Solution to the Charney Problem of Viscous Symmetric Circulation

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ABSTRACT

The classical problem of viscosity-driven, axially symmetric meridional circulation, partly solved only for the midlatitudes by Charney, is solved here analytically in the whole globe and for any value of viscosity coefficient ν. The solution satisfies Hide's theorem for any Ekman number when Ro $< 80E^2$, where Ro is the Rossby number and E is the Ekman number. For Ro $> 80E^2$, the linear solution ceases to be asymptotically valid. The nonlinear, nearly inviscid regime of Held and Hou presumably is a subset of the second regime (for $E \to 0^+$ and Ro fixed).

1. Introduction

Axially symmetric models have played an important role historically in the understanding of the general circulation of the earth's atmosphere, although we now understand that the atmosphere is far from an eddy-free state and that zonally averaged circulations need not resemble their symmetric counterparts, especially in the extratropics. An excellent review of this subject can be found in chapter 7 of Lindzen (1990). The present study concerns the viscous solution of the symmetric circulation, the so-called Charney problem, in the case of small Rossby number.

The study of *viscous* symmetric circulations is motivated by two considerations. First, if we treat the real atmospheric flow as consisting of a symmetric (zonally averaged) circulation and superimposed large-scale eddies (deviation from zonally symmetry), it often turns out that the irreversible mixing by the eddies can be parameterized as diffusion of angular momentum [see Tung (1986) for the case of isentropic diffusion]. The coefficient of "viscosity" in this case can be rather large, with the result that the circulation can be thought of as viscosity dominated. This apparently is the case for earth's atmosphere in the extratropics. In the tropical troposphere, cumulus friction may more appropriately play the role of (vertical) diffusion. With this consideration in mind (but not specifically offering a model of the real atmosphere), we seek the solution to the Charney problem of viscous symmetric circulation, which so far remains unsolved. Second, to better appreciate the role played by the large-scale eddies in forcing the mean circulation, one often attempts to seek an answer to the hypothetical question, What happens when eddies vanish? A simple-minded extrapolation of the viscous solution valid for finite viscosity to vanishing viscosity seems to suggest that the mean circulation would vanish and the atmosphere would tend to radiative equilibrium. This is in general not true. From the work of Schneider (1977), Held and Hou (1980), and Plumb and Hou (1992), we now know that while for some forms of heating profiles this is indeed the case, for the more physically relevant case of nonzero heating at the equator it is likely that a nonlinear angular momentum conserving Hadley circulation would span the tropical region. There is also the additional possibility, as we will discover, that in the case of small but nonzero viscosity there exists in the equatorial region a *viscosity-dominated* circulation, valid for the case of weaker horizontal heating contrast. These two types of circulations are both valid but pertain to distinctly different regimes of the parameter space. The Held–Hou-type nonlinear circulation is, in a relative sense, a better model of earth's tropics because the parameter regime in which the nonlinear solution is valid is physically more *relevant* than is the case for Charney's linear viscous solution. However, a lack of systematic asymptotic analysis of the problem has led to some confusions in the past, with the result that the nonlinear circulation was justified not by the relevance of its parameter regime of validity but by the implication that the linear viscous solution is somehow incorrect (e.g., that it is singular at the equator, or that it violates a viscous constraint). Some clarification will be provided in the present work.

The governing equations are nonlinear and viscous. Their solutions have been sought traditionally

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as functions of two asymptotic parameters, with the Rossby number, $\mathrm{Ro} = U/2\Omega a$, governing the order of the nonlinear inertial dynamics, and the Ekman number, $E = \nu/2\Omega H^2$, measuring the effect of viscosity. The linearized, viscous version of the problem, obtained by dropping the Rossby number terms, has been studied by Pedlosky (1969), Charney (1973), and Schneider and Lindzen (1976, 1977). The nonlinear version, with the Rossby number terms retained at the leading order, has been investigated by Schneider (1977), Held and Hou (1980), and Lindzen and Hou (1988).

In addition to a lack of asymptotic justification for the linearization procedure in formulating the Charney problem,[1] the problem is compounded by the fact that the linear viscous version of the problem has not been solved completely. Charney (1973) solved the problem for the midlatitudes and found a downward mass flux that must be balanced by a postulated upward flux near the equator, but a solution near the equator was not obtained and was assumed to be singular. Held and Hou (1980), on the other hand, assumed that "as $\nu \to 0$, this (viscosity-driven) Hadley cell disappears," and it was thought that the solution approaches the radiative equilibrium with no meridional mass circulation. And, "because of the angular momentum constraint,[2] we know that this cannot be an adequate description of this limit for the system." This observation then provided the motivation to consider the nonlinear problem. In this paper the linear viscous problem is reexamined in more detail. The complete solution to the Charney linear problem is given. Asymptotic validity of the solution is also discussed. Our solution is formally valid asymptotically for $E = O(1)$ and $\mathrm{Ro} \ll 1$. It remains asymptotically valid for $E \ll 1$ provided that Ro is even smaller (in a manner to be specified later). (It should be pointed out that the parameter regime in which Charney's linear viscous circulation is valid is rather restricted in the limit of $E \to 0^+$, and therefore the small E limit should not be taken as a relevant model for the real atmosphere. It nevertheless serves as a reference for other nonlinear models.)

One interesting and rather surprising feature of our solution is that the strength of the circulation approaches a constant as $E \to 0^+$. The limiting circulation is just the one required for the flow to satisfy Hide's theorem (Hide 1969; Schneider 1977; Lindzen 1990), which is a viscous constraint.

[1] It turns out that a regular asymptotic expansion based on the smallness of the Rossby number is not uniformly valid (it breaks down near the equator, where the Coriolis force vanishes). The same comment also applies to a regular asymptotic expansion based on the smallness of the Ekman number (it breaks down in the viscous boundary layer).

[2] Namely, Hide's theorem.

2. The governing equations

Similar to Charney (1973), steady primitive equations for a dry Boussinesq fluid on a sphere of a radius a rotating with rate Ω confined between the bottom surface and a stress-free lid at height H are considered. The classical case of heat conduction (with κ being the coefficient of heat conduction) of Charney (1973) is treated first. Results for the physically more relevant Newtonian cooling type of heating parameterization are summarized in section 7. Following most previous works, only vertical viscous diffusion (with diffusion coefficient ν) is adopted here. The more general and richer case of nonzero horizontal as well as vertical diffusion coefficients has been studied by Pedlosky (1969) but the solution cannot be expressed in a closed form.

Let θ be the potential temperature, θ_{00} the globally averaged potential temperature, and (u, v, w) the velocity of the fluid in the longitudinal, latitudinal (ϕ), and vertical (z) direction, respectively. Let Φ be the geopotential or the ratio of pressure and density in the present Boussinesq approximation.

We introduce nondimensional variables denoted by asterisks as follows:

$$z^* = \frac{z}{H}, \quad (u^*, v^*) = (u, v)/U,$$

$$w^* = \frac{w}{U}\frac{a}{H}, \quad \Phi^* = \Phi/2\Omega a U,$$

and

$$\theta^* = \frac{\theta}{\theta_{00}}\frac{gH}{2\Omega a U}.$$

Here U is a typical zonal velocity. After the nondimensionalization process we drop the asterisks without confusion. The following nondimensional fundamental equations are obtained:

$$\mathrm{Ro}\left(v\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial z} - uv\tan\phi\right) - \sin\phi\, v = E\frac{\partial^2 u}{\partial z^2} \quad (1)$$

$$\mathrm{Ro}\left(v\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial z} + u^2\tan\phi\right) + \sin\phi\, u = E\frac{\partial^2 v}{\partial z^2} - \frac{\partial \Phi}{\partial \phi} \quad (2)$$

$$\frac{1}{\cos\phi}\frac{\partial(v\cos\phi)}{\partial\phi} + \frac{\partial w}{\partial z} = 0 \quad (3)$$

$$\frac{\partial \Phi}{\partial z} = \theta \quad (4)$$

$$\sigma\,\mathrm{Ro}\left(v\frac{\partial \theta}{\partial \phi} + w\frac{\partial \theta}{\partial z}\right) = E\frac{\partial^2 \theta}{\partial z^2} \quad (5)$$

with the boundary conditions

$$z = 0: \quad w = 0; \quad u = k_c \frac{\partial u}{\partial z}; \quad v = k_c \frac{\partial v}{\partial z} \qquad (6)$$

$$z = 1: \quad w = 0; \quad \frac{\partial u}{\partial z} = 0; \quad \frac{\partial v}{\partial z} = 0. \qquad (7)$$

Two important dimensionless parameters are the Rossby number, defined as $\mathrm{Ro} \equiv U/2\Omega a$, and Ekman number, defined as $E \equiv \nu/2\Omega H^2$. Additionally, $\sigma \equiv \nu/\kappa$ is Prandtl number. We will follow Charney (1973) in assuming that Prandtl number is of order one. The top boundary condition (7) represents a stress-free lid at height H, while the lower boundary is a rigid surface at $z = 0$, where a "geophysical" viscous boundary condition (6) is applied. These are the same as those adopted by Held and Hou (1980). The no-slip boundary condition used by Charney (1973) can be obtained by setting $k_c = 0$.

