The Nonlinear Evolution of Idealized, Unforced, Conditional Symmetric Instability: A Numerical Study

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ABSTRACT

A two-dimensional version of the Penn State–NCAR mesoscale model (MM4) has been used to simulate the life cycle of conditional symmetric instability (CSI) under conditions of no deformational or planetary boundary layer forcing with the model starting from idealized initial conditions. Detailed diagnostics from the growth, decay, and post-CSI stages of the life cycle are presented, and some of these features are compared to expectations from linear theory.

The life cycle features include local areas of potential and inertial instability and specific patterns of ageostrophic zonal flow. Local areas of increased and decreased dry potential vorticity (q), including areas of negative q, develop from the initially everywhere-positive q field, principally because of the horizontally differential diabatic heating. Negative wet-bulb potential vorticity (q_w) is principally advected into the upper troposphere by the CSI updraft, though some changes in q_w do occur because of the diffusion of temperature. Model-output soundings along surfaces of constant absolute momentum (m) show that lower-tropospheric thermodynamic stabilization and a decrease in slantwise convective available potential energy occur during the simulation. Net changes produced by the CSI circulations include low-level frontogenesis, upper-level frontolysis, and local buoyant and inertial stabilization—destabilization.

The modeled updraft slope is between that of the surfaces of constant wet-bulb potential temperature (theta_u) and that of the surfaces of constant m, since the viscosity and finite grid spacing yield an unstable mode with a finite updraft width. Such a mode differs from the inviscid mode, which has an infinitely narrow updraft width and a slope along the theta_u surfaces. The cessation of the CSI is not due to the removal of the area of negative moist potential vorticity. Instead, linear stability analysis suggests that the cessation is due to the stabilization of modes with resolvable updraft widths and, possibly, to the depletion of the water vapor supply.

Idealized studies such as these do not attempt to achieve absolute realism but are necessary steps in the methodical process of linking simple theoretical treatment of CSI with the complex observations; they may be useful as aids in interpreting observational data or numerical model simulations of real-atmosphere cases.

1. Introduction

In recent years, conditional symmetric instability (CSI) has been advanced as a probable mechanism for producing banded precipitation structures in extratropical cyclones (e.g., Hoskins 1974; Bennetts and Hoskins 1979; Wolfsberg et al. 1986; Xu 1986; Reuter and Yau 1990), for producing prefrontal squall lines (Sun 1984), and for symmetrically neutralizing regions of broad cloud shields of midlatitude extratropical cyclones (Emanuel 1988). Theoretical considerations indicate that CSI is a purely mesoscale instability (Emanuel 1979), with many characteristics that agree with the observations of some types of rainbands (Parsons and Hobbs 1983; Bennetts and Ryder 1984). Aircraft soundings along surfaces of constant momentum, m (m = u − fy, where u is the zonal flow parallel to the thermal wind vector, f is the Coriolis parameter, and y is the meridional distance), have shown conditions of near neutrality to CSI (Emanuel 1988). This near neutrality to CSI may have resulted from the stabilization of CSI, though a recent study of dry symmetric instability (DSI) (Holt and Thorpe 1991) suggests the possibility that such neutral (or slightly unstable) regions may also result from the continuous local forcing of a weakly symmetrically stable atmosphere. Although some of the rainband types examined in the above studies may be caused by other mechanisms and detailed observations of the symmetric neutralization of the at-
mosphere have not been presented, it appears likely that CSI occurs in the atmosphere and may well have an impact on the distribution, if not the amount, of precipitation. It is even possible that CSI may be a ubiquitous adjustment process in saturated regions of the atmosphere.

Much of our knowledge of CSI has been obtained from linear-theory studies. The frequently used necessary criterion for CSI,

\[ q_w < 0, \]

where \( q_w \) is the wet-bulb potential vorticity, has been derived for inviscid CSI in which an infinitesimal updraft width is possible (e.g., Bennetts and Hoskins 1979, hereafter referred to as BH). In addition, this criterion assumes an atmosphere in thermal wind balance, as emphasized by Xu (1992). However, for viscous CSI or CSI with a finite updraft width, a sufficient, and probably necessary, instability criterion has the form

\[ q_w < q_{wc} < 0, \]

where \( q_{wc} \) is a critical value of \( q_w \) (BH; Xu 1986, hereafter referred to as Xu). Both BH and Xu show that the value of \( q_{wc} \) is related to the width and slope of the updraft and to the environmental parameters [e.g., moist and dry buoyancy frequencies, dry potential vorticity (\( q \)), Richardson number (\( \text{Ri} \)], though this relationship is complex and differs slightly between the two theoretical treatments. Hence, (2) is likely to be a more applicable necessary criterion for CSI than (1) when trying to assess the likelihood of CSI in the real atmosphere where viscosity is present and, particularly, in a finite-grid model where the updraft width is forced to be finite.

CSI studies using linear theory also show that the most unstable mode (the mode with the largest growth rate) for inviscid CSI has an infinitesimally small updraft width and an updraft slope equal to that of the moist isentropes (e.g., wet-bulb potential temperature, \( \theta_w \), surfaces) (Emanuel 1983; Xu). However, for inviscid modes with finite updraft widths, the growth rates are smaller than for the most unstable mode, and the updraft slope angles are therefore smaller since the updraft slope is determined by the environmental conditions and the inviscid growth rate (Xu). For a Prandtl number (\( Pr \)) of unity, the growth rate of the most unstable viscous mode is substantially smaller than that of the most unstable inviscid mode, partly because of the direct effect of viscosity on that particular viscous mode, but mostly because of the viscous stabilization of the faster-growing modes with narrower updrafts. The largest decrease in growth rates due to the inclusion of viscosity occurs for the modes with the narrowest updrafts, thereby allowing a mode with a finite updraft width to become the most unstable mode. The shallower updraft slopes of the most unstable viscous modes are a direct consequence of the finite width of its updraft and the subsequent reduction in growth rate. Xu's relations show that the reduction in growth rate directly caused by the viscosity does not change the updraft slope of a particular mode with a particular updraft width. Hence, for the hydrostatic, moderately baroclinic conditions of interest in this study, Xu's results indicate that the inclusion of viscosity is important in determining the growth rate and the updraft slope of the most unstable mode. However, the magnitude of the viscosity is not as important as the environmental conditions (e.g., stability, baroclinicity) in determining the growth rate and updraft slope of the most unstable viscous mode. This finding is in general agreement with Emanuel (1979) for DSI. Holt and Thorpe (1991) also found the kinetic energy of DSI in a numerical model to be less than that expected from classical parcel theory (which would be equivalent to the most unstable inviscid mode with an infinitely narrow updraft) but properly predicted by a parcel theory that accounted for the energy losses to the environment. For \( Pr \neq 1 \), viscous CSI may have a greater growth rate than inviscid CSI (McIntyre 1970).

