Potential Vorticity Conservation, Hydrostatic Adjustment, and the Anelastic Approximation

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ABSTRACT

An examination of the anelastic equations of Lipps and Hemler shows that the approximation requires that the temperature and potential temperature scale heights of the base state are large compared to the pressure and density scale heights. As a consequence the fractional changes of the temperature and potential temperature fields relative to their base state values are equivalent. Alternatively this equivalency requires that the ratio of the ideal gas constant to the specific heat capacity at constant pressure is small.

The anelastic equations are examined for their ability to conserve potential vorticity (PV). The equations are shown to be "PV correct" in the sense that they conserve potential vorticity in a manner consistent with Ertel's theorem and with the assumptions of the anelastic approximation.

The ability to conserve potential vorticity helps the anelastic system capture the integrated effect of the acoustic modes in Lamb's hydrostatic adjustment problem. This prototype problem considers the response of a stably stratified atmosphere to an instantaneous heating that is vertically confined but horizontally uniform. In the anelastic case, the state variables adjust instantaneously to be in hydrostatic balance with the potential temperature perturbation generated by the heating. The anelastic solutions for the pressure, density, and temperature fields are identical to those for the compressible case. In contrast there is a mutual adjustment of the pressure, density, and thermal fields in the compressible case, which is not instantaneous. The total energy in the final state for the two cases is the same.

The other versions of the anelastic approximation are examined for their PV correctness and for their ability to parameterize Lamb's acoustic hydrostatic adjustment process.

1. Introduction

Meteorological models are often based on the anelastic approximation. This approximation filters the acoustic modes from the equations of motion and produces a soundproof set of equations that are simpler to analyze theoretically than the compressible equations. Numerically, the anelastic set has the advantage of permitting a larger time step in finite-difference models. Since it retains the decrease of the ambient density field with height, the anelastic approximation represents an improvement over the Boussinesq approximation. Ogura and Phillips (1962) first used the term "anelastic." The adjectives "deep" and "shallow" are also used to distinguish the anelastic and Boussinesq approximations, respectively.

The anelastic approximation appears in many different guises in the literature. Batchelor (1953) proposed the original form of the anelastic equations. Ogura and Phillips (1962) derived an identical set using a formal scale analysis that assumes that the fractional variation in the potential temperature field is small and that the timescale of the flow is bounded by the inverse of the buoyancy frequency. Since this original anelastic set holds for an isentropic base state atmosphere, it is not directly applicable to realistic meteorological situations.

Subsequent early investigators (Dutton and Fichtl 1969; Gough 1969) sought to remove this restriction, but Wilhelmson and Ogura (1972) noted that these modified anelastic equations fail to conserve energy in a closed domain. In proposing an anelastic set that is energy conserving, Lipps and Hemler (1982) (see also Lipps and Hemler 1985; and Lipps 1990) made the assumption that the reference potential temperature field varies slowly with height. The original anelastic set is therefore a special subset of this more general formulation. Lipps and Hemler (1982) also showed that the fractional changes of the temperature and potential temperature fields relative to their base state values are equal. We refer to this property of the anelastic equations as that of thermal equivalency. Earlier, Wilhelmson and Ogura (1972) had demonstrated numerically that thermal equivalency is a good approximation and noted its utility in the calculation of the saturation vapor pressure.

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More recently Durran (1989) has introduced a pseudo-incompressible approximation to obtain an alternative soundproof set that also conserves energy in a closed domain. Nance and Durran (1994) have compared this approximation analytically and numerically with several versions of the anelastic equations.

The purpose of this paper is to examine the ability of the various anelastic equations to conserve potential vorticity (PV) and to parameterize the acoustic hydrostatic adjustment process. Potential vorticity conservation is relevant to the anelastic equations since they are often applied to synoptic-scale phenomena as well as to the cloud-scale ones for which they were first developed. In addition, Bannon (1995) has shown that the final state of Lamb's hydrostatic adjustment problem requires the conservation of potential vorticity.

Section 2 presents a heuristic derivation of the anelastic approximation of Lipps and Hemler (1982) and provides a framework for the comparison of related theories. The section gives an alternative proof of the equivalency of the perturbation thermal fields and highlights the subtle differences between the equations of Lipps and Hemler (1982) and Dutton and Fichtl (1969; see also Dutton 1986, chapters 13 and 15). Section 3 demonstrates that the Lipps–Hemler set conserves potential vorticity in a manner consistent with Ertel's theorem and the anelastic assumptions. Section 4 applies this set to Lamb's hydrostatic adjustment problem. It is shown that the pressure, density, and temperature fields, diagnosed using the anelastic set of Lipps and Hemler, adjust instantaneously to the correct solution of Lamb's problem. Thus, the Lipps–Hemler set successfully captures the role of the acoustic modes without their explicit representation. Section 5 uses these issues to examine other approximate soundproof theories, including the Boussinesq theory and the standard and modified quasigeostrophic theories. Section 6 presents the conclusions.

2. The anelastic approximation

The equations of motion for a dry, rotating atmosphere are

$$\rho \frac{D\mathbf{u}}{Dt} + \rho 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla p - \rho g \mathbf{k} + \mathbf{F}, \quad (2.1a)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (2.1b)$$

$$\frac{D\theta}{Dt} = \frac{\theta Q}{\rho C_p T}, \quad (2.1c)$$

$$\theta = T\left(\frac{p_{00}}{p}\right)^{R/C_p}, \quad (2.1d)$$

$$p = \rho R T, \quad (2.1e)$$

where $D/Dt = \partial/\partial t + (\mathbf{u} \cdot \nabla)$ is the material derivative. The notation is standard with $\mathbf{F}$ representing the frictional forces, Q the heating rate per unit volume, and p_{00} a constant reference pressure.

We introduce a static base state atmosphere denoted with a subscript s, which satisfies (2.1) with $Q = \mathbf{F} = \mathbf{u} = 0$ such that

$$\frac{dp_s}{dz} = -\rho_s g, \quad \theta_s = T_s\left(\frac{p_{00}}{p_s}\right)^{R/C_p}, \quad p_s = \rho_s R T_s. \quad (2.2)$$

This general base state is only a function of height z. Specific specialized base states (e.g., isothermal or isentropic) are introduced later. The state variables consist of the sum of the base state and a perturbation denoted with a prime. For example, $\theta(x, y, z, t) = \theta_s(z) + \theta'(x, y, z, t)$. The full velocity field is unprimed.

The anelastic approximation assumes the following:

(i) the thermodynamic state variables exhibit only small departures from their static reference values (e.g., $\theta' \ll \theta_s$), and

(ii) the density field in the momentum and continuity equations may be replaced by its static value, ρ_s, everywhere except where it multiplies gravity.