The potential temperature at the bottom and the top surface are specified:

$$z = 0: \quad \theta = \theta_s(\phi)$$

$$z = 1: \quad \theta = \theta_s(\phi) + \Delta_V(\phi). \qquad (8)$$

A special case of symmetric heating has been suggested by Charney (1973):

$$\theta_s = \theta_{s0} - \Delta_H \sin^{2n}(\phi) \quad \text{and} \quad \Delta_V = \text{const.}, \quad n = 1, 2, 3, \cdots. \qquad (9)$$

We will concentrate on the case of $n = 1$. Results for other physically less relevant cases of higher n will only be briefly mentioned. Although our solution can also include the case where the maximum heating occurs off the equator (cf. Lindzen and Hou 1988), the asymptotic validity of the solution for that case is more subtle and will not be treated here.

In some explicit calculations, we will also adopt

$$\theta_s = 1 - \Delta_H\left(\sin^2\phi - \frac{1}{3}\right) - \frac{\Delta_V}{2} \quad \text{and} \quad \Delta_V = \text{const}, \qquad (10)$$

which will result in an equilibrium solution of the potential temperature that is the same as that of Held and Hou (1980). This is the special case of $n = 1$ in (9).

Radiative equilibrium solution

We note that an exact solution to the system (1)–(5), the radiative equilibrium solution, exists for $\nu \equiv 0$, $\kappa \neq 0$:

$$\theta = \theta_E = \theta_s + \Delta_V z, \quad (v, w) = 0, \quad u = u_E, \qquad (11)$$

where u_E satisfies

$$\frac{\partial}{\partial z}(\mathrm{Ro}\tan\phi u_E^2 + \sin\phi u_E) = -\frac{\partial \theta_E}{\partial \phi} \qquad (12)$$

with the boundary condition $z = 0 : u_E = 0$. The temperature distribution is a result of heat conduction only and is a linear function of z. There is no meridional circulation ("convection"). The zonal flow is in cyclostrophic balance with the temperature. The solution is

$$u_E = \frac{\cos\phi}{2\,\mathrm{Ro}}\left\{\left[1 - 4\,\mathrm{Ro}\left(\frac{\partial\theta_s}{\partial\phi}z + \frac{\partial\Delta_V}{\partial\phi}\frac{z^2}{2}\right)\Big/(\sin\phi\cos\phi)\right]^{1/2} - 1\right\}; \qquad (13)$$

u_E has a real solution only if

$$\frac{\partial\theta_s}{\partial\phi}z + \frac{\partial\Delta_V}{\partial\phi}\frac{z^2}{2} \leqslant \frac{\sin\phi\cos\phi}{4\,\mathrm{Ro}}$$

and in particular, we note that only when

$$\frac{\partial\theta_s}{\partial\phi} = 0 \quad \text{and} \quad \frac{\partial\Delta_V}{\partial\phi} = 0 \quad \text{at the equator}$$

can a finite u_E be maintained at the equator. Therefore, the radiative equilibrium solution is automatically ruled out if heating is not centered at the equator. For the symmetric heatings given by (9),

$$u_E = \frac{\cos\phi}{2\,\mathrm{Ro}}[(1 + 8n\,\mathrm{Ro}z\sin^{2(n-1)}\phi)^{1/2} - 1]. \qquad (14)$$

Here, $U = gH\Delta_H/2\Omega a$ is used. Ruling out the equilibrium solution for $n = 1$ requires appeal to the Hide's theorem and the presence of nonzero viscous diffusion (Schneider 1977; Held and Hou 1980; Lindzen and Hou 1988).[3] A meridional circulation is required in order for the flow to satisfy Hide's theorem. Whether such a circulation is linear or nonlinear is not at all clear at this point. Hide's theorem will be discussed more in section 5.

3. Solution for finite viscosity

The solution for $E \neq 0$ is now sought for E not necessarily small. We do not assume the circulation to be a small perturbation to the radiative equilibrium solution even for a small E. Prandtl number σ is assumed to be order one.

[3] For $n > 1$ in (8), the radiative equilibrium solution does not violate Hide's theorem if

$$\mathrm{Ro} \leqslant s(n) \equiv \frac{1}{4n\cos^2\phi^*\sin^{2(n-2)}\phi^*} + \frac{1}{8n\cos^4\phi\sin^{2(n-3)}\phi^*},$$

$$\text{where} \quad \cos^2\phi^* = \frac{\sqrt{(n+1)^2 + 16} - (n-3)}{2(n+1)}.$$

Assuming Ro $\ll$ 1, we expand all variables in an asymptotic series of Ro:

$$(u, v, w, \theta, \Phi) = (u_0, v_0, w_0, \theta_E, \Phi_0) + \mathrm{Ro}(u_1, v_1, w_1, \theta_1, \Phi_1) + \cdots \quad (15)$$

and solve the equations by using the perturbation method. Expansion in E is not adopted and the viscous terms $E(\partial^2 u/\partial z^2)$, $E(\partial^2 v/\partial z^2)$ are formally taken to be O(1), as the effect of viscosity is felt at every order in the presence of rigid boundaries, no matter how small E is. The expansion for θ recognizes the fact that the leading-order deviation from the radiative equilibrium temperature is a result of meridional advection and hence should be of order Ro, whether or not the meridional circulation is driven by nonlinearity. Temperature advection by the meridional circulation is absent in the Ro $\equiv$ 0 case. In that case the temperature should be given by the radiative equilibrium, even if other quantities [such as u and (v, w)] are not the same as their radiative equilibrium counterparts.

The leading-order equations are

$$E \frac{\partial^2 u_0}{\partial z^2} + v_0 \sin\phi = 0 \quad (16)$$

$$E \frac{\partial^3 v_0}{\partial z^3} - \sin\phi \frac{\partial u_0}{\partial z} - \frac{\partial \theta_E}{\partial \phi} = 0 \quad (17)$$

$$\frac{1}{\cos\phi} \frac{\partial (v_0 \cos\phi)}{\partial \phi} + \frac{\partial w_0}{\partial z} = 0 \quad (18)$$

$$E \frac{\partial^2 \theta_1}{\partial z^2} = \sigma\left(v_0 \frac{\partial}{\partial \phi} \theta_E + w_0 \frac{\partial}{\partial z} \theta_E \right). \quad (19)$$

At this order, the equations are linear and thus are the same as in Charney's model. Charney solved this system in an asymptotic series for small E, and found thin boundary layers next to the rigid surfaces at $z = 0$ and $z = 1$ in the midlatitudes. The thickness of the viscous boundary layer was found to be $\sim \sqrt{E/\sin\phi}$, and so Charney's method of singular perturbation cannot be extended to the tropical region, where $\sin\phi$ is small.