By running numerical models on idealized conditions, attempts have been made to bridge the gap between the rainband observations and the theoretical treatments of CSI (e.g., BH; Saitoh and Tanaka 1987, 1988, henceforth ST; Knight and Hobbs 1988; Innocentini and Caetano 1992, henceforth IC92) and DSI (Thorpe and Rotunno 1989; Ducroq 1993). Unfortunately, this gap is complicated by the presence of likely interactions between CSI and large-scale forcing or smaller-scale processes, such as microphysical processes, which generally need to be included to produce rainbands with reasonably realistic attributes (e.g., Knight and Hobbs 1988; Xu 1992), by the possible quasi equilibrium between slantwise convection and the forcing, and by the difficulty of differentiating between CSI occurring in the presence of large-scale forcing and the enhancement of the large-scale forcing itself by conditions of moist symmetric near-neutrality (e.g., Emanuel 1985; Thorpe and Emanuel 1985; Emanuel et al. 1987; Xu 1989). Additional difficulties arise because of significant impacts of choices of model resolution and diffusion (e.g., Persson and Warner 1991, 1993; Ducroq 1993). Because the ultimate goal of idealized simulations of CSI is to provide an understanding of “atmospheric” structures that may be useful as guides or conceptual models in interpreting observational data or numerical modeling simulations of real-atmosphere cases, the presence of all these complications necessitates the adherence to a methodical progression from (to) theory to (from) the realistic simulations, with detailed diagnostics being important at each step.

This numerical modeling study attempts to take one such small step from the linear theory toward the more realistic simulations by examining in detail the entire
life cycle (growth, decay, post-CSI stages) of a simulation of CSI and the associated structures in frequently used parameters and by carefully comparing characteristics of the simulated CSI to the linear theory. In conjunction with previously published sensitivity tests (Persson and Warner 1993), it should provide a clean basis for comparisons to more realistic simulations. No claim is made that the simulations presented here are entirely realistic, as this study is closer to the theoretical treatments than to the observations. Interactions with larger-scale deformational forcing, forcing from the planetary boundary layer, and smaller-scale microphysical effects are absent; hence, the term unforced CSI is used. The emphasis will be on the detailed diagnostic analysis of nonlinear CSI and the quantitative comparison to linear theory rather than on realism. Only qualitative comparisons to general characteristics observed in frontal regions in extratropical cyclones are justified.

Similarities between this study and others do exist. Saitoh and Tanaka (1987, 1988) and IC92 also simulated the evolution of CSI in an atmosphere that is initially locally unstable to only CSI, though only IC92 and the present study restricted the initial instability to a limited area. However, apparently because of the specific choice of initial conditions and the inclusion of microphysical effects, IC92 was only able to obtain the growth stage of the CSI life cycle, as was the case for ST. Both ST and IC92 examine the effects of the water vapor supply on the CSI growth stage and discuss some growth-stage features. Differences in the present study therefore include 1) additional diagnostics of dynamically significant parameters (e.g., $q_w$, $q_d$, $u_x$) at all stages of the life cycle, 2) changes produced by the CSI circulation at the end of the life cycle rather than at the end of the growth stage, 3) a quantitative comparison of various simulation characteristics with linear theory, and 4) the use of variable diffusion coefficients, which has significant impacts on the nonlinear evolution of CSI (Persson and Warner 1993) and DSI (Ducrocq 1993). In the present study, the interpretation of some growth-stage features noted in other studies uses additional diagnostics to relate them to observational methods and to linear theory. The simulation of ST also had highly inconsistent vertical and horizontal resolutions (Persson and Warner 1991), which may have had a significant impact on their results.

This paper examines the results of one CSI simulation, CTRL, in detail. Although the initial conditions and the model configuration used are quite simple, fairly complex structures do develop. The results of a few other simulations are also mentioned when relevant, and a paper discussing sensitivity studies of model-simulated CSI (Persson and Warner 1993) provides results of even more simulations. Descriptions of the model and the analysis procedures are presented in section 2, and section 3 describes the initial conditions. The evolution of the CSI simulation is presented in section 4, where emphasis will be placed on the evolution of fields generally discussed in linear theory (e.g., $m$, $q$, and $q_w$), as well as the changes in the momentum and thermodynamic structure during the different stages of the CSI evolution. In section 5, a comparison is made between characteristics expected from a modified version of the linear theory of Xu and those actually present in the nonlinear simulation. This comparison also includes an assessment of the linear stability of the final state. A summary and discussion are given in section 6. The modification of Xu's linear theory is described in the appendix.

