

## Generalized Static Energy and Its Conservation

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(Manuscript received 3 August 1993, in final form 11 March 1994)

### ABSTRACT

The theoretical study presented here shows that it is possible to define an energetic parameter that generalizes the dry, saturated, and moist static energies. The properties of the generalized static energy (GSE) are similar to those of dry, saturated, and moist static energies, but GSE can be used in cloudy systems including water vapor, liquid water, and ice, as well as in nonequilibrium conditions.

It is shown that GSE is directly related to the entropy and that it is reduced to dry, saturated, and moist static energies when appropriate assumptions are made. It is also shown that GSE is a conservative parameter when irreversibility and mass flux do not exist or are ignored.

### 1. Introduction

In the last few decades, associated with the development of cumulus parameterization models, the dry, saturated, and moist static energies began to be used as stability parameters (Arakawa and Schubert 1974; Haltiner and Williams 1980). The use of these energetic magnitudes permits the design of a thermodynamic diagram (García et al. 1987) and an instability index (Rivas and García 1993) that allow an immediate use of the dry, saturated, and moist static energies in operative analysis and prediction of static instability and convective phenomena. Recently, static energy has been used as basis for a forecasting index of forest fires (García et al. 1994).

However, these definitions of static energies are useful if they are used in connection with a simple thermodynamic model: a closed system of dry air, water vapor, and liquid water under equilibrium conditions. Evidently, when the atmospheric thermodynamic models are refined in order to be more realistic, it is necessary to include the possibility of open systems, irreversible processes, and ice. When these complex systems are considered, the aforementioned static energies may not be accurate.

In an earlier analysis, Hauf and Holler (1987) introduced the entropy temperature. The entropy temperature is a unified potential temperature that generalizes the potential temperatures previously defined. The closed relationship between potential temperature and static energy suggests the possibility of extending the

analysis of Hauf and Holler and defining a generalized static energy.

In this work a more general definition of static energy is proposed that has a much more general validity and allows its application to cloudy systems with water in the three phases, including nonequilibrium conditions. In consequence, this general static energy can be used as an energetic parameter in connection with more realistic thermodynamic processes in the atmosphere.

The generalized static energy (GSE) will be defined in section 3, where we will show the close relationship between GSE and entropy. In section 4 it is shown that, under appropriate assumptions, GSE is reduced to dry, saturated, and moist static energies. In section 5 the conservation of GSE is considered.

### 2. Basic assumptions and notation

We consider a volume of air consisting of dry air, water vapor, liquid water, and ice (i.e., a cloudy system). If the system has a total mass  $M$ , then

$$M = M_d + M_v + M_l + M_f, \quad (1)$$

where  $M_d$ ,  $M_v$ ,  $M_l$ , and  $M_f$  are, respectively, the dry air, water vapor, liquid water, and ice masses. After division by  $M$ , Eq. (1) becomes

$$1 = m_d + m_v + m_l + m_f, \quad (2)$$

with  $m_k = M_k/M$  ( $k = d, v, l, f$ ).

It is assumed that (i) both dry air and water vapor are ideal gases completely mixed, (ii) the volume occupied by liquid water and ice is very small compared with those of gases (dry air and water vapor), (iii) the liquid water and ice are not mixed either with each other or with the gases, and (iv) all effects that arise from drop surface curvature, temperature and pressure

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differences, radius-dependent fall speeds, etc., are ignored. Under these assumptions, the specific entropy of each component is (e.g., Iribarne and Godson 1973)

$$\begin{aligned} S_d &= C_{pd} \ln T - R_d \ln p_d + b_d \\ S_v &= C_{pv} \ln T - R_v \ln e_v + b_v \\ S_l &= C_l \ln T + b_l \\ S_f &= C_f \ln T + b_f, \end{aligned} \quad (3)$$

where, as above, the subscripts  $d, v, l,$  and  $f$  are, respectively, dry air, water vapor, liquid water, and ice;  $R_d$  and  $R_v$  are the ideal gas constants for dry air and water vapor;  $C_{pd}$  and  $C_{pv}$  the specific heats at constant pressure for dry air and water vapor;  $C_l$  and  $C_f$  the specific heats for liquid water and ice; and  $p_d$  and  $e_v$  the partial pressures for dry air and water vapor. The quantities  $b_k$  ( $k = d, v, l, f$ ) are the reference constant values for  $S_k$ .