It turns out that equations (16)–(19) can be solved exactly in a closed form, without using an asymptotic expansion in E. It is also not necessary to assume E to be small. Following Schneider (1976), a streamfunction Ψ_0 is introduced so that

$$v_0 = \frac{-1}{\cos\phi} \frac{\partial \Psi_0}{\partial z}, \quad w_0 = \frac{1}{\cos\phi} \frac{\partial \Psi_0}{\partial \phi}. \quad (20)$$

Then (16)–(18) can be combined into, with $\mu = \sin\phi$

$$E^2 \frac{\partial^4 \Psi_0}{\partial z^4} + \mu^2 \Psi_0 = -E(1 - \mu^2) \frac{\partial \theta_E}{\partial \mu} \quad (21)$$

with the boundary conditions

$$z = 0: \quad \Psi_0 = 0; \quad \frac{\partial \Psi_0}{\partial z} = k_c \frac{\partial^2 \Psi_0}{\partial z^2}; \quad u_0 = k_c \frac{\partial u_0}{\partial z} \quad (22)$$

$$z = 1: \quad \Psi_0 = 0; \quad \frac{\partial^2 \Psi_0}{\partial z^2} = 0; \quad \frac{\partial u_0}{\partial z} = 0. \quad (23)$$

The solution for Ψ_0 is

$$\Psi_0 = -\frac{E(1 - \mu^2)}{\mu^2} \frac{\partial \theta_s}{\partial \mu} \{1 + qz + e^{-\lambda z}(a \cos\lambda z + b \sin\lambda z) + e^{\lambda(z-1)} \times [c \cos\lambda(z - 1) + d \sin\lambda(z - 1)]\}, \quad (24)$$

where $\lambda \equiv \sqrt{|\mu|/2E}$. We assume that (i) $\partial\theta_E/\partial\mu$ is finite at the poles, $\mu = \pm 1$, and (ii) $q \equiv (\partial\Delta\theta/\partial\mu)/(\partial\theta_s/\partial\mu)$ is finite for any ϕ. The symmetric form (9) of Charney (1973) belongs to our case, and the form (10) of Held and Hou (1980) is a special case.

Since at $z = 0$, $\Psi_0 = 0$, so $u_{0|z=0} = 0$ because

$$\left.\frac{\partial u_0}{\partial z}\right|_{z=0} = 0.$$

Integrating (16), we obtain

$$u_0 = -\frac{(1 - \mu^2)^{1/2}}{2\lambda\mu} \frac{\partial \theta_s}{\partial \mu} \{2\lambda z + \lambda q z^2 + e^{-\lambda z}[(a - b) \sin\lambda z - (a + b) \cos\lambda z] + e^{\lambda(z-1)}[(c - d) \cos\lambda(z - 1) + (c + d) \times \sin\lambda(z - 1)] + (a + b) - e^{-\lambda}[(c - d) \cos\lambda - (c + d) \sin\lambda]\}, \quad (25)$$

which is very different from the radiative equilibrium solution u_E.

Applying the boundary conditions, the coefficients of the solution are obtained. These are given in appendix A.

The leading-order deviation from the radiative equilibrium temperature θ_E for the temperature is

$$\theta_1 = -\frac{\sigma}{E} \int_0^z \int_0^{z'} J[\Psi_0, \theta_E] dz' dz'', \quad (26)$$

where

$$J[A, B] = \frac{\partial A}{\partial \mu} \frac{\partial B}{\partial z} - \frac{\partial A}{\partial z} \frac{\partial B}{\partial \mu}$$

is the Jacobian for A and B. (The integrals are not needed if the Newtonian cooling parameterization for heating is adopted instead of the heat conduction form used here.)

The solutions for some sample parameters are shown in Fig. 1. The radiative equilibrium temperature distri-

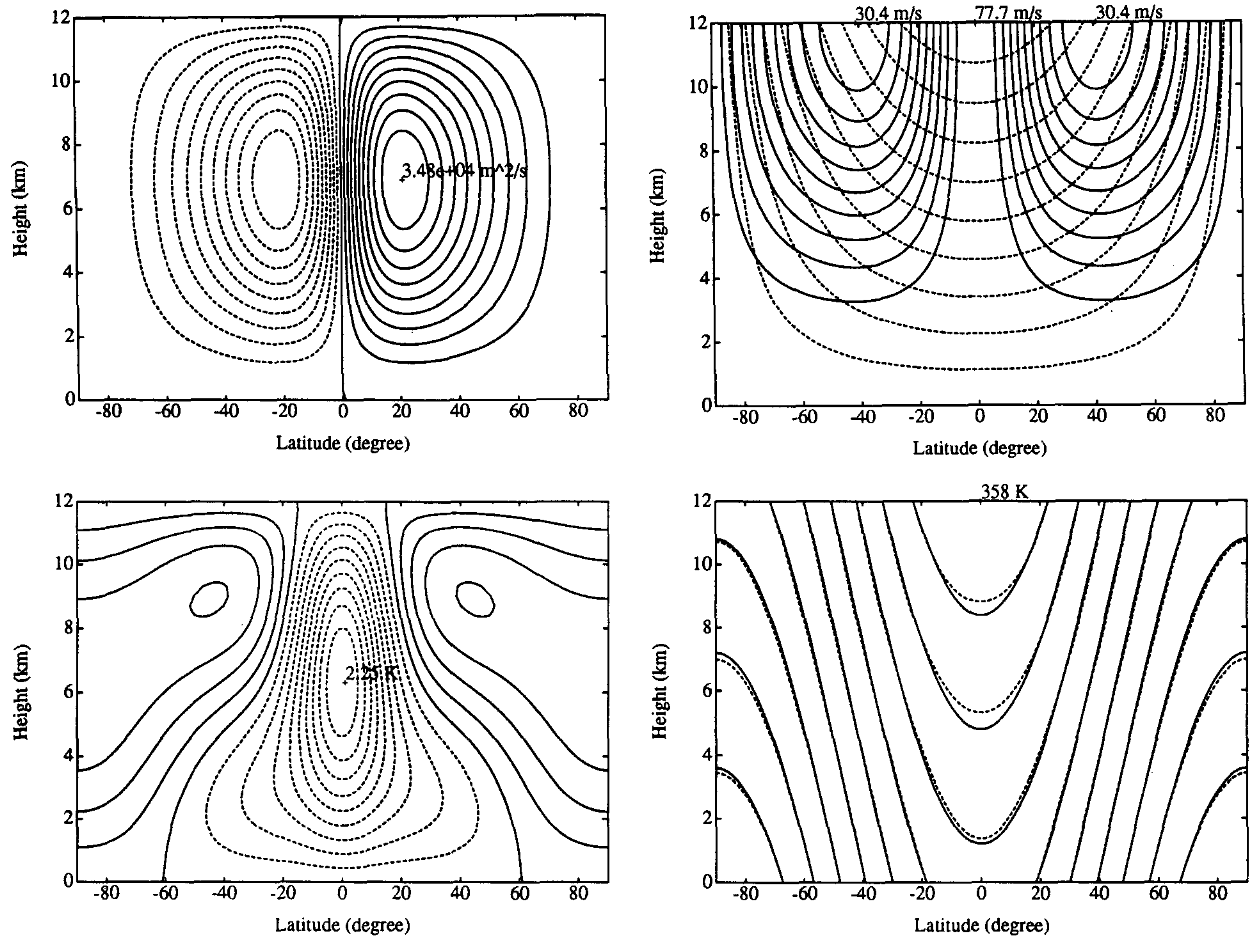

FIG. 1. The solution for $n = 1$ and $E = 0.03$: (a) the streamfunction. Solid lines represent positive values while the dashed lines negative values; (b) the zonal velocity (in solid lines) and the radiative equilibrium solution of zonal velocity (in dashed lines); (c) the leading-order correction of potential temperature to the radiative equilibrium distribution where positive values are in solid lines and negative values in dashed lines; (d) potential temperature (dashed lines) and the radiative equilibrium potential temperature (solid lines). Contour interval is one-tenth of the maximum value.

bution is specified as in (10) and in Held and Hou (1980). Here E is chosen to be 0.032. In converting to dimensional units, we may choose $H = 12$ km, $\Delta_H = 1/3$ and $\Delta_V = 1/6$, $U = gH\Delta_H/(2\Omega a) = 42.1$ m s^{-1}, $\theta_{00} = 300$ K, and Ro $\equiv U/2\Omega a = 0.0452$.

