - v_0\dfrac{\partial f}{\partial x}\right)\end{pmatrix} + \begin{pmatrix} -f\dfrac{\partial(v_0/f, u_0/f)}{\partial(y, z)} \\ f\dfrac{\partial(v_0/f, u_0/f)}{\partial(x, z)} \\ -f\dfrac{\partial(v_0/f, u_0/f)}{\partial(x, y)}\end{pmatrix}. \quad (63)$$

Note that f indicates $f(X, Y)$ everywhere in (63). The first term in this expression is the geostrophic vorticity, with an additional contribution to the vertical component associated with variations in f. Following Hoskins (1975), it is common practice to omit the second term, which introduces nonlinear terms into the potential vorticity inversion problem. By transforming each of the terms of this equation to isentropic and geostrophic coordinates, then nondimensionalizing using the scaling described earlier, it can be shown that the first term of (63) contains contributions at $O(\epsilon)$, while the second term is $O(\epsilon^2)$ and thus is negligible. If this term is omitted, the resulting potential vorticity is only approximately conserved, but the errors are of the same order as in the original semigeostrophic equations.

In a recent paper, Synder et al. (1991) analyze the errors in the semigeostrophic vorticity and potential vorticity equations using a conventional Rossby number expansion in physical coordinates. But since the semigeostrophic approximation is not a consistent O(Ro) truncation in this scaling, it is not clear that the $O(\mathrm{Ro}^2)$ terms provide an accurate description of the errors. It turns out, however, that their results have close analogs in the present scaling. In particular, with the hydrostatic approximation, the isentropic coordinate form of the primitive equation potential vorticity is

$$q_{\mathrm{IPE}} = \left(\frac{\partial z}{\partial\theta}\right)_{x,y}^{-1}\left[f(x, y) + \left(\frac{\partial v}{\partial x}\right)_{y,\theta} - \left(\frac{\partial u}{\partial y}\right)_{x,\theta}\right] \quad (64)$$
$$= \left(\frac{\partial z}{\partial\theta}\right)_{x,y}^{-1}\Bigg[\underbrace{f(x, y)}_{O(1)} + \underbrace{\left(\frac{\partial v_0}{\partial x}\right)_{y,\theta} - \left(\frac{\partial u_0}{\partial y}\right)_{x,\theta}}_{O(\epsilon)} + \underbrace{\frac{2}{f(X, Y)}\frac{\partial(v_0, u_0, \theta)}{\partial(x, y, \theta)}}_{O(\epsilon^2)} + O(\epsilon^3)\Bigg]. \quad (65)$$

Meanwhile, the semigeostrophic potential vorticity can be written as

$$q_{\mathrm{SG}} = \left(\frac{\partial z}{\partial\theta}\right)_{x,y}^{-1}\Bigg[\underbrace{f(x, y)}_{O(1)} + \underbrace{\left(\frac{\partial v_0}{\partial x}\right)_{y,\theta} - \left(\frac{\partial u_0}{\partial y}\right)_{x,\theta}}_{O(\epsilon)} - \underbrace{\frac{1}{f(X, Y)}\frac{\partial(v_0, u_0, \theta)}{\partial(x, y, \theta)}}_{O(\epsilon^2)}\Bigg], \quad (66)$$

where the final term in the square brackets is $O(\epsilon^2)$. Therefore, the contributions of the $O(\epsilon^2)$ term to the semigeostrophic potential vorticity are of the correct form, but have the wrong magnitude and sign. The principal difference between these relations and those obtained by Snyder et al. (1991) is that the derivatives in the vorticity are evaluated on isentropic surfaces. This implies that the error terms will be similar except where there are strong horizontal θ gradients.

c. Comparision with the quasigeostrophic and two-dimensional semigeostrophic equations

Despite the similarity in the leading terms of the vorticity, it can be demonstrated that the semigeostrophic equations are formally more accurate than the quasigeostrophic system. The omission of ageostrophic advection in the quasigeostrophic approximation leads to errors at leading order in the advective derivative. Transforming the quasigeostrophic derivative to isentropic and geostrophic coordinates results in the following form,

$$\frac{L}{U}\left(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right) = \epsilon\left(\frac{\partial}{\partial \tilde{t}} + \tilde{u}_0\frac{\partial}{\partial \tilde{X}} + \tilde{v}_0\frac{\partial}{\partial \tilde{Y}}\right) - \epsilon\left[\left(\frac{\partial \tilde{\Pi}}{\partial \tilde{\theta}}\right)^{-1}\left(\tilde{u}_g\frac{\partial \tilde{\Pi}}{\partial \tilde{X}} + \tilde{v}_g\frac{\partial \tilde{\Pi}}{\partial \tilde{Y}}\right)\frac{\partial}{\partial \tilde{\theta}}\right] + O(\epsilon^2). \quad (67)$$