The second assumption retains the buoyancy force in the momentum equation. Neglect of the density perturbations in the continuity equation requires a constraint on the dynamical timescale (see, e.g., Dutton 1986). With these assumptions, (2.1) simplifies to

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho_s}\nabla p' - \frac{\rho'}{\rho_s} g \mathbf{k} + \frac{\mathbf{F}}{\rho_s}, \quad (2.3a)$$

$$\nabla \cdot (\rho_s \mathbf{u}) = 0, \quad (2.3b)$$

$$\frac{D\theta'}{Dt} + w\frac{d\theta_s}{dz} = \frac{\theta_s Q}{\rho_s C_p T_s}, \quad (2.3c)$$

$$\frac{\theta'}{\theta_s} = \frac{p'}{\gamma p_s} - \frac{\rho'}{\rho_s}, \quad (2.3d)$$

$$\frac{p'}{p_s} = \frac{\rho'}{\rho_s} + \frac{T'}{T_s}, \quad (2.3e)$$

where $\gamma = C_p/C_v = 7/5$ is the ratio of the specific heats. The absence of a time derivative in the continuity equation (2.3b) filters the acoustic modes. However, the set (2.3) fails to conserve energy for adiabatic inviscid flow in a closed domain.

To obtain an energy-conserving set, additional assumptions are required. We note that the pressure and gravity terms in (2.3a) may be written in the form

$$-\frac{1}{\rho_s}\nabla p' - \frac{\rho'}{\rho_s} g \mathbf{k} = -\nabla\left(\frac{p'}{\rho_s}\right) + \frac{g\theta'}{\theta_s}\mathbf{k} + \frac{p'}{\rho_s H_\theta}\mathbf{k}, \quad (2.4a)$$

or, equivalently,

$$-\frac{1}{\rho_s}\nabla p' - \frac{\rho'}{\rho_s} g\mathbf{k} = -\nabla\left(\frac{p'}{\rho_s}\right) + g\left(-\frac{\rho'}{\rho_s} + \frac{p'}{\rho_s g H_\rho}\right)\mathbf{k}, \quad (2.4b)$$

where the density and potential temperature scale heights are defined as

$$H_\rho^{-1} = \frac{-1}{\rho_s}\frac{d\rho_s}{dz}, \quad H_\theta^{-1} = \frac{1}{\theta_s}\frac{d\theta_s}{dz}, \quad (2.5)$$

respectively. Lipps and Hemler (1982) use (2.4a) but neglect the last term. A scale analysis of the vertical component of the first and third terms on the rhs of (2.4a),

$$-\frac{\partial}{\partial z}\left(\frac{p'}{\rho_s}\right) + \frac{p'}{\rho_s}\frac{d\ln\theta_s}{dz} = -\frac{1}{\rho_s}\frac{\partial p'}{\partial z} + \frac{p'}{\rho_s}\left(\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} + \frac{1}{\theta_s}\frac{\partial\theta_s}{\partial z}\right), \quad (2.6)$$

indicates that the neglect of the last term in (2.4a) requires that H_θ is large in the sense that

$$(H_\rho, D) \ll H_\theta, \quad (2.7)$$

where D is the characteristic depth scale of the flow.

Alternatively, Dutton and Fichtl (1969) retain (2.4b) but make a similar assumption regarding H_θ to simplify (2.3d). Specifically, logarithmic differentiation of the definition of the base state potential temperature field yields

$$\frac{1}{\theta_s}\frac{d\theta_s}{dz} = \frac{1}{\gamma p_s}\frac{dp_s}{dz} - \frac{1}{\rho_s}\frac{d\rho_s}{dz}, \quad (2.8a)$$

or, equivalently,

$$\frac{1}{H_\theta} = \frac{-1}{\gamma H_p} + \frac{1}{H_\rho}, \quad (2.8b)$$

where $H_p^{-1} = -d\ln p_s/dz$. Then assuming H_θ is large in the sense that

$$(H_\rho, H_p) \ll H_\theta, \quad (2.9)$$

(2.8b) implies $\gamma H_p \approx H_\rho$. Then (2.3d) may be written using (2.2) as

$$\frac{\theta'}{\theta_s} = \frac{p'}{\rho_s g H_\rho} - \frac{\rho'}{\rho_s}, \quad (2.10)$$

and (2.3a) becomes

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\nabla\left(\frac{p'}{\rho_s}\right) + \frac{g\theta'}{\theta_s}\mathbf{k} + \frac{\mathbf{F}}{\rho_s}. \quad (2.11)$$

Thus, Dutton and Fichtl (1969) and Lipps and Hemler (1982) both arrive at a momentum equation in the form (2.11).

Lipps and Hemler (1982) further derive the relation for the equivalence of the temperature and potential temperature perturbations,

$$\frac{T'}{T_s} = \frac{\theta'}{\theta_s}, \quad (2.12)$$

based on a scale analysis that requires the first-order buoyancy and vertical accelerations to be the same order of magnitude. We now demonstrate that this assumption is overly restrictive by deriving (2.12) in an alternative fashion. Logarithmic differentiation of the equation of state for the base state yields

$$\frac{1}{p_s}\frac{dp_s}{dz} = \frac{1}{\rho_s}\frac{d\rho_s}{dz} + \frac{1}{T_s}\frac{dT_s}{dz}, \quad (2.13a)$$

or, equivalently,

$$\frac{1}{H_p} = \frac{1}{H_\rho} + \frac{1}{H_T}, \quad (2.13b)$$

where $H_T^{-1} = -d\ln T_s/dz$. Then the reasonable assumption that the temperature scale height H_T is large in the sense that

$$(H_\rho, H_p) \ll H_T \quad (2.14)$$

implies $H_p \sim H_\rho$. Thus, (2.10) reduces to

$$\frac{\theta'}{\theta_s} = \frac{p'}{p_s} - \frac{\rho'}{\rho_s}. \quad (2.15)$$

Combination of (2.3e) and (2.15) yields (2.12).