The entropy  $S$  of the total system can be put in the form (Hauf and Holler 1987)

$$\begin{aligned} S &= m_d S_d + (m_v + m_l + m_f) S_l \\ &+ m_v(L_v + A_v)/T - m_f(L_l + A_l)/T, \end{aligned} \quad (4)$$

where  $L_v$  ( $L_l$ ) and  $A_v$  ( $A_l$ ) are, respectively, the latent heat and affinity of vaporization (fusion). It should be noted that affinities vanish only if equilibrium conditions are assumed.

We will also use the notation

$$\begin{aligned} q &= M_v/M_d = m_v/m_d \\ l &= M_l/M_d = m_l/m_d \\ f &= M_f/M_d = m_f/m_d, \end{aligned} \quad (5)$$

in which  $q$  is the mixing ratio and  $l$  and  $f$  some kind of mixing ratio for liquid water and ice, respectively. After division of (2) by  $m_d$  we get

$$1/m_d = 1 + q + l + f. \quad (6)$$

### 3. Generalized static energy (GSE)

Under the assumptions established, we define the generalized static energy ( $h_g$ ) as

$$h_g = C^*T + gz + (L_v + A_v)q - (L_l + A_l)f - \epsilon, \quad (7)$$

where  $g$  is the gravity (assumed constant),  $z$  the dry air geopotential height defined as  $gdz = -R_d T d(\ln p_d)$ ,  $C^*$  is some kind of specific heat defined as  $C^* = C_{pd} + C_l(q + l + f)$ , and

$$\epsilon = \int_{T_0}^T [(L_v + A_v)q - (L_l + A_l)f] dT/T, \quad (8)$$

with  $T_0$  a constant temperature of reference.

The term  $\epsilon$  can be roughly taken as 0 if the variations of  $T$ ,  $(L_v + A_v)$ , and  $(L_l + A_l)$  are smaller than the variations of  $q$  and  $f$ .

It is clear that  $h_g$  has dimensions of energy per mass unit. It is also seen from (7) that  $h_g$  involves the more significant energetic contributions except the kinetic energy.

It should be noted that the significance of static energies (dry, saturated, and moist) arises from their close relationship to the entropy. Then, static energies can be directly used to determine entropic and energetic budgets and static instabilities. In consequence, if we wish to understand the physical meaning and meteorological applications of  $h_g$ , we must fix its relationship to the entropy of the complex cloudy system that we are considering.

After differentiation of (6) we get

$$dm_d/m_d^2 + dq + dl + df = 0. \quad (9)$$

Changes in Eq. (9) may be caused by phase transitions,  $d_i$ , or by exchange of mass through the system boundaries  $d_e$ . Since dry air does not change phase,  $d_i q + d_i l + d_i f = 0$ . Now, we make the assumption

$$0 = d_e m_d/m_d^2 + d_e q + d_e l + d_e f \approx d_e q + d_e l + d_e f, \quad (10)$$

for which  $d_e m_d \ll m_d^2$  is a sufficient condition. Equation (10) is exact in closed systems. However, the condition  $d_e m_d \ll m_d^2$  can be considered valid for the entrainment in great convective clouds, in which the dry air that entrains is less than the preexisting dry air. Using (9) and (10) we get

$$dq + dl + df \approx 0. \quad (11)$$

Differentiating (7) and using (11) we obtain

$$\begin{aligned} dh_g &= Td \left[ C^* \ln T - R_d \ln p_d \right. \\ &\left. + \frac{L_v + A_v}{T} q - \frac{L_l + A_l}{T} f \right]. \end{aligned} \quad (12)$$