2. Model and analysis methods

a. Model description

The model is a two-dimensional version of the Penn State–NCAR mesoscale model (MM4) that uses the hydrostatic, viscous primitive equations. The equations are formulated in flux form to conserve energy. The vertical coordinate is $\sigma = (p - p_\ell)/p^*$, where $p^* = (p - p_\ell)$. $p$ is the pressure, $p_\ell = 150$ mb is the pressure at the top of the model, and $p_\ell$ is the surface pressure. The two-dimensional model equations of motion, the continuity equation, the thermodynamic equation, and the hydrostatic equation are obtained from Anthes et al. (1987) by setting $\partial f/\partial x$ equal to 0. The Coriolis parameter, $f$, is calculated from the latitude, which varies from $41.2^\circ$ to $53.4^\circ$ across the south–north-oriented domain. The vertical velocity, $\dot{\sigma}$, is diagnosed by integrating the two-dimensional continuity equation, and the pressure-coordinate vertical velocity, $\omega$, is defined by

$$\omega = \frac{dp}{dt} = p^*\dot{\sigma} + \sigma(dp^*/dt),$$

(3)

where $dp^*/dt = \partial p^*/\partial t + s_m\partial p^*/\partial y$. The budget equation for the water vapor mixing ratio, $q_v$, is

$$\frac{\partial p^*q_v}{\partial t} = s_m^2 \frac{\partial p^*q_v/s_m}{\partial y} - \frac{\partial p^*q_v}{\partial \sigma}$$

$$- p^*P_{con} + F_Hq_v + F_Vq_v,$$

(4)

where $P_{con}$ represents the condensation of water vapor. The map-scale factor, $s_m$, has only a very minor influence on the simulation since its value ranges from 1.026 to 1.035. The addition of the latent heat of condensation is included in the thermodynamic equation, though evaporation, melting, and freezing effects are not included. No parameterization of subgrid-scale convection or precipitation is used.

At the lateral boundaries, all variables are fixed, except that $p^*u$ and $p^*v$ are extrapolated on outflow, and the commonly used five-point ’’sponge’’ zone (e.g., Perkey and Kreitzberg 1976) is used. However, tests show that the form of the lateral boundary conditions have little effect on the simulations since the circula-
tions are distant from the lateral boundaries. The subgrid-scale vertical flux divergences of momentum, heat, and moisture are zero in the lowest layer, and the subgrid-scale vertical fluxes of these quantities are zero at the top of the domain. The lower boundary conditions are equivalent to the potential vorticity conserving free-slip, thermally insulated (FI) boundary conditions discussed by Thorpe and Rotunno (1989). If the no-slip, thermally conducting boundary conditions (NC) are used (i.e., imposing zero surface fluxes), a minor (~10%) weakening of the circulations and a 2-h shortening of the life cycle occur. The choice of the NC rather than the FI upper boundary conditions has no effect on the tropospheric circulations.

The horizontal and vertical friction term operators, $F_h$ and $F_v$, respectively, are defined by

$$F_h \alpha = \rho^* K_h \frac{\partial \alpha}{\partial y}$$

$$F_v \alpha = \rho^* \frac{\partial}{\partial z} \left( \rho K_v \frac{\partial \alpha}{\partial z} \right),$$

(5a, 5b)

where $\rho$ is the density and $\alpha = u, v, T, \text{or} q_e$. The fourth-order horizontal diffusion coefficient, $K_h$, is the sum of a background horizontal diffusion coefficient and a coefficient that is dependent on the horizontal deformation (Smagorinsky 1963). The vertical diffusion coefficient, $K_v$, is dependent on the local dry Richardson number, $R_i$, where the local value of the diffusion critical Richardson number, $R_{ic}$, is dependent on the local layer depth and ranges from 0.61 near the surface to 0.73 near 300 mb in CTRL. Here $R_i$ is calculated using both the acrossfront and alongfront vertical shears. Following Durrant and Klemp (1982), a moist-$R_i$ ($R_{i_m}$) criterion for diffusion with a smaller critical value ($R_{icr} = 0.33$) was tested, leading to slower development of the CSI evolution. Also note that the CSI criterion (1) can be rewritten to show instability for $R_{i_m}$ less than approximately 1, which implies the explicit representation of some CSI instabilities and a parameterization of those for which $R_{i_m} < R_{icr}$. As it is the more standard formulation, the simulation with the dry Ri criterion is presented here, reducing the link between the vertical diffusion parameterization and the CSI criterion. The Prandtl number used is 1. Further details of the model characteristics can be obtained from Anthes and Warner (1978), Anthes et al. (1987), and Persson and Warner (1993).

The two-dimensional domain is a vertical cross section oriented in a south–north direction. Uniformly spaced horizontal and vertical grid spacings of $\Delta y = 15$ km and $\Delta z = 0.02$ (50 layers), respectively, are used, which are consistent with the sloping thermal structures in the simulation (Persson and Warner 1991).

### b. Diagnostic relations

The diagnostic parameters used in this study include the Cartesian coordinate vertical velocity, $w$, calculated from the model output of $\omega$ by the approximation

$$w = -\frac{\omega}{g \rho},$$

(6)

which neglects the horizontal advection of pressure and the local time rate of change of pressure, and the geostrophic momentum field calculated from

$$m = u - f v,$$

(7)

where $u$ is the geostrophic zonal flow. Accounting for the two-dimensionality assumption, Ertel's dry potential vorticity expressed in sigma coordinates is calculated from

$$q = \frac{g}{\rho^*} \left[ -\frac{\partial u}{\partial \sigma} \frac{\partial \theta}{\partial y} + \left( \frac{\partial u}{\partial y} - f \frac{\partial \theta}{\partial \sigma} \right) \right],$$

(8)

where $\theta$ is the potential temperature and $g$ is the acceleration due to gravity. The rate of change of $q$ following a parcel is obtained by taking the substantive derivative of (8), substituting the equations of motion, continuity equation, and thermodynamic equation, and rearranging. This process leads to

$$\frac{D q}{D t} = -\frac{g}{\rho^*} \left\{ \mathbf{i} \cdot \left[ \nabla \times \nabla_{\sigma} \left( \frac{1}{\rho^*} F_h v + \frac{1}{\rho^*} F_v u \right) \right] + \mathbf{j} \cdot \frac{\partial}{\partial \sigma} \left( \mathbf{C_o} \mathbf{Q} \right) \right\},$$