In summary, the anelastic approximation further requires that

(iii) the temperature and potential temperature scale heights are large compared with the pressure and density scale heights and with the depth scale of the flow

$$(H_\rho, H_p, D) \ll (H_T, H_\theta). \quad (2.16)$$

Then the anelastic equations are

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\nabla\left(\frac{p'}{\rho_s}\right) + \frac{g\theta'}{\theta_s}\mathbf{k} + \frac{\mathbf{F}}{\rho_s}, \quad (2.17a)$$

$$\nabla\cdot(\rho_s\mathbf{u}) = 0, \quad (2.17b)$$

$$\frac{D\theta'}{Dt} + w\frac{d\theta_s}{dz} = \frac{\theta_s Q}{\rho_s C_p T_s}, \quad (2.17c)$$

$$\frac{\theta'}{\theta_s} = \frac{T'}{T_s}, \quad (2.17d)$$

$$\frac{p'}{p_s} = \frac{\rho'}{\rho_s} + \frac{T'}{T_s}. \quad (2.17e)$$

The set (2.17) is identical with that of Lipps and Hemler (1982). Comparison of (2.3d) and (2.15) indicates that the assumptions (2.16) are equivalent to replacing γ^{-1} with unity in (2.3d). Thus, since $\gamma^{-1} = 1 - R/C_p$, (2.16) implies

$$\kappa \ll 1. \tag{2.18}$$

This assumption holds only marginally since $\kappa \equiv R/C_p = 2/7$.

We now show that (2.17) may also be derived using assumptions (i), (ii), and (2.18). The first two assumptions yield (2.3a) and (2.17b,c,e) from (2.1). The definition of potential temperature (2.1d) may be expanded as

$$T_s\left(1 + \frac{T'}{T_s}\right) = \theta_s\left(\frac{p_s}{p_{00}}\right)^\kappa\left(1 + \frac{p'}{p_s}\right)^\kappa\left(1 + \frac{\theta'}{\theta_s}\right). \tag{2.19}$$

Assumptions (i) and (2.18) imply that the presence of the pressure perturbation may be ignored and linearization of (2.19) yields (2.17d). Then (2.17d) and (2.17e) may be used to write (2.4b) as

$$-\frac{1}{\rho_s}\nabla p' - \frac{\rho'}{\rho_s}g\mathbf{k} = -\nabla\left(\frac{p'}{\rho_s}\right) + g\frac{\theta'}{\theta_s}\mathbf{k} - \kappa g\left(\frac{\Gamma}{\Gamma_d}\right)\frac{p'}{p_s}\mathbf{k}, \tag{2.20}$$

where $\Gamma = -dT_0/dz$ is the lapse rate and $\Gamma_d = g/C_p$ is the dry adiabatic lapse rate. The last term may be ignored using (2.18) and the fact that $\Gamma/\Gamma_d \leqslant O(1)$. Then (2.3a) simplifies to (2.17a).

It is important to note the energy properties of (2.17). Lipps (1990) has shown that the set conserves energy in flux form as

$$\frac{\partial E}{\partial t} + \nabla\cdot[(E + p')\mathbf{u}] = Q + \rho_s\mathbf{u}\cdot\mathbf{F}, \tag{2.21}$$

where

$$E = \rho_s\frac{\mathbf{u}\cdot\mathbf{u}}{2} + \rho_s C_p(T_s + T') + \rho_s gz. \tag{2.22}$$

The flux form assures that the volume integral of the energy E is conserved for adiabatic inviscid flow in a closed domain for which the normal component of the velocity field vanishes on the boundaries. The interpretation of the terms in (2.21) and (2.22) is standard except for the term C_pT' in (2.22). There the use of the specific heat capacity at constant pressure rather than that at constant volume reveals that, in the anelastic equations, the heating occurs at constant pressure. This behavior is consistent with assumption (i) that $p' \ll p_s$ and with (2.18), which implies that $C_v = C_p(1 - \kappa) \approx Cp$.

The set (2.17) is similar to that advanced by Dutton and Fichtl (1969), who use (2.10) in place of (2.17d) since they do not make assumption (2.14). In addition, their heat equation takes the form

$$\frac{D}{Dt}\left(\frac{\theta'}{\theta_s}\right) + \frac{w}{\theta_s}\frac{d\theta_s}{dz} = \frac{Q}{\rho_s C_p T_s}. \tag{2.23}$$

However, use of (2.23) in place of (2.17c) does not lead directly to a statement for energy conservation in flux form such as (2.21) without further assumptions (Dutton and Fichtl 1969).

Henceforth, the three assumptions (i)–(iii) are collectively referred to as the anelastic approximation and the set (2.17) as the anelastic equations. The present argument clarifies the equivalency of the assumptions (2.16) and (2.18).

3. Conservation of potential vorticity

In this section the statement of potential vorticity conservation is derived for the anelastic equations and compared with Ertel's theorem. Ertel's theorem for the full equation set (2.1) is

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}_a\cdot\nabla\theta}{\rho}\right) = \frac{\boldsymbol{\omega}_a\cdot\nabla\dot{\Theta}}{\rho} + \frac{\nabla\theta}{\rho}\cdot\left(\nabla\times\frac{\mathbf{F}}{\rho}\right), \tag{3.1}$$

where the absolute vorticity is $\boldsymbol{\omega}_a = 2\boldsymbol{\Omega} + \nabla\times\mathbf{u}$, the diabatic warming is

$$\dot{\Theta} = \frac{\theta Q}{\rho C_p T}, \tag{3.2}$$

and the potential vorticity is

$$\mathrm{PV}_{\mathrm{Ertel}} = \frac{\boldsymbol{\omega}_a\cdot\nabla\theta}{\rho}. \tag{3.3}$$

Potential vorticity (3.3) is conserved in the absence of friction and heating.

The anelastic form of Ertel's theorem may be derived in two different ways. The first approach is to derive the anelastic analog to (3.1) directly from the anelastic set. The curl of (2.17a) yields the vorticity equation

$$\frac{D\boldsymbol{\omega}_a}{Dt} = (\boldsymbol{\omega}_a\cdot\nabla)\mathbf{u} - \boldsymbol{\omega}_a(\nabla\cdot\mathbf{u}) + \nabla\left(g\frac{\theta'}{\theta_s}\right)\times\mathbf{k} + \left(\nabla\times\frac{\mathbf{F}}{\rho_s}\right). \tag{3.4}$$

Since the pressure term in (2.17a) is in the form of a gradient, there is no solenoidal term in (3.4). Substituting for the velocity divergence from the anelastic continuity equation (2.17b) and using the fact that $\nabla\theta_s \times \mathbf{k} = 0$ yields

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}_a}{\rho_s}\right) = \left(\frac{\boldsymbol{\omega}_a}{\rho_s}\cdot\nabla\right)\mathbf{u} + \frac{1}{\rho_s}\nabla\left(g\frac{\theta}{\theta_s}\right)\times\mathbf{k} + \frac{1}{\rho_s}\left(\nabla\times\frac{\mathbf{F}}{\rho_s}\right). \tag{3.5}$$