Figure 1a shows the contours of the streamfunction. The strength of the circulation (the maximum value of the streamfunction) is 0.069 (3.48×10^4 m^2 s^{-1} in dimensional variables), as compared to the observed Hadley circulation strength of 2.46×10^4 m^2 s^{-1}, and the centers of the cells are located at $\pm 21°$. Figure 1b shows the zonal velocity distribution. The maximum jet value is 0.7213 (30.43 m s^{-1} in dimensional variables), which is less than half of the radiative equilibrium value of 77.7 m s^{-1}. The locations of the jet maximum are at $\pm 41°$. The two zonal jets are separated by a zero wind line over the equator. The radiative equilibrium u is drawn in dashed lines; it is westerly everywhere, including over the equator. Figure 1c shows θ_1, the leading-order correction of potential temperature from its radiative equilibrium value divided by Ro. The maximum potential temperature correction occurs at the equator in midheight with the dimensional value -49.7 K $\times$ Ro, or about -2.25 K. The viscous circulation serves to smooth out the temperature gradient in the tropics, as can be seen in Fig. 1d, where the solution is plotted in dotted lines, compared with the radiative equilibrium solution in solid lines.

For a flatter radiative equilibrium potential temperature distribution ($\sim \mu^{2n}$ and $n \geq 2$), the solution is close to the radiative equilibrium solution. The circulation is weaker because the horizontal gradient of the potential temperature, which drives the circulation, is smaller. The center of the circulation is situated more poleward (not shown).

4. Dependence of solution on E

We now investigate the solution obtained as E ranges from O(1) to $E \ll 1$ satisfying the no-slip boundary condition, with θ_E given by (9). We have

$$\Psi_0 = \frac{-(1-\mu^2)}{2\mu}\frac{\partial\theta_s}{\partial\mu}\,\mathrm{sgn}(\mu)\tilde{\Psi}(\lambda, \lambda z), \quad (27)$$

where

$$\tilde{\Psi} = \frac{1}{\lambda^2}\{1 + e^{-\lambda z}(a\cos\lambda z + b\sin\lambda z) + e^{\lambda(z-1)} \times [c\cos\lambda(z-1) + d\sin\lambda(z-1)]\}. \quad (28)$$

The function $\tilde{\Psi}$ is a function of λ and z only, and has a maximum at (λ_*, z_*), which does not depend on E explicitly. These are numerical constants and are found to be $\lambda_* = 2.757$, $z_* = 0.578$. Figure 2 shows $\tilde{\Psi}(\lambda, z)$ at $z = 0.578$. We find that the maximum value is 0.083.

Figure 3 shows the calculated streamfunction (in solid line) and the zonal velocity (in dashed line) for $E = 0.002$, 0.004, and 0.008. From these results, we see that the maximum value of the streamfunction for small E is always around 0.083. The location of the center of the circulation cell is $\phi \approx \sin^{-1}(2\lambda_*^2 E) \approx 7.6E$ off the equator for small E. (The location of maximum of Ψ is slightly different from that of $\tilde{\Psi}$ when E is not small.) The strength of the circulation *decreases* (see Fig. 4) and the center of the circulation cell moves poleward as E increases. The horizontal extent of the tropical boundary layer is O(E). The upward motion is confined in a narrow region $-\phi_* \leq \phi \leq \phi_*$, where $\phi_* = \sin^{-1}(2\lambda_*^2 E)$.

The zonal velocity for the no-slip boundary condition and constant vertical equilibrium potential temperature gradient is

$$u_0 = \frac{-1}{\sin\phi}\frac{\partial\theta_s}{\partial\phi}(z - \hat{u}(\lambda, z)), \quad (29)$$

where

$$\hat{u}(\lambda, z) = -\frac{1}{2\lambda}[e^{-\lambda z}((a-b)\sin\lambda z - (a+b)\cos\lambda z) + e^{\lambda(z-1)}((c-d) \times \cos\lambda(z-1) + (c+d)\sin\lambda(z-1)) + (a+b) - e^{-\lambda}((c-d)\cos\lambda - (c+d)\sin\lambda)]. \quad (30)$$

The maximum of u_0 is located at $z = 1$. At $z = 1$,

$$u_0 = \frac{-1}{\sin\phi}\frac{\partial\theta_s}{\partial\phi}(1 - \tilde{u}(\lambda)), \quad (31)$$

where $\tilde{u}(\lambda) = A(\lambda)/2\lambda(1 - e^{4\lambda} + 2e^{2\lambda}\sin 2\lambda)$ and $A(\lambda) \equiv -3 - 4e^{2\lambda} - 3e^{4\lambda} + 8e^{\lambda}\cos\lambda + 8e^{3\lambda}\cos\lambda - 6e^{2\lambda} \times \cos 2\lambda$ is the deviation from radiative equilibrium solution. The function $\tilde{u}(\lambda)$ decreases monotonously from 1 to 0 as λ ranges from 0 to ∞. Even as $E \to 0^+$, the zonal velocity will never approach u_E in the tropical region. The region where u deviates from u_E becomes wider and the maximum value of the zonal wind decreases as E increases. There is no westerly flow over the equator for *any* $E > 0$. The zonal velocity profile on the upper boundary for various E and $n = 1$ is shown in solid lines in Fig. 5. The equilibrium solution u_E at $z = 1$ is plotted in a dashed line.

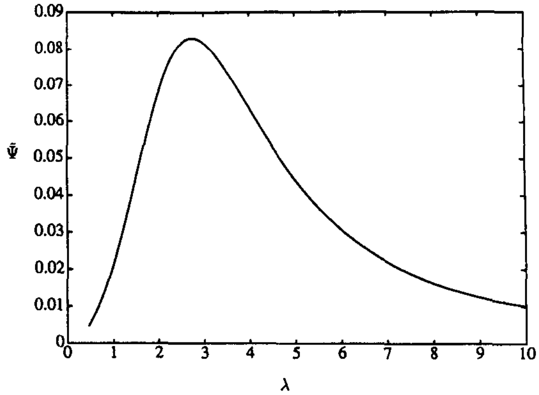

FIG. 2. $\tilde{\Psi}(\lambda, z)$ at $z = 0.578$.

In the midlatitudes, $\sin\phi \sim$ O(1). For small E, we have $\lambda = \sqrt{\sin\phi/2E} \gg 1$. For $k_c = 0$, we have, to leading order (see appendix A),

$$D_0(\lambda) \sim e^{4\lambda}, \quad a(\lambda) \approx -1, \quad b(\lambda) \approx -1,$$
$$c(\lambda) \approx -1, \quad d(\lambda) \approx 0.$$

By using the fact that $\lambda \gg 1$, we obtain the approximate solution in the midlatitudes, as was found by Charney (1973).

The meridional flow in the upper boundary layer at ϕ is

$$V_{\text{upper}} = \int_{1-}^{1}\cos\phi v_0 dz \approx 2n(1-\mu^2)\mu^{(2n-3)}E, \quad (32)$$

which turns out to be equal and opposite to the meridional flux in the lower boundary layer: $V_{\text{lower}} = \int_0^{0+}\cos\phi v_0 dz$, which is also the same as the *total* downward flux poleward of a chosen latitude ϕ in the interior:

$$W_{\text{down}} = \int_\phi^{\pi/2} w_0\cos\phi d\phi = -V_{\text{upper}} = V_{\text{lower}} \approx -2n(1-\mu^2)\mu^{(2n-3)}E. \quad (33)$$

The result in (32) is the same as that found by Charney. Because of the explicit linear dependence of the mass flux on E in (32) and (33), it is easy to mistake (32) and (33) to mean that the circulation vanishes as $E \to 0$. However, the total downward flux should be calcu-