(9)

where

$$\mathbf{C_o} = \left( \frac{1000}{\rho} \right) \mathbf{K_{o cr}},$$

and

$$\mathbf{Q} = \frac{\partial u}{\partial \sigma} \mathbf{j} + \left( f - \frac{\partial u}{\partial y} \right) \mathbf{k}.$$

The terms in (9) represent the generation of potential vorticity by momentum diffusion (friction, FRI), thermal diffusion (TDIFF), and differential latent heating (DLH).
For later use, it is convenient to express the last term in (9) as

\[
- \frac{g}{p^*} \nabla \cdot \left( \frac{C_v Q}{c_p} \right) = - \frac{g}{p^*} \left[ \frac{\partial u}{\partial \sigma} \frac{\partial}{\partial y} \left( \frac{C_v Q}{c_p} \right) \right]
\]

HDLH

\[
+ \left[ f - \frac{\partial u}{\partial y} \frac{\partial}{\partial \zeta} \left( \frac{C_v Q}{c_p} \right) \right] = \frac{1}{\rho} \left[ \frac{\partial u}{\partial \zeta} \frac{\partial}{\partial y} \left( \frac{C_v Q}{c_p} \right) \right]
\]

VDLH

\[
+ \left[ f - \frac{\partial u}{\partial y} \frac{\partial}{\partial \zeta} \left( \frac{C_v Q}{c_p} \right) \right],
\]

(10)

where the equality $\partial \sigma / \partial \zeta = - \rho g / p^*$ and the hydrostatic approximation have been used. The terms HDLH and VDLH represent the horizontally differential diabatic heating and the vertically differential diabatic heating, respectively.

The wet-bulb potential vorticity, $q_w$ (also referred to as moist potential vorticity), is calculated by replacing $\theta$ by $\theta_v$ in (8), where $\theta_v$ is calculated using an iterative technique based on the thermodynamic equations of Bolton (1980). The resulting equation is

\[
q_w = \frac{g}{p^*} \left[ - \frac{\partial u}{\partial \sigma} \frac{\partial \theta_w}{\partial y} \right]_\sigma
+ \left\{ \frac{\partial u}{\partial \zeta} \left( f - \frac{\partial}{\partial \sigma} \theta_v \right) \right\}.
\]

(11)

An equation for $Dq_w/ Dt$,

\[
\frac{Dq_w}{Dt} \approx \frac{g}{p^*} \left\{ 1 \left[ \nabla \cdot \nabla \left( \frac{C_v}{p^*} \frac{F_{\theta} \theta + F_{\nu} \nu}{C_{vw}} \right) \right] \right. \\
FRI

\[
+ \nabla \cdot \left( \frac{C_v}{p^*} \frac{F_{\theta} \theta + F_{\nu} \nu}{C_{vw}} \right) \right. \\
TDIFF

\[
\left. \times \left( \frac{C_v}{p^*} (F_{\theta} \theta + F_{\nu} \nu) \right) \right. \\
QVDIFF

\[
+ \nabla \cdot \left( \frac{C_v}{p^*} \frac{F_{\theta} \theta + F_{\nu} \nu}{C_{vw}} \right) \right. \\
QVDIFF

\[
\left. \times \left( \frac{2675}{T_L} (F_{\theta} q_w + F_{\nu} q_w) \right) \right\},
\]

(12)

The effects of momentum diffusion and temperature diffusion are again represented by the terms FRI and TDIFF, respectively. The exponential factors result from the approximate relationship between $\theta$ and equivalent potential temperature, $\theta_e$ [see Bolton 1980, Eq. (35)]. The temperature at the lifting condensation level, $T_L$, is determined from Bolton's Eq. (21), and the factor $C_{vw}$ approximately converts $Dq_w / Dt$ to $Dq_w^e / Dt$, where $q_w^e$ is a function of the local value of $\theta_v$ and the saturation mixing ratio, and $q_w$ is the equivalent potential vorticity. The relation for $C_{vw}$ was derived from Bolton's Eq. (40).

Note that $q$ and $q_w$ are calculated from the total winds and not the geostrophic ones so that they are pseudo-conserved quantities. This will limit the usefulness of $q$ and $q_w$ as predictors of instability to the times when geostrophy is valid; that is, to the initial time and at the end of the simulation.

Growth rates are calculated from the model output for the total perturbation circulation (GR) and its zonal (GRz) and transverse (GRv) components, where the perturbations (denoted by primes) are changes from the initial state. These growth rates are calculated using the relationships

\[
GR = \frac{1}{2} \frac{\Delta \ln \langle (u')^2 + (v')^2 \rangle}{\Delta t},
\]

(13a)

\[
GR_z = \frac{1}{2} \frac{\Delta \ln \langle (u')^2 \rangle}{\Delta t},
\]

(13b)

\[
GR_v = \frac{1}{2} \frac{\Delta \ln \langle (v')^2 \rangle}{\Delta t},
\]

(13c)

where $\Delta t = 1$ h and the angled brackets refer to tropospheric averages. The growth rates are related to the rate of change of perturbation kinetic energy.

### 3. Initial conditions

The initial conditions are not selected to match any particular observational study. The main physical bases are to 1) create initial conditions unstable only to CSI, 2) avoid having the entire domain unstable to CSI, and 3) use the most simple conditions possible. To satisfy these constraints, especially 1), is not trivial. Because we use a broad baroclinic zone rather than a front, to avoid regions of potential upright instability and negative dry potential vorticity, which are both present in observations (e.g., Thorpe and Clough 1991), the initial conditions are not as realistic as they might be but are rather chosen to facilitate comparisons to linear theory.