The gradient of the anelastic heat equation (2.17c) yields

$$\frac{D\nabla\theta}{Dt} = -\frac{\partial u_j}{\partial x_i}\frac{\partial \theta}{\partial x_j} + \nabla\dot{\Theta}_s, \tag{3.6}$$

where

$$\dot{\Theta}_s = \frac{\theta_s Q}{\rho_s C_p T_s} \tag{3.7}$$

is the anelastic diabatic warming. Taking the dot product of (3.5) with the gradient of θ and using (3.6) yields the anelastic form of Ertel's theorem

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}_a\cdot\nabla\theta}{\rho_s}\right) = \frac{\boldsymbol{\omega}_a\cdot\nabla\dot{\Theta}_s}{\rho_s} + \frac{\nabla\theta}{\rho_s}\cdot\left(\nabla\times\frac{\mathbf{F}}{\rho_s}\right), \tag{3.8}$$

where

$$\text{PV}_{\text{anelastic}} = \frac{\boldsymbol{\omega}_a\cdot\nabla\theta}{\rho_s} \tag{3.9}$$

is the anelastic potential vorticity. This derivation will be helpful in comparing the alternative forms of the anelastic equations (see section 5 below).

The second approach is to apply the assumptions (i)–(iii) to (3.1) directly. The result is again (3.8). Thus, the anelastic set conserves potential vorticity in a manner consistent with the full equations (2.1) and with the assumptions (i)–(iii).

4. Anelastic solution of Lamb's adjustment problem

Lamb's problem (Bannon 1995) consists of the calculation of the response of an atmosphere initially at rest to an instantaneous heating that is vertically confined but horizontally infinite. It represents the prototype for hydrostatic adjustment by acoustic modes. Since the anelastic set filters out these modes, the question arises as to whether the anelastic set can parameterize their effects accurately.

The linearized form of the anelastic set (2.17) with no horizontal motion is

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{p'}{\rho_s}\right) + g\frac{\theta'}{\theta_s}, \tag{4.1a}$$

$$\frac{\partial}{\partial z}(\rho_s w) = 0, \tag{4.1b}$$

$$\frac{\partial}{\partial t}\left(g\frac{\theta'}{\theta_s}\right) + wN_s^2 = B_s, \tag{4.1c}$$

where $N_s^2 = g\, d\ln\theta_s/dz$ is the square of the buoyancy frequency of the base state atmosphere. In this section we specialize to the case of an isothermal base state with $T_s = T_0 =$ constant. Then $H_\rho = H_s = RT_0/g$ is the constant density scale height. For Lamb's problem, the forcing term in (4.1c) corresponds to a uniform heating over the layer $-a < z < +a$. It has the form

$$B_s \equiv g\frac{\dot{\Theta}_s}{\theta_s} = \frac{\Delta p}{\gamma\rho_s H_s}[H(z+a) - H(z-a)]\delta(t), \tag{4.2}$$

where δ is the Dirac delta function and H is the Heaviside step function (and not a scale height). For definiteness, the constant Δp is assumed positive; then (4.2) describes a warming. Note that the origin of the coordinate system, $z = 0$, lies at the center of the region of heating and does not correspond to a bounding lower surface.

The solution for the potential temperature perturbation immediately after the heating (e.g., $t = 0^+$) can be found by integrating (4.1c) with (4.2) from $t = -\Delta t$ to $t = +\Delta t$ and taking the limit as Δt tends to zero. We find

$$g\frac{\theta'}{\theta_s}(t = 0^+) = \frac{\Delta p}{\gamma\rho_s H_s}[H(z+a) - H(z-a)]. \tag{4.3}$$

A diagnostic equation for the perturbation pressure is found by multiplying the momentum equation (4.1a) by ρ_s, taking the derivative of the result with respect to height, and using continuity (4.1b) to eliminate the time derivative term. The result is

$$\frac{\partial}{\partial z}\left[\rho_s\frac{\partial}{\partial z}\left(\frac{p'}{\rho_s}\right)\right] = \frac{\partial}{\partial z}\left(\rho_s g\frac{\theta'}{\theta_s}\right), \tag{4.4}$$

or, using (4.3),

$$\frac{d^2p'}{dz^2} + \frac{1}{H_s}\frac{dp'}{dz} = \frac{\Delta p}{\gamma H_s}[\delta(z+a) - \delta(z-a)]. \tag{4.5}$$

Equation (4.5) is identical to the equation for the final pressure perturbation in the fully compressible case found from a statement for the conservation of potential vorticity [see Bannon (1995) and his (4.7)]. The solution is

$$p'(t = 0^+) = \frac{\Delta p}{\gamma}\begin{cases} 2\sinh\left(\dfrac{a}{H_s}\right)e^{-z/H_s}, & z \geqslant +a, \\ 1 - e^{-(z+a)/H_s}, & -a \leqslant z \leqslant +a, \\ 0, & z \leqslant -a, \end{cases} \tag{4.6}$$

and, using (2.17d) and (2.17e),

$$\rho'(t = 0^+) = \frac{\Delta\rho}{\gamma}\begin{cases} 2\sinh\left(\dfrac{a}{H_s}\right)e^{-z/H_s}, & z \geqslant +a, \\ -e^{-(z+a)/H_s}, & -a \leqslant z \leqslant +a, \\ 0, & z \leqslant -a, \end{cases} \tag{4.7}$$

$$T'(t = 0^+) = \frac{\Delta T}{\gamma}e^{+z/H_s}\begin{cases} 0, & z \geqslant +a, \\ 1, & -a \leqslant z \leqslant +a, \\ 0, & z \leqslant -a, \end{cases} \tag{4.8}$$

where $\Delta\rho = \Delta p/gH_s$ and $\Delta T = T_o(\Delta p/p_{s(z=0)})$. Inspection of (4.6)–(4.8) indicates that the characteristic depth scale of the final state is the scale height H_s.

These anelastic results are now compared to the fully compressible results of Bannon (1995), whose equations are denoted with a prefix B. Equations (4.6)–(4.8) are identical to B(4.8)–B(4.10), respectively. Thus, the anelastic set predicts an instantaneous adjustment of the pressure, density, and temperature fields to a structure identical to that achieved in the final state in the compressible case. However, the potential temperature perturbation (4.3) differs with B(4.11). Equation (4.3) indicates that the anelastic potential temperature perturbation is identical to B(2.12) for the potential temperature perturbation immediately following the heating but not B(4.11) for the final adjusted perturbation. This result implies that the anelastic adjustment to (4.2) occurs with no vertical displacements of the fluid. Consistent with (2.10), the pressure, density, and potential temperature perturbations are in hydrostatic balance in the anelastic sense that $d(p'/\rho_s)/dz = g\theta'/\theta_s$, as well as in the compressible sense that $dp'/dz = -\rho' g$. Since there is a force balance, the adjusted anelastic atmosphere remains at rest and the vertical motion vanishes.