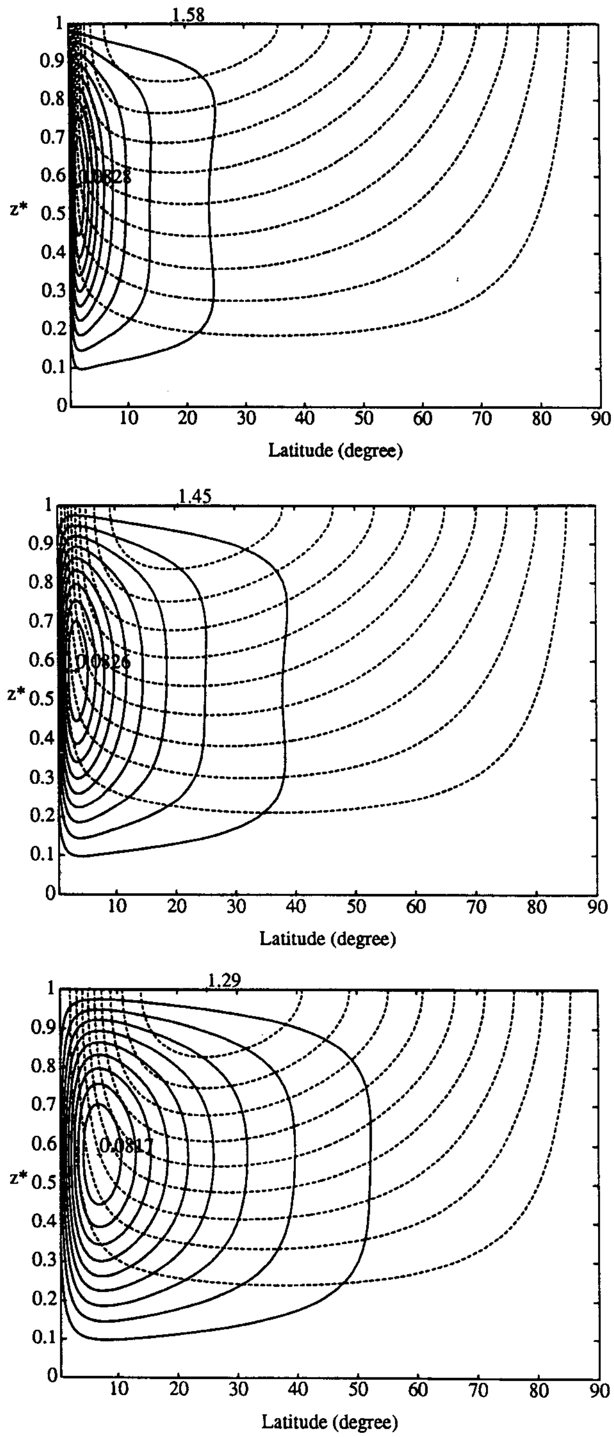

FIG. 3. The streamfunction and the zonal velocity (dotted line) for various E and for $n = 1$: (a) $E = 0.002$; (b) $E = 0.004$; (c) $E = 0.008$. Contour interval is one-tenth of the maximum value.

lated by setting $\phi = \phi_* = \sin^{-1}(2\lambda_*^2 E)$ in, say, (33) to obtain, for $n = 1$, $W_{\text{down}}^{\text{total}} \sim -1/\lambda_*^2$, which is *independent* of E. Quantitatively this is in excess of the exact value, but it has the correct order, O(1). The exact value cannot be obtained by using an asymptotic solution valid only at the midlatitudes in the limit of small E because some of the downward flux occurs in the tropics. From our exact solution, the total upward flux in the region $0 \leqslant \phi \leqslant \phi_*$ is

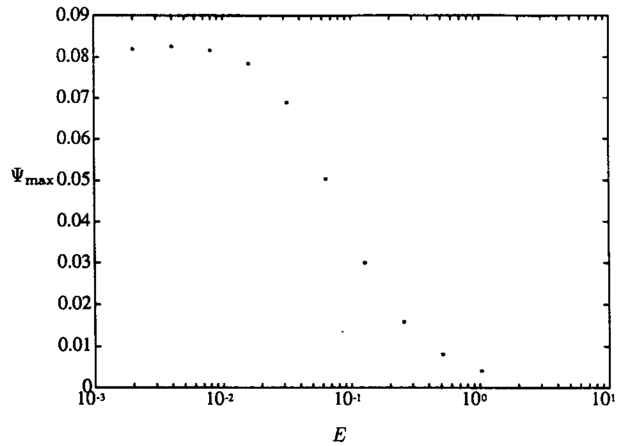

FIG. 4. The maximum of the streamfunction for $n = 1$ as a function of E.

$$W_{\text{up}} = \int_0^{\phi_*} w_0 \cos\phi d\phi = \Psi_0(\phi_*, z_*)$$

$$= \Psi_{0|\max} = 0.083 \quad \text{for} \quad n = 1. \qquad (34)$$

An important point to be made here is that the circulation does not vanish as $E \to 0^+$. There is a finite viscous circulation even as the viscosity coefficient $\nu \to 0^+$ and the boundary layers become thinner. Figure 6 shows the viscosity-driven circulation schematically.

For $n \geqslant 2$, the viscous circulation vanishes as $E \to 0^+$, as can be deduced from (32) and (33). The solu-

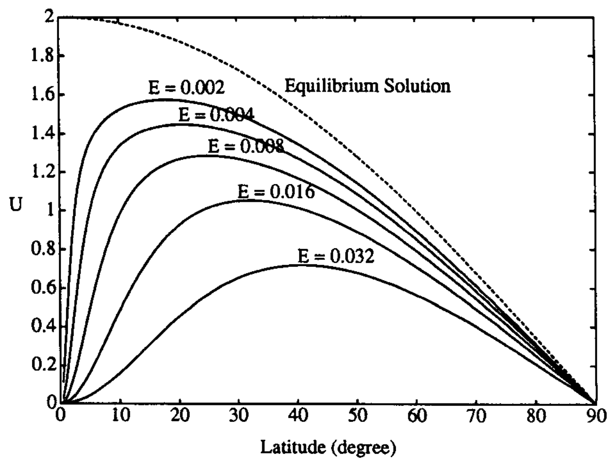

FIG. 5. Zonal velocity on the upper boundary for $n = 1$.

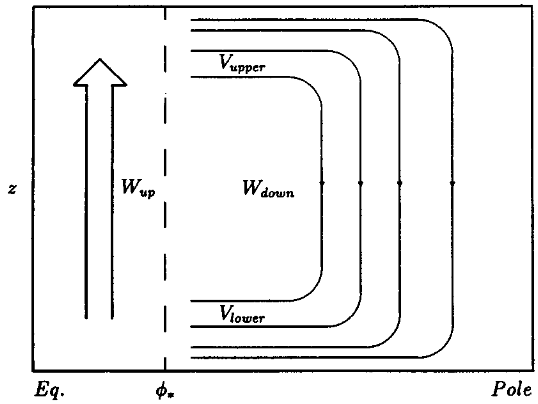

FIG. 6. The schematic diagram for the viscosity-driven circulation.

tion reduces to the radiative equilibrium, which, as we have pointed out earlier, satisfies Hide's theorem.

5. Hide's theorem and asymptotic validity

Schneider (1977), Held and Hou (1980), and Lindzen (1990) have discussed the implication of Hide's theorem in detail. Hide's constraint is a property of the exact equation for the steady-state absolute zonal angular momentum, $M \equiv \Omega a^2 \cos^2\phi + ua \cos\phi$ (in dimensional form) in the presence of nonzero viscosity:

$$\nabla \cdot (\mathbf{u}M) = \nabla \cdot (\nu \nabla M). \qquad (35)$$

It forbids the presence of a maximum of M in the interior of the fluid, because if M did have such a maximum, one can then always find a closed contour surrounding the maximum along which M is a constant. Integrating Eq. (35) over the area bounded by this contour makes the left-hand side vanish, while the right-hand side is negative because the viscous flux is downgradient. Similar considerations also rule out the presence of a maximum in a stress-free upper boundary, and show that M can attain its maximum only at the lower boundary provided that the surface vertical wind shear is easterly. Therefore, an upper bound[4] for M is

$$M_{\max} \leqslant \Omega a^2. \qquad (36)$$

[4] One important case not considered by previous discussions of Hide's theorem is the possibility of nonisolated maximum of M. In particular, the presence of a vertical interior maximum line joining the top and bottom boundaries cannot be ruled out if only vertical diffusion is present [i.e., replace the right-hand side of Eq. (35) by $(\partial/\partial z)\nu(\partial/\partial z)M$, as was done by Schneider (1976), Held and Hou (1980), Lindzen (1990), and in this paper]. In fact, in both the nonlinear case of Held and Hou (1980) and the linear case here, the maximum of M turns out to occur as a vertical line over the equator, and not at the lower surface with easterly shear. The maximum, Ωa^2, is an attainable maximum, hence the "equal to or less than" sign in (36).