If it is assumed that the variations of  $T$ ,  $(L_v + A_v)$ , and  $(L_l + A_l)$  are smaller than the variations of  $q$  and  $f$ , then

$$\begin{aligned} d \left[ \frac{L_v + A_v}{T} q \right] &\approx \frac{L_v + A_v}{T} dq \\ d \left[ \frac{L_l + A_l}{T} f \right] &\approx \frac{L_l + A_l}{T} df, \end{aligned} \quad (13)$$

and, under these conditions, it is clear that to obtain (12) it is not necessary to include  $\epsilon$  in the definition (7) of  $h_g$ .

Using (3) and (4) we can write (12) as

$$dh_g = Td \left[ C_{pd} \ln T - R_d \ln p_d + (q + l + f) C_l \ln T + \frac{L_v + A_v}{T} q - \frac{L_l + A_l}{T} f \right] = Td \left[ \frac{S - m_d b_d - (m_v + m_l + m_f) b_l}{m_d} \right], \tag{14}$$

and also

$$dh_g = C^* T d(\ln \theta_s), \tag{15}$$

where

$$\theta_s = \exp \left[ \frac{S - m_d b_d - (m_v + m_l + m_f) b_l}{C^* m_d} \right] = T(p_0/p_d)^{R_d/C^*} \exp \left[ \frac{L_v + A_v}{C^* T} q - \frac{L_l + A_l}{C^* T} f \right] \tag{16}$$

is the entropy temperature (Hauf and Holler 1987). It should be noted that entropy temperature is the most general and exact potential temperature.

Equation (14) shows that the variations of GSE are directly related to the entropy variations of the complex cloudy system that is considered in this work. Moreover, Eq. (15) shows that the GSE variations are also related with the variations of the logarithm of the entropy temperature. In conclusion, we have found that the relationship between GSE, entropy, and entropy temperature is the same as the well-known relationship between dry static energy, dry air entropy, and potential temperature of dry air.

Equations (14) or (15) can be taken as a formal definition of  $h_g$ . Thus, the approximation (10) is not strictly necessary in order to assure the validity of the concept of generalized static energy. However, in this case  $dh_g$  is not an exact differential, and, consequently, (14) or (15) have to be integrated for each particular evolution of the system.

It should be noted that in  $h_g$  the height  $z$  is defined in terms of dry air geopotential instead of related to total pressure as is usual in static energies. If  $z$  in (7) is defined in terms of total pressure  $p$ , then (15) is not exact. In this case, to obtain the general equation (15), further assumptions are needed. For example, if  $z$  is defined as  $gdz = -R_d T d(\ln p)$  instead of (15), we obtain  $dh_g = C^* T d(\ln \theta_s) + R_d T d \ln(p_d/p)$ . To obtain (15) it is necessary to assume  $d \ln(p_d/p) \approx 0$ . In consequence, it is more general to define  $z$  in (7) in terms of dry air geopotential.

Entropy temperature includes as particular cases all potential temperatures defined in bibliography as potential temperature of dry air, equivalent potential temperature, liquid water potential temperature, ice liquid water potential temperature, and wet equivalent potential temperature (Hauf and Höller 1987). Thus, using

Eqs. (7), (15), and (16) it can be defined as static energy connected to each particular potential temperature, being conservative under the same conditions. In the next section we will show several examples.

#### 4. Some particular cases of $h_g$

In this section, we show that GSE is a generalization of dry, saturated, and moist static energies and that, in fact,  $h_g$  includes them.

##### a. Dry static energy

If only dry air is considered, then  $q = l = f = 0$  and  $h_g$  reduces to

$$h_g = C_{pd} T + gz = M, \tag{17}$$

and the GSE is equivalent to the dry static energy. Using Eq. (16) we find

$$\theta_s = T(p_0/p_d)^{R_d/C_{pd}} = \theta; \tag{18}$$

that is,  $\theta_s$  reduces to potential temperature of dry air. In these conditions Eq. (15) is

$$dh_g = C_{pd} T d(\ln \theta) = dM, \tag{19}$$

which is the well-known equation relating dry static energy and potential temperature of dry air.