The energetics are described by

$$\frac{\partial}{\partial t}\left[\frac{\rho_s w^2}{2} + \frac{\rho_s}{2N_s^2}\left(g\frac{\theta'}{\theta_s}\right)^2\right] = -\frac{\partial(p'w')}{\partial z} + \frac{\rho_s}{N_s^2}\left(g\frac{\theta'}{\theta_s}\right)B_s. \quad (4.9)$$

After the heating, all the energy is available potential energy with the value

$$\mathrm{APE} \equiv \int \frac{\rho_s}{2N_s^2}\left(g\frac{\theta'}{\theta_s}\right)^2 dz = 2E_0\sinh(a/H_s), \quad (4.10)$$

where $E_0 = (\Delta p)^2/[2\kappa\gamma^2 g\rho_s(z=0)]$. This amount is identical to that for the total energy in the final state of the compressible case, B(5.11), though the compressible energy is composed of both potential and elastic energies. It is also identical to the initial APE in the compressible case. The result (4.10) is consistent with the anelastic generation, G,

$$G \equiv \int_0^{0^+}\int \frac{\rho_s}{N_s^2}\left(g\frac{\theta'}{\theta_s}\right)B_s\,dz\,dt = 2E_0\sinh(a/H_s), \quad (4.11)$$

provided the principal value for θ' [i.e., half of (4.3)] is used in evaluating the integral in time. Thus, the anelastic set conserves energy and all the energy generated by the heating goes into available potential energy with none converted to kinetic energy. The energy generated is a fraction $\gamma^{-1} \sim 71\%$ of the energy generated by the heating in the compressible case. The difference in energy is "lost" to the acoustic waves in the compressible case. In the anelastic case, the energy that would have gone to the acoustic modes is filtered out at the initial time and there is no excess generation.

5. Comparison with other theories

a. Original anelastic theory

The original (Batchelor 1953; Ogura and Phillips 1962) form of the anelastic equations uses an isentropic base state such that $\theta_s = \theta_a =$ constant. Then H_θ is infinite and assumption (2.7) is satisfied automatically. In this special case (2.3a) and (2.11) are equivalent and a consistent potential vorticity equation, (3.8), exists.

The shortcoming of this unrealistic base state has been circumvented in practical applications (e.g., Clark and Peltier 1977; Nance and Durran 1994) by incorporating the features of a more realistic base state into the perturbation fields. Formally, one sets

$$\theta = \theta_a + \theta' = \theta_a + \tilde{\theta}(z) + \theta''(x, y, z, t), \quad (5.1)$$

etc., where the tilde variables describe, say, the departures of a representative sounding from the isentropic base state. Potential vorticity correctness is unaffected by the partitioning (5.1).

We apply this approach to the hydrostatic adjustment problem and let the tilde fields describe an isothermal atmosphere. Then the set (4.1) is replaced by

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{p''}{\rho_a}\right) + g\frac{\theta''}{\theta_a}, \quad (5.2a)$$

$$\frac{\partial}{\partial z}(\rho_a w) = 0, \quad (5.2b)$$

$$\frac{\partial}{\partial t}\left(g\frac{\theta''}{\theta_a}\right) + w\frac{g}{\theta_a}\frac{d\tilde{\theta}}{dz} = \frac{gQ}{\rho_a C_p T_a}, \quad (5.2c)$$

where the base state variables, now referenced by a subscript a, are

$$p_a(z) = p_{00}(1 - z/H_a)^{1/\kappa},$$

$$\rho_a(z) = \frac{p_{00}}{R\theta_a}(1 - z/H_a)^{(1-\kappa)/\kappa},$$

$$T_a(z) = \theta_a(1 - z/H_a),$$

with $H_a = C_p\theta_a/g$.

The solution for the potential temperature perturbation is

$$\frac{g\theta''}{\theta_a} = \frac{g\Delta p}{\gamma\rho_a R T_a}[H(z+a) - H(z-a)], \quad (5.3)$$

and the diagnostic pressure equation associated with (5.2) is

$$\frac{d^2p''}{dz^2} + \frac{d}{dz}\left(\frac{p''}{\gamma R T_a(z)/g}\right) = \frac{d}{dz}\left(\rho_a g\frac{\theta''}{\theta_a}\right). \quad (5.4)$$

We have not solved (5.4), but the difference with the anelastic case (4.5) is revealed by neglecting the variation of T_a in the limit of small z. Then the solution is (4.6) with the coefficient γ^{-1} set to unity and with $H_s = RT_0/g$ replaced with $\gamma R\theta_a/g$. This result suggests that the practical use of (5.1) with the original anelastic theory overestimates the amplitude and vertical scale of the hydrostatic adjustment. Note that the tilde variables have no effect on (5.3) or the solution to (5.4).

b. *Theory for deep convection*

Dutton and Fichtl (1969) employed assumptions (i), (ii), and (2.9) to obtain a soundproof set of equations for deep convection consisting of (2.10), (2.17a,b,e), and (2.23). Use of the latter equation in place of (2.17c) leads to a potential vorticity equation in the form

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}_a\cdot\nabla\lambda}{\rho_s}\right) = \frac{\boldsymbol{\omega}_a}{\rho_s}\cdot\nabla\left(\frac{\dot{\Theta}_s}{\theta_s}\right) + \frac{\nabla\lambda}{\rho_s}\cdot\left(\nabla\times\frac{\mathbf{F}}{\rho_s}\right), \quad (5.5)$$

where $\lambda = \ln\theta_s + \theta'/\theta_s$ is the linearized form of the function $\ln\theta$ consistent with assumption (i). Thus, (5.5) is consistent with the alternative form of Ertel's theorem given by

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}_a\cdot\nabla\ln\theta}{\rho}\right) = \frac{\boldsymbol{\omega}_a}{\rho}\cdot\nabla\left(\frac{\dot{\Theta}}{\theta}\right) + \frac{\nabla\ln\theta}{\rho}\cdot\left(\nabla\times\frac{\mathbf{F}}{\rho}\right). \quad (5.6)$$

Application of the deep equations of motion to Lamb's problem yields a set identical to (4.1) for the anelastic set since the linearized form of the heat equation (2.23) and (2.17c) are the same. Moreover, since Lamb's problem assumes an isothermal base state, relation (2.10) is consistent with (2.15). Thus, the solution of Lamb's problem using the deep convection equations is identical to the anelastic solution presented in section 4.