Hide's constraint requires that $\Omega a^2 \cos^2\phi + ua \cos\phi \leqslant \Omega a^2$. In dimensionless form this is

$$u \leqslant u_M, \quad \text{where} \quad u_M \equiv \frac{\sin^2\phi}{2\,\text{Ro}\cos\phi}. \qquad (37)$$

The radiative equilibrium u_E for $n = 1$ in (14) violates (37) because it is westerly over the equator. Therefore, it cannot be a valid solution in the presence of viscous diffusion, no matter how small. A meridional circulation is required. But since Hide's theorem is a *viscous* constraint, without regard to whether the full nonlinear term on the left-hand side of (35) is retained or not, it, by itself, cannot be used to justify the existence of a "nonlinear, nearly inviscid" Hadley circulation. The Held and Hou circulation needs to be justified on separate grounds (e.g., that it is asymptotically valid in a parameter regime that is more *relevant* to the earth's tropical troposphere).

The following result is self evident:

Hide's constraint is satisfied by any *asymptotically valid approximate viscous solution of (35).*

In particular, our linear viscous solution should satisfy Hide's constraint in the parameter regime in which that solution is asymptotically valid.

The value of our viscous solution u_0 at $z = 1$ is plotted in Fig. 7 for various values of E against u_M (in dashed lines) for different values of Ro. It shows that there exists a parameter regime

$$\text{Ro} < 80E^2, \qquad (38)$$

where our linear viscous solution satisfies Hide's theorem everywhere.[5] The inequality can also be obtained analytically by expanding (31) near the equator.

Asymptotic validity of our solution can also be established easily. Our solution is formally valid for $E = O(1)$ and Ro $\ll 1$. For $E \ll 1$, the regime of asymptotic validity is

$$\frac{\text{Ro}}{E^2} = \frac{gH^5\Delta_H}{a^2\nu^2} \ll 1. \qquad (39)$$

However, in practice, the strict condition (39) is not needed. One can show a posteriori that, in arriving at our leading-order approximation (16)–(19), the neglected terms are small compared to the retained terms when (38), Hide's constraint, is satisfied. This is done in appendix B.

[5] Mathematically, since Ro was assumed to be asymptotically small, $u_M \rightarrow \infty$ asymptotically and our solution satisfies Hide's theorem for all $E > 0$. Figure 7 shows what happens when Ro is not taken to be asymptotically small.

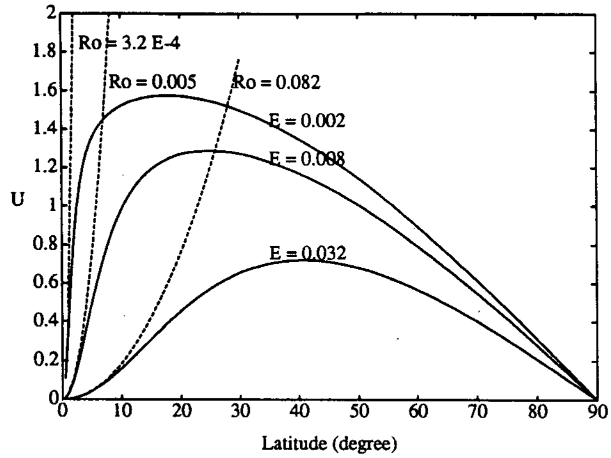

FIG. 7. Latitude plot of u_M for various Ro and u at the top for various E and $n = 1$.

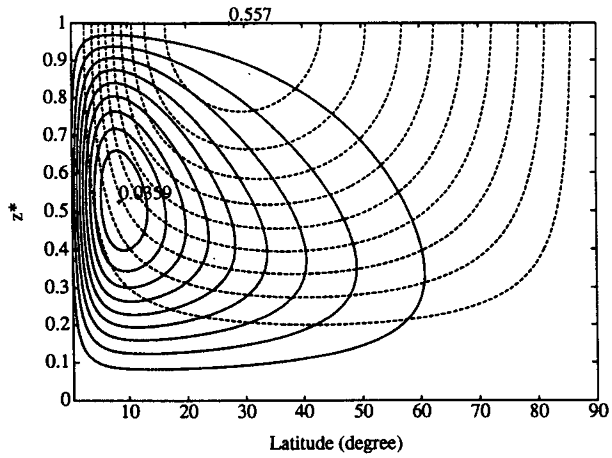

FIG. 8. The streamfunction (solid lines) and zonal velocity (dotted lines) for $q = -1$, $E = 0.01$, and for $n = 1$. Contour interval is one-tenth of the maximum value.

6. Meridional circulation driven by lower surface temperature gradient

We now consider the case where the imposed temperature gradients are different at the top and the bottom surfaces. Specifically, a sample result is given for the case where the potential temperature has no horizontal gradient along the upper boundary, motivated by the situation in the atmosphere where the top of the troposphere is bounded approximately by the 342 K potential temperature contour, which is almost horizontal. At the lower boundary, a horizontal temperature gradient is imposed by, say, a specified sea surface temperature gradient. This is the case of $q = -1$ in section 3.

Figure 8 shows the solution for such a case with $E = 0.01$. The meridional circulation still has a single cell and is now driven entirely by the temperature gradient along the bottom surface. The sense of the circulation is countergradient, in the sense that near the bottom boundary the meridional circulation is directed from cold to warm. This is a distinct characteristic of a rotating viscous flow (Ekman boundary layer flow) (see Holton 1979, section 5.2). The solution still has the same structure as the $q = 0$ case previously treated (see Fig. 3) where the horizontal temperature gradient on the top boundary is the same as that on the bottom boundary. The strength of the circulation approaches a finite value 0.0364 (in nondimensional variable) as $E \to 0^+$. If we use $H = 12$ km and $\Delta_H = 1/3$, as in Held and Hou (1980), this strength is 1.84×10^4 m^2 s^{-1}, implying a mass flux of 15.1×10^{12} kg s^{-1} in dimensional variable, and is close to the observed value of 19.7×10^{12} kg s^{-1} reported by Oort (1983). The maximum value is attained at a latitude that is $O(E)$ off the equator. The maximum zonal wind is also reduced since the temperature gradient near the top surface is reduced. We can multiply 42.1 m s^{-1} to the dimensionless unit to convert it to the dimensional value. These range from 23.4 m s^{-1} for $E = 0.01$ to 42.1 m s^{-1} as $E \to 0^+$, while in $q = 0$ case the zonal wind could be up to 84.1 m s^{-1} for $E \to 0^+$.

If the temperature gradient at the bottom surface is interpreted as that of the sea surface temperature, then the equator–pole contrast should be taken to be smaller than the value adopted above by a factor of 2 ($\Delta_H = 1/6$). Therefore, it appears that the strength of the meridional circulation driven by the sea surface temperature gradient is about half of the observed values. Figure 9 shows the leading-order correction of potential temperature from its radiative equilibrium value under the prescribed conditions ($\Delta_H = 1/6$, $\Delta_V = 1/6$, $H = 12$ km

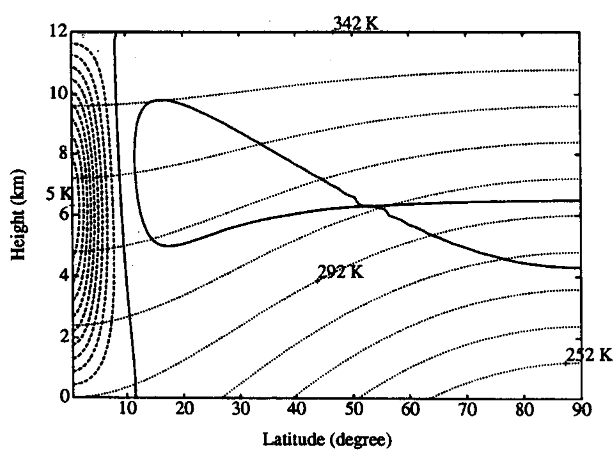

FIG. 9. The leading-order correction for potential temperature, with $\Delta_H = 1/6$, $\Delta_V = 1/6$, Ro = 0.0226, $E = 0.01$, and $H = 12$ km. The solid lines are for the positive values and the dashed lines are for the negative values. The background dotted lines are for the radiative equilibrium potential temperature θ_E. Contour interval is one-tenth of the maximum value.

and Ro = 0.023, $E = 0.01$). (Asymptotic validity for the present case is somewhat different from the previous case discussed in section 5. The condition is now $\mathrm{Ro} < 180E^2$.) The maximum correction in the tropics is −5.1 K, which is weaker (for the same E value) than that in the $q = 0$ case due to a weaker circulation and a weaker vertical gradient of the potential temperature in the tropics in the present case.