The initial conditions consist of a broad, moderately strong, baroclinic zone with a constant south–north potential temperature gradient of $\partial \theta / \partial y = -3.3$ K (100 km)$^{-1}$ (Fig. 1a). A nearly uniform, tropospheric, upright stability is achieved along the southern boundary by specifying $\partial \theta / \partial Z = 4.8$ K km$^{-1}$, where $Z$ is the
Fig. 1. Synthetic initial conditions for CTRL. Shown are south–north vertical cross sections of (a) potential temperature (solid; isopleth interval of 5 K) and $u$-component wind speed (dashed; isolach interval of 10 m s$^{-1}$), (b) wet-bulb potential vorticity (solid; isopleth interval of 0.1 pvu) and relative humidity (RH, dashed; isopleth interval of 15%), and (c) dry potential vorticity (solid; isopleth interval of 0.2 pvu) and wet-bulb potential temperature (dashed; isopleth interval of 2°C). In-plane wind vectors combining the initial values of $v$ and $\omega = dp/dt$ are also shown in (b) and (c), with the scale vectors shown at lower right (the vertical scale represents 10 mb s$^{-1}$ and the horizontal scale 10 m s$^{-1}$). The vertical wind vectors in these and later cross sections are scaled relative to the horizontal wind vectors so as to be consistent with the respective dimensions of the cross sections. The slopes of the vectors are then correct relative to the scales of the axes and to the isopleths within the diagrams. In (b), the heavy isopleth is $q_w = 0$, and RH $> 100\%$ is shaded.

pseudoelevation vertical coordinate (e.g., Hoskins and Bretherton 1972). Aloft, a stratosphere is defined in which the vertical temperature gradient is zero. To complete the specification of the temperature field, a temperature in the lowest layer along the southern boundary, $T_{1,XX} = 292$ K, is chosen. The relative humidity field is specified so that the central saturated region in the lower troposphere is about 120 km wide and 170 mb deep (Fig. 1b). Once the temperature and moisture fields are defined, the $u$-component of the wind is determined by assuming a thermal wind balance, giving a fairly strong vertical wind shear (Fig. 1a). The weak horizontal wind shear results from using virtual temperature in calculating the $u$-component of the wind.

By adjusting the humidity and temperature fields, conditions are found that have a large area of negative moist potential vorticity approximately one-half the size of the tropospheric part of the domain (Fig. 1b) but positive dry potential vorticity everywhere (Fig. 1c). The use of a fairly small saturated area further limits the area actually unstable to CSI. When defining a broad baroclinic zone like this, it is difficult to avoid creating an area of potential instability ($\partial \theta_w / \partial p > 0$). Such an area is present in a small region near the southern boundary in the lower troposphere of Fig. 1e. How-
ever, because the initial model atmosphere is substanta-
ially sub saturated in the vicinity of the potential in-
stability, no saturation ever occurs in this area during
the simulation. Therefore, the initial conditions are unsta-
bile to CSI in part of the domain but stable to DSI and
upright convection everywhere. The value of the wet-
bulb potential vorticity at the center of the perturbation
is \(-0.2\) potential vorticity units (pvu, where 1 pvu
\(= 1 \times 10^{-6} \text{ m}^2 \text{ s}^{-1} \text{ K kg}^{-1}\)). These initial conditions
appear to be reasonably realistic although substantially
smoothed. The initial area of negative \(q_a\) is fairly large,
but Thorpe and Clough (1991) show that areas of nega-
tive \(q_a\) can be 6–7 km deep and a few hundred kilo-
meters wide in the real atmosphere.

Moderately large values of the initial perturbation
amplitude \((v_{\text{max}} = 2.0 \text{ m s}^{-1}, w_{\text{max}} = 3.2 \text{ cm s}^{-1}; \text{Fig.}\
1b)\) are used to conserve computer resources. In a simu-
lation using an amplitude one order of magnitude
smaller, the “startup period” of the simulation is ex-
tended by about 10 h, though the subsequent evolution
is nearly identical to the evolution in CTRL.

The requirement of using initial conditions unstable
to only CSI produces strong limitations on the initial
conditions that can be used. If, for instance, slightly
warmer temperatures are used, initial conditions that
include substantial potential upright instability are cre-
ated; if a stronger baroclinic zone is used, substantial
areas unstable to dry symmetric instability will be in-
cluded. By restricting our initial conditions to be un-
stable to only CSI, we also severely limit the range of
possible sensitivity tests to the initial conditions. Some
such tests have been done, and some of these had to be
aborted early because upright convection developed,
which cannot be handled properly in a hydrostatic
model. This appears to have been a difficulty for IC92
as well. All the detailed diagnostic analyses done for
CTRL have not been done on these tests, although
those with only slantwise convection show no obvious
qualitative differences from the simulation discussed
here. Simulations of CSI using a broader range of initial
conditions in a nonhydrostatic model are currently
being run and will be the topic of a future paper.

4. CSI evolution

a. General evolution

CTRL is run for 96 hours, though no significant
changes occur beyond 60 hours. The transverse growth
rate, \(GR\), is used to define the transition from the
growth stage to the decay stage (Fig. 2), and the char-
acter of the transverse flow is used to define the tran-
sition to the post-CSI stage (see below). During the
growth stage, a sloping updraft and two flanking com-
pen sating downdrafts produce one thermally indirect
and one thermally direct circulation. The temporally
maximum \(GR\), (max \(GR\)) of \(4.3 \times 10^{-5} \text{ s}^{-1}\) occurs at
12 hours. Near the end of the CSI growth period (24
h; Fig. 3a), the updraft strength and the \(v\)-component
are at approximately their maximum values (11.2
\text{ cm s}^{-1} and 9.2 \text{ m s}^{-1}, respectively), and the thermally
direct circulation has become dominant. Strong hori-
zontal and vertical shears are produced at the interfaces
between the sloping updraft and the two downdrafts.
The lower downdraft appears to consist of two parts;
cross-isentropic flow occurs principally in the upper
half of the downdraft, and flow approximately parallel
to the isentropes exists in the lower part. Most of the
sloping updraft is contained within the sloping cloud,
but some of it is outside of the cloud in the upper tro-
posphere and along the lower flanks of the cloud. The
horizontal extent of the cloud is about 390 km. Al-
though the velocity magnitudes decrease during the de-
cay period, the extent of the cloud and the updraft con-
tinues to increase, reaching almost to the tropopause by
36 hours (Fig. 3b). Also by 36 hours, the saturated area
starts becoming quite narrow, the sloping updraft and
two sloping downdrafts are weaker but still present, and
the lower downdraft is very nearly parallel to the is-
entropes over its entire depth. By 48 hours (Fig. 3c),
the cloud is extremely narrow, being represented by
only one grid point at some locations, and by 60 hours
(Fig. 3d), the central portions of the once continuous
cloud are no longer saturated.