c. *Modified anelastic theory*

The early (Gough 1969; Wilhelmson and Ogura 1972) modification of the anelastic system employs the set (2.3). The momentum equation is (2.3a) or

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{u} = -C_p\theta_s\nabla\pi' + \frac{g\theta'}{\theta_s}\mathbf{k} + \frac{\mathbf{F}}{\rho_s}, \quad (5.7)$$

where π' is the perturbation of the Exner pressure function $\pi = (p/p_{00})^{R/C_p}$ and is related to the perturbation pressure by $p' = \rho_s C_p\theta_s\pi'$. The modified theories invoke only assumptions (i) and (ii). As a consequence, a closed form to the energy equation is impossible, as is a consistent statement of potential vorticity conservation. The latter can be seen as follows. Use of (5.7) leads to a solenoidal term on the right-hand side of the vorticity equation (3.4) of the form

$$+\frac{1}{\rho_s^2}\nabla\rho_s\times\nabla p' = C_p\nabla\pi'\times\nabla\theta_s. \quad (5.8)$$

As a consequence, an extra term in the potential vorticity equation (3.8),

$$+\frac{\nabla\theta\cdot(\nabla\rho_s\times\nabla p')}{\rho_s^3} = \frac{C_p\nabla\theta\cdot(\nabla\pi'\times\nabla\theta_s)}{\rho_s}, \quad (5.9)$$

arises and, in general, does not vanish. Note, however, that (5.8) and (5.9) vanish for an isentropic base state since $\nabla\theta_s = \nabla\theta_a = 0$.

Application of the modified anelastic theory to Lamb's problem is straightforward. The potential temperature perturbation is again given by (4.3) but the diagnostic pressure equation is, for an isothermal base atmosphere,

$$\frac{d^2p'}{dz^2} + \frac{1}{\gamma H_s}\frac{dp'}{dz} = \frac{\Delta p}{\gamma H_s}[\delta(z+a) - \delta(z-a)]. \quad (5.10)$$

The solution [see (5.16) below] is again (4.6) but with the coefficient γ^{-1} set to unity and with H_s replaced by $\gamma H_s = (7/5)H_s$. Thus, the amplitude and the vertical scale of the pressure adjustment is slightly overestimated. These discrepancies with the anelastic solution are consistent with the approximation (2.18).

d. *Pseudo-incompressible theory*

Durran (1989) advanced a soundproof set of equations given by (2.1) with the sole assumption that the density ρ in the continuity equation is replaced by the pseudo-incompressible density ρ^*. Thus, (2.1b) becomes

$$\frac{\partial\rho^*}{\partial t} + \nabla\cdot(\rho^* u) = 0, \quad (5.11a)$$

where

$$\rho^* = \rho_s\left(\frac{\theta_s}{\theta}\right). \quad (5.11b)$$

No other simplifications are made. In particular, the pressure gradient and frictional terms in the momentum equation (2.1a) have not been approximated. Durran (1989) discusses the energy conservation of this system.

The potential vorticity equation for the system is

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}_a\cdot\nabla\theta}{\rho^*}\right) = \frac{\boldsymbol{\omega}_a\cdot\nabla\dot{\Theta}}{\rho^*} + \frac{\nabla\theta}{\rho^*}\cdot\left(\nabla\times\frac{\mathbf{F}}{\rho}\right). \quad (5.12)$$

Thus, the pseudo-incompressible potential vorticity

$$\mathrm{PV}_{\text{pseudo-incompressible}} = \frac{\boldsymbol{\omega}_a\cdot\nabla\theta}{\rho^*} \quad (5.13)$$

is conserved for adiabatic inviscid flow.

The pseudo-incompressible form of the hydrostatic adjustment problem consists of the solution (4.3) for the potential temperature perturbation and the diagnostic pressure equation, which reduces to

$$\frac{d^2p'}{dz^2}+\frac{1}{H_s}\frac{dp'}{dz}+\frac{\kappa}{\gamma H_s^2}p'=\frac{d}{dz}\left(\rho_s g\frac{\theta'}{\theta_s}\right)+\frac{\kappa}{H_s}\left(\rho_s g\frac{\theta'}{\theta_s}\right)-\frac{\rho_s}{g}\frac{\partial B_s}{\partial t}\qquad(5.14)$$

for an isothermal atmosphere. The temporal derivative term can be eliminated if we evaluate the equation at a time just after the heating has occurred. We note that the homogeneous solution to (5.14) has the form

$$p'\sim e^{+rz},\qquad(5.15\text{a})$$

where

$$r=\frac{1}{2H_s}[-1\pm(1-4\kappa/\gamma)^{1/2}]=\frac{1}{H_s}\left(-\kappa,-\frac{1}{\gamma}\right).\qquad(5.15\text{b})$$

Then, the solution to (5.14) just after the heating is

$$p'(t=0^+)=\Delta p\begin{cases}2\sinh\left(\dfrac{a}{\gamma H_s}\right)e^{-z/\gamma H_s}, & z\geqslant +a,\\ 1-e^{-(z+a)/\gamma H_s}, & -a\leqslant z\leqslant +a,\\ 0, & z\leqslant -a,\end{cases}\qquad(5.16)$$

and is identical to that for the modified anelastic theory (see subsection c). The density and thermal perturbations are also identical and may be found from the linearized relations (2.3d) and (2.3e) to be

$$\rho'(t=0^+)=\frac{\Delta\rho}{\gamma}\begin{cases}2\sinh\left(\dfrac{a}{\gamma H_s}\right)e^{-z/\gamma H_s}, & z\geqslant +a,\\ -e^{-(z+a)/\gamma H_s}, & -a\leqslant z\leqslant +a,\\ 0, & z\leqslant -a,\end{cases}\qquad(5.17)$$

$$T'(t=0^+)=\Delta T e^{+z/H_s}\times\begin{cases}2\kappa\sinh\left(\dfrac{a}{\gamma H_s}\right)e^{-z/\gamma H_s}, & z\geqslant +a,\\ 1-\kappa e^{-(z+a)/\gamma H_s}, & -a\leqslant z\leqslant +a,\\ 0, & z\leqslant -a,\end{cases}\qquad(5.18)$$

respectively. Comparison of (5.16)–(5.18) with (4.6)–(4.8) indicates that the pseudo-incompressible system overestimates the vertical scale by a factor of γ. The amplitude of the pressure and temperature perturbations are overestimated by a factor of γ and the temperature perturbation contains additional terms of order κ. These discrepancies with the anelastic solution are consistent with the approximation (2.18). Solutions for the parameter setting a/H_s are compared in Fig. 1. A major qualitative difference is that the modified and pseudo-incompressible systems predict a temperature perturbation that grows exponentially with height for $z>a$.