7. Newtonian cooling

The solution that we have obtained is for the Charney problem of thermal diffusion. If a Newtonian cooling parameterization to radiative heating and cooling is adopted, the thermal equation (5) can be rewritten as

$$\mathrm{Ro}\left(v\frac{\partial\theta}{\partial\phi} + w\frac{\partial\theta}{\partial z}\right) = \frac{\theta_E - \theta}{2\Omega\tau}, \qquad (40)$$

where τ is the dimensional Newtonian cooling damping time. The governing equations are (1)–(4) and (40) now. Similar to the derivation in section 3, we know that for small Ro the leading order equations are (16)–(18) and $\theta_0 = \theta_E$ provided that the Newtonian radiative relaxation time τ is such that $2\Omega\tau$ is order one. If the radiative equilibrium potential temperature θ_E is specified to be of (9) or (10), governing equation for the streamfunction will be exactly the same as Eq. (21) for the Charney problem of thermal diffusion. The restriction that $2\Omega\tau$ is order one is not physically realistic for the earth's lower atmosphere, where τ is of the order of a few tens of days. The more realistic case of $2\Omega\tau\,\mathrm{Ro} = O(1)$ will be dealt with in a separate paper.

Thus, under the assumption $\mathrm{Ro} \ll 1$, $\mathrm{Ro} < 80E^2$, and $2\Omega\tau \sim O(1)$, the previously obtained solution for Ψ_0 and u_0 continues to hold for the present Newtonian cooling case. The only difference is that the deviation of θ from its radiative equilibrium value θ_E is to be calculated from the simpler equation:

$$\theta_1 = -2\Omega\tau\left(v_0\frac{\partial\theta_E}{\partial\phi} + w_0\frac{\partial\theta_E}{\partial z}\right) = 2\Omega\tau J[\Psi_0, \theta_E], \qquad (41)$$

instead of (26).

8. Conclusions

We have provided a complete analytic solution to the Charney problem of linear, viscous, axially symmetric circulation on a rotating sphere. To the extent that linearization can be justified, our solution is valid for any Ekman number, including the small Ekman number case partially solved by Charney (1973). As the Ekman number $E \to 0^+$, the solution either (i) approaches the radiative equilibrium solution with no meridional circulation, provided that the radiative equilibrium solution satisfies the viscous constraint of Hide (1968)[6]; or (ii) in the case where the radiative equilibrium solution violates Hide's theorem, a meridional circulation is present for any Ekman number. In particular, this circulation does not vanish even as $E \to 0^+$ (the "nearly inviscid limit"). As a result, the zonal flow is always zero at the equator, thus satisfying Hide's theorem, which forbids the presence of westerly zonal flow over the equator. For the case where the imposed surface temperature varies as $\sin^2\phi$ near the equator, the strength of the meridional circulation is almost independent of E for $E \leqslant 10^{-2}$, that is,

$$|\Psi_{0|\max}| \approx 0.083 + O(E). \qquad (42)$$

In dimensional units it is about $4.1 \times 10^4\ \mathrm{m^2\,s^{-1}}$, which is stronger than the nonlinear circulation of Held and Hou (1980) for the same heating distribution. It happens to be of the same order as, but is somewhat larger than, the observed strength of the Hadley circulation. However, when a more realistic (zero) upper potential temperature gradient and a weaker temperature gradient that corresponds to the sea surface temperature are used, the circulation strength becomes only half the observed value.

One may call into the question the validity of the linear solution when the circulation becomes so strong in case (ii) for small E. In Charney's problem, the nonlinear terms are dropped under the assumption of small Rossby number, Ro. This in some sense presumes that the Rossby number is smaller than the Ekman number. As it turns out, asymptotic validity requires that $\mathrm{Ro} < 80E^2$ (or $\mathrm{Ro} < 180E^2$, when the temperature gradient near the tropopause level is zero). Therefore, when we take the limit as $E \to 0^+$, it is understood that $\mathrm{Ro} \to 0^+$ also, in such a way that the above constraint is maintained. When this is true, Hide's theorem is satisfied everywhere. It should be pointed out that this simultaneous limit is not realistic for earth's atmosphere. Using realistic parameters,[7] the smallest value of E we can use and still have Hide's theorem satisfied is about 10^{-2}, which appears to be near the top range of what is considered as realistic for the vertical diffusivity ($10^2\ \mathrm{m^2\,s^{-1}}$) in earth's atmosphere.

The nonlinear, nearly inviscid regime of Held and Hou (1980) presumably applies in (and is a subset of) the parameter range $\mathrm{Ro} > 80E^2$ for case (ii), which is more applicable to the real atmosphere, although Held and Hou did not specify such a condition for the validity of their solution.

[6] This case corresponds to $n > 1$ in our model, and is probably less realistic than the $n = 1$ case of (ii).

[7] See the last paragraph of section 6. $H = 12$ km, $\Delta_H = 1/6$, $\Delta_V = 1/6$, and $U = gH\Delta_H/2\Omega a = 21\ \mathrm{m\,s^{-1}}$. $\mathrm{Ro} = U/2\Omega a = 0.023$. The potential temperature on the upper boundary is specified as a constant.

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APPENDIX A

Solution Details

Applying the boundary conditions $z = 0: \Psi_0 = 0$, $\partial\Psi_0/\partial z = k_c(\partial^2\Psi_0/\partial z^2)$ and $z = 1: \Psi_0 = 0$, $\partial^2\Psi_0/\partial z^2 = 0$, we solve for a, b, c, d algebraically:

$$\begin{aligned} aD(\lambda) = {} & \lambda[-e^{4\lambda} - 2e^{4\lambda}k_c\lambda + e^{\lambda}\cos(\lambda) \\ & + e^{3\lambda}\cos(\lambda) - 2e^{\lambda}k_c\lambda\cos(\lambda) \\ & + 2e^{3\lambda}k_c\lambda\cos(\lambda) - e^{2\lambda}\cos(2\lambda) \\ & + 2e^{2\lambda}k_c\lambda\cos(2\lambda) + e^{2\lambda}\sin(2\lambda)] \\ & + q[\lambda e^{\lambda}\cos(\lambda) + \lambda e^{3\lambda}\cos(\lambda) \\ & - 2k_c\lambda^2 e^{\lambda}\cos(\lambda) + 2k_c\lambda^2 e^{3\lambda}\cos(\lambda) \\ & - e^{2\lambda}\sin(2\lambda)] \end{aligned} \tag{A.1}$$

$$\begin{aligned} bD(\lambda) = {} & \lambda[-e^{4\lambda} + 2e^{3\lambda}\cos(\lambda) - e^{2\lambda}\cos(2\lambda) \\ & + e^{\lambda}\sin(\lambda) + e^{3\lambda}\sin(\lambda) - 2e^{\lambda}k_c\lambda\sin(\lambda) \\ & - 2e^{3\lambda}k_c\lambda\sin(\lambda) - e^{2\lambda}\sin(2\lambda) \\ & + 2e^{2\lambda}k_c\lambda\sin(2\lambda)] + q[-e^{4\lambda} \\ & + 2\lambda e^{3\lambda}\cos(\lambda) + e^{2\lambda}\cos(2\lambda) \\ & + \lambda e^{\lambda}\sin(\lambda) + \lambda e^{3\lambda}\sin(\lambda) - 2k_c\lambda^2 e^{\lambda} \\ & \times \sin(\lambda) - 2k_c\lambda^2 e^{3\lambda}\sin(\lambda)] \end{aligned} \tag{A.2}$$