Recalling that $u_0 = u_g$ to leading order, it can be seen that the first group of terms on the right-hand side gives the equation correct to $O(\epsilon)$. However, there is an additional term of the same magnitude (in square brackets) that is not present in either the semigeostrophic or primitive equations. This will result in errors of $O(\epsilon)$ in the quasigeostrophic momentum equations, and at leading order in the potential vorticity equation. In the quasigeostrophic approximation, advection is constrained to occur on isobaric surfaces. The error term in (67) is associated with a spurious cross-isentropic flow that occurs in regions such as fronts where the isentropes are strongly tilted with respect to pressure surfaces.

Finally, it is interesting to compare the accuracy of the three-dimensional semigeostrophic equations with the two-dimensional version that was originally derived from a different scale analysis (Hoskins and Bretherton 1972). It can be seen from Eqs. (65) and (66) that the semigeostrophic definition of potential vorticity is accurate to $O(\epsilon^2)$ for straight flow. It can further be shown that the full equations for conservation of potential vorticity and potential temperature are correct through $O(\epsilon^2)$ in two dimensions. This is not true of the momentum equations, but the $O(\epsilon^2)$ error terms in these equations cancel out when they are combined to form the potential vorticity equation. Since conservation of potential vorticity, in combination with conservation of potential temperature on the boundaries, is sufficient to specify the evolution of the flow (Hoskins et al. 1985), the two-dimensional semigeostrophic equations are formally more accurate than their three-dimensional generalization.

Acknowledgments. The author would like to thank Dr. Craig Bishop and Prof. Brian Hoskins for reading and commenting on the first draft of the manuscript.

REFERENCES

Gill, A. E., 1981: Homogeneous intrusions in a rotating stratified fluid. *J. Fluid Mech.*, **103**, 275–295.

Haynes, P. H., and T. G. Shepherd, 1989: The importance of surface-pressure changes in the response of the atmosphere to zonally-symmetric thermal and mechanical forcing. *Quart. J. Roy. Meteor. Soc.*, **115**, 1181–1208.

Hoskins, B. J., 1975: The geostrophic momentum approximation and the semigeostrophic equations. *J. Atmos. Sci.*, **32**, 233–242.

——, and F. P. Bretherton, 1972: Atmospheric frontogenesis models: Mathematical formulation and solution. *J. Atmos. Sci.*, **29**, 11–37.

——, and N. V. West, 1979: Baroclinic waves and frontogenesis. Part II: Uniform potential vorticity jet flows—Cold and warm fronts. *J. Atmos. Sci.*, **36**, 1663–1680.

——, M. E. McIntyre, and A. W. Robertson, 1985: On the use and significance of isentropic potential vorticity maps. *Quart. J. Roy. Meteor. Soc.*, **111**, 877–946.

McWilliams, J. C., and P. R. Gent, 1980: Intermediate models of planetary circulations in the atmosphere and ocean. *J. Atmos. Sci.*, **37**, 1657–1678.

Magnusdottir, G., and W. H. Schubert, 1990: The generalization of semigeostrophic theory to the β-plane. *J. Atmos. Sci.*, **47**, 1714–1720.

Moore, G. K. W., and W. R. Peltier, 1989: Frontal cyclogenesis and the geostrophic momentum approximation. *Geophys. Astrophys. Fluid Dyn.*, **45**, 183–197.

Ogura, Y., and N. A. Phillips, 1962: Scale analysis of deep and shallow convection in the atmosphere. *J. Atmos. Sci.*, **19**, 173–179.

Purser, R. J., and M. J. P. Cullen, 1987: A duality principle in semigeostrophic theory. *J. Atmos. Sci.*, **44**, 3449–3468.

Salmon, R., 1985: New equations for nearly geostrophic flow. *J. Fluid Mech.*, **153**, 461–477.

——, 1988: Semigeostrophic theory as a Dirac-bracket projection. *J. Fluid Mech.*, **196**, 345–358.

Schär, C., and H. C. Davies, 1990: An instability of mature cold fronts. *J. Atmos. Sci.*, **47**, 929–950.

Snyder, C., W. C. Skamarock, and R. Rotunno, 1991: A comparison of primitive equation and semigeostrophic simulations of baroclinic waves. *J. Atmos. Sci.*, **48**, 2179–2194.