e. Boussinesq theory

The Boussinesq equations (e.g., Chandrasekhar 1961; Speigel and Veronis 1960) can be obtained from the anelastic set with the additional assumption that:

(iv) the depth scale, D, of the flow is small compared to the density and pressure scale heights,

$$D\ll(H_\rho,H_p).\qquad(5.19)$$

For this parameter regime, undifferentiated base state variables may be replaced with constants, denoted with an asterisk subscript. Then one finds that the anelastic equations reduce to the following Boussinesq set

$$\frac{D\mathbf{u}}{Dt}+2\boldsymbol{\Omega}\times\mathbf{u}=-\frac{1}{\rho_*}\nabla p+\frac{g\theta'}{\theta_*}\mathbf{k}+\frac{\mathbf{F}}{\rho_*},\qquad(5.20\text{a})$$

$$\nabla\cdot\mathbf{u}=0,\qquad(5.20\text{b})$$

$$\frac{D}{Dt}\left(\frac{g\theta'}{\theta_*}\right)+w\frac{g}{\theta_*}\frac{d\theta_s}{dz}=\frac{gQ}{\rho_*C_pT_*}\equiv B_*,\qquad(5.20\text{c})$$

$$\frac{\theta'}{\theta_*}=\frac{T'}{T_*},\qquad(5.20\text{d})$$

$$\frac{\rho'}{\rho_*}=-\frac{T'}{T_*}.\qquad(5.20\text{e})$$

Furthermore, since $T_*=\theta_*$, $T'=\theta'$ and there is no distinction between the temperature and potential temperature perturbations. Note that (5.20) implies that the buoyancy force is related to the potential temperature perturbation by

$$-\rho' g=\rho_* g\frac{\theta'}{\theta_*}.$$

An analogous relation does not hold for the anelastic equations. It is well known that this set has consistent energy and potential vorticity equations.

The Boussinesq form of the hydrostatic adjustment problem is

$$\frac{g\theta'}{\theta_*}=\frac{\Delta p}{\gamma\rho_* H_s}[H(z+a)-H(z-a)],\qquad(5.21)$$

and

$$\frac{d^2p'}{dz^2}=\frac{d}{dz}\left(\rho_* g\frac{\theta'}{\theta_*}\right).\qquad(5.22)$$

The solution for the pressure perturbation

$$p'=\frac{\Delta p}{\gamma}\begin{cases}2a/H_s, & z\geqslant +a,\\ (z+a)/H_s, & -a\leqslant z\leqslant +a,\\ 0, & z\leqslant -a,\end{cases}\qquad(5.23)$$