$$\begin{aligned} cD(\lambda) = {} & \lambda[-e^{2\lambda} - e^{4\lambda} - 2e^{4\lambda}k_c\lambda + e^{\lambda}\cos(\lambda) \\ & + e^{3\lambda}\cos(\lambda) - 2e^{\lambda}k_c\lambda\cos(\lambda) \\ & + 2e^{3\lambda}k_c\lambda\cos(\lambda) + 2e^{2\lambda}k_c\lambda\cos(2\lambda) \\ & - e^{\lambda}\sin(\lambda) + e^{3\lambda}\sin(\lambda) + e^{2\lambda}\sin(2\lambda)] \\ & + q[-\lambda e^{2\lambda} - \lambda e^{4\lambda} - 2k_c\lambda^2 e^{4\lambda} \\ & + 2k_c\lambda^2 e^{2\lambda}\cos(2\lambda) + e^{\lambda}\sin(\lambda) \\ & + e^{3\lambda}\sin(\lambda) + \lambda e^{2\lambda}\sin(2\lambda)] \end{aligned} \tag{A.3}$$

$$\begin{aligned} dD(\lambda) = {} & \lambda[e^{2\lambda} - e^{\lambda}\cos(\lambda) - e^{3\lambda}\cos(\lambda) \\ & + e^{2\lambda}\cos(2\lambda) - e^{\lambda}\sin(\lambda) + e^{3\lambda}\sin(\lambda) \\ & + 2e^{\lambda}k_c\lambda\sin(\lambda) + 2e^{3\lambda}k_c\lambda\sin(\lambda) \\ & - 2e^{2\lambda}k_c\lambda\sin(2\lambda)] + q[\lambda e^{2\lambda} + e^{\lambda}\cos(\lambda) \\ & - e^{3\lambda}\cos(\lambda) + \lambda e^{2\lambda}\cos(2\lambda) \\ & + 2k_c\lambda^2 e^{2\lambda}\sin(2\lambda)], \end{aligned} \tag{A.4}$$

where

$$\begin{aligned} D(\lambda) = {} & \lambda[-1 + e^{4\lambda} - 2e^{2\lambda}\sin(2\lambda) \\ & + 2k_c\lambda(1 + e^{4\lambda} - 2e^{2\lambda}\cos(2\lambda))]. \end{aligned} \tag{A.5}$$

The no-slip lower boundary condition used in Charney (1973) corresponds to the case here with $k_c = 0$, and $\Delta_V = \text{const}$ ($q = 0$). Setting $k_c = 0$ and $q = 0$, we find that

$$\begin{aligned} aD_0(\lambda) = {} & -e^{4\lambda} + e^{\lambda}\cos(\lambda) + e^{3\lambda}\cos(\lambda) \\ & - e^{2\lambda}\cos(2\lambda) + e^{2\lambda}\sin(2\lambda) \end{aligned} \tag{A.6}$$

$$\begin{aligned} bD_0(\lambda) = {} & -e^{4\lambda} + 2e^{3\lambda}\cos(\lambda) - e^{2\lambda}\cos(2\lambda) \\ & + e^{\lambda}\sin(\lambda) + e^{3\lambda}\sin(\lambda) - e^{2\lambda}\sin(2\lambda) \end{aligned} \tag{A.7}$$

$$\begin{aligned} cD_0(\lambda) = {} & -e^{2\lambda} - e^{4\lambda} + e^{\lambda}\cos(\lambda) + e^{3\lambda}\cos(\lambda) \\ & - e^{\lambda}\sin(\lambda) + e^{3\lambda}\sin(\lambda) + e^{2\lambda}\sin(2\lambda) \end{aligned} \tag{A.8}$$

$$\begin{aligned} dD_0(\lambda) = {} & e^{2\lambda} - e^{\lambda}\cos(\lambda) - e^{3\lambda}\cos(\lambda) \\ & + e^{2\lambda}\cos(2\lambda) - e^{\lambda}\sin(\lambda) + e^{3\lambda}\sin(\lambda), \end{aligned} \tag{A.9}$$

where

$$D_0(\lambda) = -1 + e^{4\lambda} - 2e^{2\lambda}\sin(2\lambda). \tag{A.10}$$

APPENDIX B

Asymptotic Validity

Since the circulation is weaker ($\sim O(E)$) in midlatitudes, the most stringent test occurs near the equator, where for E small our solution for $n = 1$ is

$$\Psi_0 = \frac{\cos^2\phi\,\sin\phi}{24E} z^2(z-1)(2z-3) + o(\sin\phi) \tag{B.1}$$

$$u_0 = \frac{\cos\phi\,\sin^2\phi}{480E^2} z^3(8z^2 - 25z + 20) + o(\sin^2\phi). \tag{B.2}$$

For the zonal momentum equation, the nonlinear momentum advection terms are, near the equator:

$$\mathrm{Ro}v_0\frac{\partial u_0}{\partial\phi} \approx -\frac{\mathrm{Ro}\cos\phi\,\sin^2\phi}{24\times 480E^3}(2 - 3\sin^2\phi)f_1(z) \tag{B.3}$$

$$\mathrm{Ro}w_0\frac{\partial u_0}{\partial z} \approx \frac{\mathrm{Ro}\cos\phi\,\sin^2\phi}{24^2E^3}(1 - 3\sin^2\phi)f_2(z) \tag{B.4}$$

$$\mathrm{Ro}u_0v_0\tan\phi \approx -\frac{\mathrm{Ro}\cos\phi\,\sin^4\phi}{24\times 480E^3}f_1(z), \tag{B.5}$$

which are to be compared to the viscous term

$$E\frac{\partial^2 u_0}{\partial z^2} \approx \frac{\cos\phi\,\sin^2\phi}{24E}f_3(z), \tag{B.6}$$

where

$$f_1(z) = z^4(8z^2 - 15z + 6)(8z^2 - 25z + 20) \tag{B.7}$$

$$f_2(z) = z^4(z-1)(2z-3)(2z^2 - 5z + 3) \tag{B.8}$$

$$f_3(z) = z(8z^2 - 15z + 6). \tag{B.9}$$

Since

$$\max_{0\leq z\leq 1} f_1(z) = 0.0676; \quad \max_{0\leq z\leq 1} f_2(z) = 0.378;$$

$$\max_{0\leq z\leq 1} f_3(z) = 0.688,$$

the neglected momentum advection terms are small when (40) is satisfied.

If we write $\theta = \theta_E + \Delta\theta$, then the energy equation is

$$\sigma \operatorname{Ro}\left[\left(v\frac{\partial\theta_E}{\partial\phi} + w\frac{\partial\theta_E}{\partial z}\right) + \left(v\frac{\partial\Delta\theta}{\partial\phi} + w\frac{\partial\Delta\theta}{\partial z}\right)\right] = E\frac{\partial^2\Delta\theta}{\partial z^2}. \quad \text{(B.10)}$$

The terms in the second set of parentheses were dropped in (19), and we need to show that this is justifiable. Calculating $\Delta\theta$ from (19), we find

$$\Delta\theta = \frac{\sigma \operatorname{Ro}\Delta_V}{24E^2}\left(\frac{z^6}{15} - \frac{z^5}{4} + \frac{z^4}{4} - \frac{z}{15}\right) + O(\sin^2\phi). \quad \text{(B.11)}$$

In the tropics, the meridional velocity is weaker ($\sim\sin^2\phi$). Therefore, the advection is dominated by the vertical advection and the ratio of the advection due to the vertical gradient of $\Delta\theta$ and the vertical gradient of θ_E is approximately given by

$$\frac{\sigma \operatorname{Ro}}{24 \times 180E^2} f_4(z), \quad \text{(B.12)}$$

where

$$f_4(z) = (72z^5 - 225z^4 + 180z^3 - 12)$$

$$\text{and} \quad \max_{0\leq z\leq 1} f_4(z) = 15. \quad \text{(B.13)}$$

Therefore, (19) is a quantitatively valid approximation to (B.10) if (38) is satisfied: $\operatorname{Ro} \leq 80E^2$, which is the same as Hide's constraint.

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