##### b. Saturated static energy

We consider now a cloudy system without ice, where the water vapor is saturated and under equilibrium conditions. In this case  $f = 0$  (no ice),  $q = q^*$  (saturated mixing ratio), and  $A_v = 0$  (equilibrium conditions). In consequence (7) reduces to saturated static energy:

$$h_g = [C_{pd} + (q^* + l) C_l] T + gz + L_v q^* - \int_{T_0}^T L_v q^* dT/T = h^{*'} \tag{20}$$

Using (16) the entropy temperature reduces to equivalent potential temperature:

$$\theta_s = T(p_0/p_d)^{R_d/C_{pd} + (q^* + l) C_l} \times \exp \left\{ \frac{L_v q^*}{[C_{pd} + (q^* + l) C_l] T} \right\} = \theta_e. \tag{21}$$

Consequently Eq. (15) now is

$$dh_g = [C_{pd} + (q^* + l) C_l] T d(\ln \theta_e) = dh^{*'} \tag{22}$$

Equation (22) shows that saturated static energy is related to equivalent potential temperature.

It should be noted that  $h^{*'}$  does not exactly agree with the usual definition of saturated static energy,  $h^* = [C_{pd} + (q^* + l) C_l] T + gz + L_v q^*$  (Betts 1974). The definition of  $h^*$  equals  $h^{*'}$  only if the integral term of (20) is ignored. As was mentioned above, that ap-

proximation is roughly true when the variation of  $q$  is smaller than those of  $L_v$  and  $T$ , as indeed occurs in the saturated adiabatic evolution, in which  $\theta_e$  is constant. In consequence, it is consistent to affirm that  $h^*$  is constant in saturated adiabatic evolutions.

### c. Moist static energy

Now we consider a cloudy system without ice. In this case, Eq. (7) and (16) reduce to

$$h_g = [C_{pd} + (q + l)C_l]T + (L_v + A_v)q - \int_{T_0}^T (L_v + A_v)q dT/T = h', \quad (23)$$

$$\theta_s = T(p_0/p_d)^{R_d/C_{pd} + (q+l)C_l} \exp\left\{\frac{(L_v + A_v)q}{[C_{pd} + (q + l)C_l]T}\right\}, \quad (24)$$

where  $h'$  is the moist static energy.

The affinity  $A_v$  is written as

$$A_v = R_v T \ln(e_v^*/e_v), \quad (25)$$

in which  $e_v^*$  is the saturated pressure over a plane surface of liquid water. Inserting (25) in (24) yields

$$\theta_s = T(p_0/p_d)^{R_d/C_{pd} + (q+l)C_l} \times \exp\left\{\left[\frac{L_v}{T} + R_v \ln\left(\frac{e_v^*}{e_v}\right)\right] \times \frac{q}{C_{pd} + (q + l)C_l}\right\} = \theta_q. \quad (26)$$

that is, under our assumptions  $\theta_s$  reduces to Paluch's (1974) wet equivalent potential temperature. Thus we find

$$dh_g = [C_{pd} + (q + l)C_l]Td(\ln\theta_q) = dh'. \quad (27)$$

Equation (27) shows that moist static energy is related to wet equivalent potential temperature. As before,  $h'$  does not exactly agree with the usual definition of moist static energy,  $h = [C_{pd} + (q + l)C_l]T + gz + L_v q$ . If we wish that  $h'$  equal  $h$ , two supplementary approximations must be assumed: (i) the water vapor must be sufficiently near to saturation in equilibrium that  $A_v \approx 0$ , and (ii) the integral term in (23) must be ignored. However, it seems clear that approximation (ii) cannot, in general, be assumed. With assumptions (i) and (ii) Eqs. (23), (24), and (27) reduce to