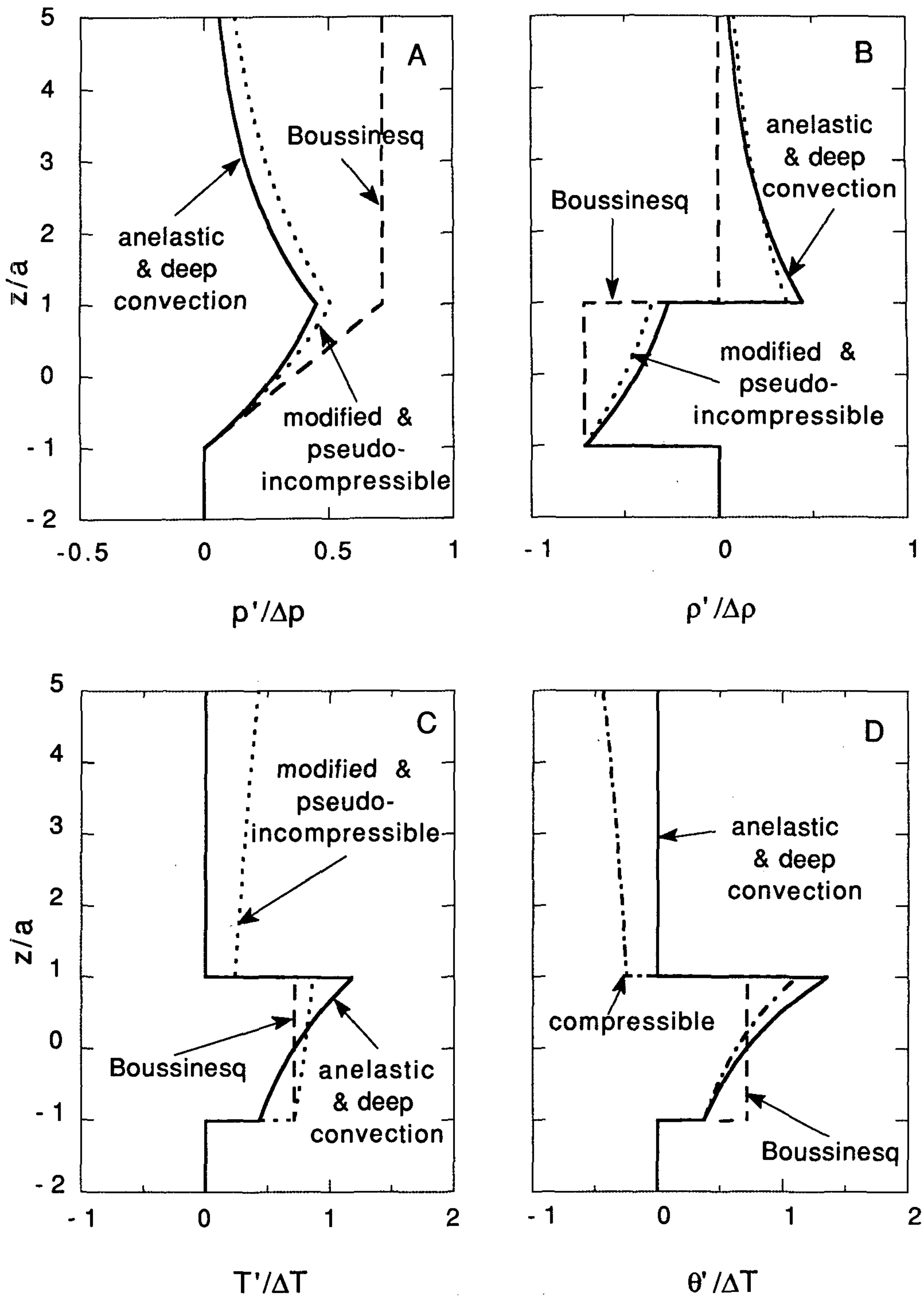

FIG. 1. Vertical profiles of the solutions to Lamb's problem for (a) perturbation pressure p', (b) perturbation density ρ', (c) perturbation temperature T', and (d) perturbation potential temperature θ' for $a/H_s = 1/2$. Panels (a)–(c) depict the anelastic and deep convection (solid), the modified and pseudoincompressible (dotted), and the Boussinesq (dashed) solutions. Panel (d) depicts the anelastic (solid), the steady-state compressible (dot–dashed), and the Boussinesq (dashed) solutions. The anelastic and deep convection solutions in (a)–(c) are identical to the final perturbations in the compressible case; in (d) it is identical to the initial perturbation in the compressible case. The solution for θ' is the same for the anelastic, deep convection, modified, and pseudoincompressible systems. The solution for the standard quasigeostrophic system agrees with that for the anelastic system; the solution for the modified quasigeostrophic system agrees with the final solution in the compressible case.

represents the limit of the compressible and anelastic result (4.6) in the limit $z/H_s \ll 1$. This result is therefore consistent with assumption (iv). A plot (see Fig. 1) of the Boussinesq solution for $a/H_s = 1/2$ reveals significant differences with the anelastic solution when (iv) holds only weakly.

f. Quasigeostrophic theory

The quasigeostrophic potential vorticity equation is

$$\frac{D}{Dt_g}\left[\nabla^2\psi + \beta y + \frac{f_0^2}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N_s^2}\frac{\partial\psi}{\partial z}\right)\right] = \frac{f_0}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N_s^2}B_s\right), \quad (5.24)$$

where $\psi = p'/(\rho_s f_0)$ is the geostrophic streamfunction and the constants f_0 and β denote the Coriolis parameter and its meridional variation. Equation (5.24) holds for both the standard (Pedlosky 1982) and the modified (White 1977) formulations. The subscript g on the material derivative indicates that only the geostrophic wind contributes to the convective derivative.

Equation (5.24) with the forcing (4.2) of the adjustment problem reduces to

$$\frac{\partial}{\partial t}\left[\frac{f_0}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N_s^2}\frac{\partial\psi}{\partial z}\right)\right] = \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{N_s^2}B_s\right). \quad (5.25)$$

Integrating with respect to time and specializing to an isothermal base atmosphere yields (4.5) for the perturbation pressure. The density and thermal perturbations may be diagnosed from the pressure perturbation using the quasigeostrophic relations

$$g\frac{T'}{T_s} = \frac{\partial}{\partial z}\left(\frac{p'}{\rho_s}\right) - \frac{1}{T_s}\frac{\partial T_s}{\partial z}\left(\frac{p'}{\rho_s}\right), \quad (5.26a)$$

$$g\frac{\theta'}{\theta_s} = \frac{\partial}{\partial z}\left(\frac{p'}{\rho_s}\right) - \frac{1}{\theta_s}\frac{\partial \theta_s}{\partial z}\left(\frac{p'}{\rho_s}\right), \quad (5.26b)$$

$$g\frac{\rho'}{\rho_s} = \frac{-1}{\rho_s}\frac{\partial p'}{\partial z}. \quad (5.26c)$$

The last terms in (5.26a) and (5.26b) are retained in the modified theory but are dropped in the standard theory. This simplification is consistent with assumption (2.16). Note that adding (5.26a) and (5.26c) yields (2.17e) for either the standard or modified theories. Adding (5.26b) and (5.26c) yields (2.10) in the standard theory but (2.3d) in the modified theory.

It is readily shown that the solution for the standard theory agrees with the anelastic results of section 4. The solutions for the modified theory also agree except that the predicted potential temperature perturbation is that for the compressible theory of Bannon (1995). Thus, the standard quasigeostrophic theory yields the anelastic result while the modified theory yields the fully compressible solution. In each quasigeostrophic theory, the adjustment is instantaneous.

6. Conclusions

This study has presented an examination of the anelastic equations of motion. The anelastic approximation assumes that (i) the dynamical variations of the thermodynamical state variables are small, (ii) the dynamical variation of the density field may be ignored in the momentum and continuity equations except for the buoyancy force, and (iii) the temperature and potential temperature scale heights are large compared to either the density or pressure scale heights. The resulting anelastic set (2.17) is identical to that of Lipps and Hemler (1982). Assumption (iii) also justifies the equivalency of the temperature and the potential temperature perturbations given by (2.17d) and the equality of the pressure and density scale heights. Alternatively assumption (iii) is equivalent to assuming that the ratio of the ideal gas constant to the specific heat capacity at constant pressure is small.

It has been shown that this anelastic set yields a potential vorticity equation that is consistent with Ertel's theorem. The other versions of the soundproof equations are also PV correct except for the modified equations of Gough (1969) and Wilhelmson and Ogura (1972). This failure is traced to the vorticity equation where the modified anelastic equations incorrectly retain a solenoidal term. This finding is consistent with that of White (1977) for quasigeostrophic theory.

The anelastic equations have been used to solve Lamb's hydrostatic adjustment problem where a horizontal layer of atmosphere is instantaneously heated. Though idealized, this problem is a good test of the anelastic equations since it retains several terms that separate the various theories. These terms include the pressure gradient force and the buoyancy term in the vertical momentum equation as well as the vertical divergence of mass flux in the continuity equation.

In all the anelastic theories, the adjustment is instantaneous since the acoustic modes have been filtered from the equations and the sound speed is infinite. The pressure perturbation adjusts to be in hydrostatic balance with the potential temperature perturbation, whose structure is dictated by that of the heating, and there are no vertical displacements of the fluid. This result differs from that of the compressible case where vertical displacements occur as the pressure, density, and thermal fields mutually adjust over a finite time. Only the anelastic theories of Lipps and Hemler (1982) and Dutton and Fichtl (1969) and the quasigeostrophic theories predict a steady-state solution for the pressure, density, and temperature in agreement with the exact compressible solution (Bannon 1995). The modified quasigeostrophic theory also predicts the correct steady-state solution for the potential temperature.

Thus, the total energy in the final state in the anelastic and compressible cases is the same. The energy generated by the heating and lost to the acoustic waves in the compressible case is filtered from the anelastic system. The prediction of the Boussinesq equations agrees with the exact solution in the limit of shallow heating (i.e., in the limit as the depth of the heating is much less than a density scale height).

These findings, together with the analysis of Lipps (1990) regarding the energetics and wave spectrum of small amplitude perturbations, highlight the advantages of using the anelastic equations (2.17).

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