By 48 hours, only weak transverse circulations hav-
ing oscillatory characteristics remain. These oscillatory
circulations appear to represent inertia–gravity waves
in a sheared environment because they propagate in
response to weak ageostrophy in the zonal flow. The
phase velocity is toward the center of the decayed up-
draft, and the group velocity is directed away from the
updraft, indicating that the updraft is the energy source
of these waves. The period (22–24 hours), phasing,
and amplitudes of the waves in the model output are in
good agreement with those theoretically determined for
the stable waves of subinertial frequency described by Stone's (1966) dispersion relation Eq. (2.27a). As it is not obvious that these weak waves have any direct effect on the CSI evolution and its impact on the environment, no further analysis of these waves is presented here.

During the first 36 hours of the simulation, the surface precipitation occurs in a region up to 600 km wide. Peak precipitation rates are a modest 2.7 mm (12 h)\(^{-1}\) (0.225 mm h\(^{-1}\)) (not shown), so a peak accumulated precipitation of only about 5.7 mm has occurred after 60 hours. These small precipitation amounts agree with those from the modeling studies of Knight and Hobbs (1988) and IC92, who attribute increases of a few tenths of millimeters per hour in the surface precipitation rate to CSI. Clearly, if CSI is to account for the observed precipitation-rate increases of several millimeters per hour in rainbands 50–100 km wide, other
initial conditions and/or interactions with smaller or larger scales need to be considered.

b. Evolution of $\theta_u$, $\theta_{wu}$, $m$, $m_x$, and $u_x$ fields

Surfaces of constant $\theta_u$ and $m$ may be used for assessing the upright and inertial stability, respectively, and constant $\theta_{wu}$ and $m$ (or $m_x$) surfaces are frequently calculated from observations to assess symmetric stability. According to linear theory, the initial slopes of constant $\theta_u$, surfaces and constant $m_x$ ($= m$ initially, in this case) surfaces should be used to compare to the updraft slopes. Also, the local difference between $m$ and $m_x$ is a direct measure of the ageostrophic zonal flow ($u_x = m - m_x$), which is proportional to the meridional acceleration. Hence, the evolutions of the $\theta_u$, $\theta_{wu}$, $m$, $m_x$, and $u_x$ fields in CTRL are of interest.

Some surfaces of constant $\theta_u$ ($\theta_{wu}$), $m$ and $m_x$ become “buckled” during the growth stage (Figs. 4a and 5a). The buckling of the $\theta_u$ ($\theta_{wu}$) field produces a region of potential upright instability (PI) along the upper edge of the sloping updraft, as noted by numerous other studies (e.g., BH, ST; IC92). The buckling of the $m$ surfaces, suggested by Thorpe and Rotunno (1989) to be characteristic of DSI, indicates regions of negative absolute vorticity along the underside of the updraft. Such regions were also noted in the simulations of BH and IC92 and in the rainband observations of Hertzmann and Hobbs (1988). The buckling of $\theta_u$ and $m$ surfaces is due to the differential advection of these quasi-conserved quantities, with the differences in the slopes of the updraft and downdrafts and the $\theta_u$ and $m$ surfaces being the underlying cause. As discussed in the introduction, these slope differences can be explained by viscous linear theory. Comparisons of simulation slopes with those from linear theory are made in section 5. During the decay stage (36 h), the vertical gradient of $\theta_u$ is not as strong as at 18 hours, though a shallow area of PI still does exist and the buckling of the $m$ surfaces is still evident (Fig. 4b and 5b). By 48 hours, the former area of PI has become approximately neutral to upright moist convection (not shown).

Although the geostrophic absolute momentum, $m_x$, is not conserved by the flow and is at least slightly different than $m$ except at the initial time, the $m_x$ surfaces also buckled during the development of CSI, though by different amounts at different locations compared to the $m$ surfaces (Fig. 5a). The $m_x$ surfaces buckled more than the $m$ surfaces above the updraft near the southern outflow from the updraft, and the $m$ surfaces buckled more below and to the north of the updraft near the northern outflow from the updraft. This indicates that the mass field has changed more rapidly than the zonal momentum field in the former area, and the zonal momentum field has changed more rapidly in the latter area. Strongly buckled $m$ and $m_x$ surfaces persist into the decay stage of the CSI evolution (Fig. 5b) but are no longer evident by 48 hours. Because of the difficulty in using observations to determine the mesoscale-scale mass field, the slope of $m$ surfaces rather than that of the more technically correct $m_x$ surfaces is often used for comparison to $\theta_{wu}$ surfaces in determin-

![Figure 4](image_url)

**Fig. 4.** Cross sections of $\theta_u$ (solid; isopleth interval of 2°C) and $m_x$ (dashed; isopleth interval of 8 m s$^{-1}$) for CTRL at (a) 18 and (b) 36 h. The in-plane wind vectors are also shown as in Fig. 1. The heavy solid isopleth in (a) marked "t = 0 h" is the $\theta_u = 6°C$ isopleth at the initial time of CTRL.
Fig. 5. Cross sections of $m$ (solid) and $m_e$ (dashed) for CTRL at (a) 18 and (b) 36 h. The isopleth interval is 8 m s$^{-1}$, and the in-plane wind vectors are shown as in Fig. 1. The heavy solid isopleth in (a) marked 't = 0 h' is the $m = -34$ m s$^{-1}$ isopleth at the initial time of CTRL.

ing symmetric instability (e.g., Thorpe and Clough 1991). Figure 5 indicates that using $m$-surface slopes rather than $m_0$-surface slopes could lead to inaccurate stability assessments during the CSI growth stage. Despite the suggestion in other studies (e.g., IC92), neither buckled $m$ nor $m_0$ surfaces necessarily indicate inertial instability ($\partial m_0/\partial y > 0$ and assuming geostrophic zonal flow) since geostrophy is also required. Meridional acceleration is probably better diagnosed by the field of $u_\alpha$, which is a measure of the difference between the $m$ and $m_0$ surfaces and is directly proportional to the meridional acceleration for inviscid flow.