$$h = [C_{pd} + (q + l)C_l]T + gz + Lq$$

$$\theta_q = T(p_0/p_d)^{R_d/C_{pd} + (q+l)C_l} \exp\left\{\frac{L_v q}{[C_{pd} + (q + l)C_l]T}\right\}$$

$$dh = [C_{pd} + (q + l)C_l]Td(\ln\theta_q). \quad (28)$$

## 5. Conservation of the generalized static energy

In this section we consider the conditions in which the GSE is a conservative parameter. From (15)

$$\frac{dh_g}{dt} = C^* T \frac{d \ln \theta_s}{dt}. \quad (29)$$

Using the result of Hauf and Holler (1987) on the prognostic equation for the entropy temperature, in connection with (11), we find

$$\frac{dh_g}{dt} = \frac{T}{\rho m_d} \left\{ \rho \frac{dS}{dt} + [(b_d - b_l)m_d + S - (m_v + m_l + m_f)b_l] \frac{\nabla \cdot \mathbf{F}_d}{m_d} \right\}$$

$$\mathbf{F}_d = \rho_d (\mathbf{V}_d - \mathbf{V}), \quad (30)$$

where  $\rho$  is the total density of the system,  $\rho_d$  the density of the dry air,  $\mathbf{V}_d$  the velocity of the dry air, and  $\mathbf{V}$  the velocity of the center of mass. Since  $\mathbf{V}_d - \mathbf{V}$  is the velocity of the dry air with respect to the center of mass, then  $\mathbf{F}_d$  is the mass flux of the dry air.

Equation (30) is a prognostic equation for GSE in the most general case. We can see that variations of  $h_g$  arise from (i) variations of  $S$  and (ii) mass flux divergence of dry air. Therefore, a constant entropy does not generally imply conservation of GSE. We can only assume that GSE is constant when there is simultaneously a constant entropy and the divergence of mass flux of dry air vanishes. It should be noted that  $\mathbf{F}_d$  always vanishes if the dry air moves with the center of mass velocity.

Since the dry air mass is greater than other component masses, it could be assumed that the dry air guides the movement of the system, that is,  $\mathbf{V} \approx \mathbf{V}_d$ . Under this approximation Eq. (30) reduces to

$$dh_g/dt = (T/m_d)dS/dt; \quad (31)$$

that is, the GSE is a conservative parameter if the entropy is also a constant.

## 6. Conclusions

The dry, saturated, and moist static energies are appropriate energetic parameters to use in atmospheric thermodynamic models. However, they involve rather simple thermodynamic processes. Consequently, they are not completely realistic parameters to describe the complex thermodynamic processes that arise in cloudy systems. The aim of this study is to deduce a new energetic parameter with properties analogous to those of dry, saturated, and moist static energies that generalize them.

It is a major finding of this work to show that it is possible to define a generalized static energy (GSE) in such a way that the relationship between GSE and the entropy temperature of Hauf and Holler is analogous

to the well-known relationship between the dry static energy and the potential temperature of the dry air. Then, for a complex model of atmosphere, we can use  $h_g$  in an way analogous to that of dry static energy in a dry atmosphere. Consequently, we have extended the result of Hauf and Holler (1987).

In the generalized static energy the height is defined in terms of dry air geopotential. However, in usual definitions of static energies, the height is related to total pressure. If we define the height in terms of total pressure, then the generalized static energy is not related to the most general definition of potential temperature and, consequently, further assumptions are needed. Thus, the static energy is more general if the height is defined in terms of dry air geopotential.

It is also shown that the variations of the generalized static energy arise from variations of the entropy and from the mass flux divergence of dry air. If it is assumed that dry air mass is so great that its mass flux is negligible, then when entropy is a constant the GSE is a conservative parameter. Similarly, if GSE is not a constant, then irreversible fluxes have to be included.

*Acknowledgments.* Financial support came from Spanish CICYT (FOR 91-0818).

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