Fig. 6. The ageostrophic $u$-component, $u_\alpha$, for CTRL at (a) 18 and (b) 36 h. The isopleth interval is 2 m s$^{-1}$, and subgeostrophic values are dashed. In-plane wind vectors are shown as in Fig. 1.
Figure 6a shows that $u$ is subgeostrophic within the updraft during the growth stage ($u_u = -4 \text{ m s}^{-1}$), and it is supergeostrophic by about the same amount above and below the northern end of the updraft. Subgeostrophic zonal flow will produce northward accelerations within the updraft, and supergeostrophic zonal flow will produce southward accelerations above and below the northern part of the updraft. The tilt of the updraft–downdraft system is maintained by the convergence–divergence and resulting vertical motions implied by the horizontal variation of the meridional accelerations. The subgeostrophic zonal flow within the updraft has essentially ceased by 36 hours (Fig. 6b), though supergeostrophic flow of 1–3 m s$^{-1}$ persists in three areas: near the southern and northern outflows of the updraft, in the sloping zones of weak meridional flow between the updraft and the downdrafts, and locally in small areas below the northern end of the updraft. By 48 hours, the ageostrophic flow is less than 1 m s$^{-1}$ everywhere (not shown).

The evolution of the $m$-surface structure in this simulation suggests that strong buckling, such as that seen by Thorpe and Clough (1991) in conjunction with active cold fronts, may indicate a currently active CSI updraft (in either the growth or decay stage). In addition, a sloping updraft coinciding with a sloping area of subgeostrophic front-parallel flow, with small areas of supergeostrophic flow above and below the top of the subgeostrophic area, may indicate a CSI circulation in its growth stage. If mainly the supergeostrophic areas are present, then the transverse CSI circulation may be in its decay stage.

c. Potential vorticity evolution

1) DRY POTENTIAL VORTICITY

Initially, the dry potential vorticity ($q$) is positive everywhere (Fig. 1c). During the growth stage, $q$ decreases to become negative along the underside of the sloping updraft near the northern outflow (Fig. 7a), while an increase in $q$ occurs above and toward the southern end of the updraft. The potential vorticity budget (Fig. 8) indicates that the increase in $q$ along the top of the updraft and the decrease in $q$ along the underside of the updraft are caused by the dominance of horizontally differential latent heating [HDLH in Eq. (10)] over vertically differential latent heating [VDLH in Eq. (10)]. The VDLH also contributes to the decreasing $q$ in the areas in which negative absolute vorticity ($f - (\partial u/\partial y) < 0$) has developed along the underside of the updraft toward the latter part of the growth stage (e.g., at 18 hours, not shown). The other terms in Eq. (11) are small compared to HDLH and VDLH. The area of negative $q$ encompasses the area of negative absolute vorticity mentioned in section 4b, though the area of negative $q$ is larger, indicating that the first term in Eq. (8) is strongly negative.

The areas of positive and negative $q$ anomalies in the CSI simulation persist throughout the simulation and remain even after the CSI circulation has ceased (compare Fig. 1c to Fig. 7b). An increase in $q$ of up to 0.3 puv is seen in the lower troposphere and on the warm side of the updraft, whereas a decrease in $q$ of up to $-0.4$ puv occurs in the upper troposphere and along the underside of the updraft and near the updraft outflow. The dry symmetric destabilization of the area underneath the updraft and the dry symmetric stabilization of the area along the topside of the updraft is the likely reason for the later dominance of the thermally direct circulation. This argument incorporates the momentum advection arguments of Thorpe and Rotunno (1989), which will produce inertial stabilization along the topside of the updraft and inertial destabilization along the underside, and this argument is further supported by the fact that the downdraft of the thermally direct circulation tends to orient itself along $\theta$ surfaces (Fig. 3b).

2) WET-BULB POTENTIAL VORTICITY

The wet-bulb potential vorticity is initially negative over one-half of the tropospheric domain (Fig. 1b). As CSI evolves, some changes in the $q_w$ field occur. The nonadvective changes are given by (12) and are due entirely to diffusional processes. By the middle of the growth stage (Fig. 7c), an area of slightly more negative $q_w$ has developed near the base of the updraft, and an area of slightly less negative $q_w$ has developed near the center of the updraft and within the two downdrafts. These features are also noticeable in Fig. 9 of IC92 at the end of the growth stage. An examination of the terms in (12) shows that the slight increase in $q_w$ in the center of the updraft results because the positive temperature diffusion term (TDIFF) is slightly larger in magnitude than the negative advection term. The increase in $q_w$ in the lower downdraft is due to advection of larger $q_w$ from above. Both advection and TDIFF give positive $q_w$ tendencies in the upper downdraft, causing an increase in $q_w$. In general, the frictional term (FRI) and the moisture diffusion term (QVDIFF) in Eq. (12) are secondary to TDIFF, though QVDIFF increases the $q_w$ in the center of the updraft, while FRI decreases it. The upward extension of negative $q_w$ values (primarily by advection) penetrates the area of initially positive $q_w$ near the end of the growth stage, in accordance with IC92; it reaches the top of the troposphere during the decay stage, where it remains until the end of the simulation (Fig. 7d). However, the diffusional processes, principally TDIFF with some help from QVDIFF, have apparently increased $q_w$ within the updraft somewhat. Values of $q_w \approx -0.15$ to $-0.25$ puv are present near the initial perturbation, while the same air parcels aloft by 48 hours have values of $q_w \approx -0.05$ to 0.0 puv. By 48 hours, the $q_w$ has decreased within the entire updraft area (compare Figs. 1b and 7d), with
Fig. 7. Dry potential vorticity (solid; isopleth intervals of 0.2 pvu) at (a) 18 and (b) 48 h. and wet-bulb potential vorticity (solid; isopleth intervals of 0.1 pvu) at (c) 18 and (d) 48 h for CTRL. The relative humidity (%; RH > 100% is shaded) and the in-plane wind vectors are shown as in Fig. 1. The stars in (d) show the locations of the five points for which linear growth rate curves are calculated in section 5 and Table 1.

decreases of up to 0.2 pvu concentrated near the top and the bottom of the updraft. Throughout the rest of the troposphere, $q_w$ increased by 0.0–0.07 pvu. The tropospherically integrated mass-weighted wet-bulb potential vorticity at 48 hours is about the same as that at 0 hours.

Apparently, the cessation of the CSI transverse circulation is not caused by the removal of the negative $q_w$. The circulation has stretched the negative area upward and northward into a region that initially had positive $q_w$, enlarging the area of negative $q_w$ along the updraft while slightly decreasing the area of negative $q_w$ near the downdrafts. One possible reason for the cessation is the depletion of the water vapor supply since water vapor is necessary for condensation within the updraft for CSI. An-
other possible explanation is discussed in section 5d.

d. Soundings along momentum surfaces

Thermodynamic soundings obtained from aircraft along m surfaces have been used to assess the stability of the atmosphere to CSI (Emanuel 1988). Our CSI simulation provides an opportunity to assess the evolution of the thermodynamic soundings along m surfaces during the life cycle of CSI. Thermodynamic soundings from CTRL, taken at the initial time, at the end of the growth stage and after the CSI circulation has ceased, indicate that the model atmosphere along the m = -34 m s⁻¹ surface below about 700 mb has become thermodynamically less unstable during the simulation, whereas the model atmosphere between 700 and 360 mb has destabilized (Fig. 9). Values of slantwise convective available potential energy (SCAPE) along the m = -34 m s⁻¹ surface are 1759 J kg⁻¹ for the 0-h sounding and 1383 J kg⁻¹ for the 48-h sounding, indicating that potential energy has been converted to kinetic energy during the simulation and that the final state is thermodynamically less unstable than the initial state.

However, the change in stability as determined from m-surface soundings may not be reflected in the local qₘ field, since the magnitude of a nonzero qₘ is depen-
This example shows that the local value of \( q_w \) may be a misleading indicator of the thermodynamic stability of the surface sounding. The stabilization of the lower troposphere, as suggested by the surface sounding, is more reasonable than the partial destabilization suggested by the local changes in \( q_w \). We expect the model atmosphere to stabilize during the course of the simulation, and surface soundings are a more direct measure of the degree of slantwise instability because they provide a measure of the energy available for the instability. Note also that the 0 h surface sounding indicates that a parcel originating near the surface could move to 300 mb before encountering negative buoyancy forces, explaining why the CSI circulation extends outside the part of the domain with negative \( q_w \) at the initial time. In fact, if a buoyantly unstable region is defined to be a region "within which parcels can enter and transit under their own buoyancy" (Stull 1991), the region of instability to CSI in the initial conditions is larger than the region of negative \( q_w \).

\( e. \) Net changes of heat, momentum, and moisture

Cross sections of mass-weighted heat, moisture, and momentum change between 0 and 48 hours for CTRL are examined to reveal the net effect of CSI on the model atmosphere, suggesting the net transport of these quantities (Fig. 11). These cross sections suggest that the CSI has caused the following:

- low-level frontogenesis,
- upper-level frontolysis,
- low-level stabilization to upright convection,
- upper-level destabilization to (shallow) upright convection,
- inertial stabilization in the lower troposphere by increasing cyclonic horizontal shear,
- inertial destabilization in the upper troposphere by creating anticyclonic horizontal shear,
- changes suggesting a northward heat transport and a downward zonal momentum transport.

The low-level frontogenesis and upper-level frontolysis is also suggested by the cross-isentropic flow near the bottom and top of the updraft (e.g., Fig. 3a).

The horizontally averaged vertical profiles of \( \Delta(p^* \theta) \) and \( \Delta(p^*u) \) (Fig. 11) indicate that net heating has occurred in the lower to midtroposphere—strong net cooling has occurred aloft and weak net cooling has occurred in the lowest layers—while momentum decreases above about 700 mb and increases below 700 mb. Vertically averaged horizontal profiles (Fig. 11) indicate that a net warming occurs at all latitudes, with the largest warming occurring at the latitudes approximately corresponding to the southern limit of the CSI updraft, and that momentum decreases near the center of the updraft and increases near the southern and northern updraft extremities. These average changes in
5. Comparisons with linear theory

In this section, a comparison is made between the nonlinear CSI evolution documented in section 4 and the linear theory of Xu. There are two reasons for making this comparison. First, we want to show that because viscosity and a finite grid length impose a finite width on the CSI updraft in CTRL, commonly used CSI characteristics (e.g., growth rate and updraft slope) obtained in the simulation are in good agreement with the viscous, linear-theory characteristics of modes with finite updraft widths. These characteristics differ from those of the frequently discussed most unstable, inviscid, linear-theory modes. Second, this agreement then gives us some confidence to use the linear theory as an aid in interpreting the symmetric stability of the final conditions of the numerical simulation.

5. Linear theory characteristics

The linear theory used in this study is a modified version of Xu’s (1986) linear theory. The modifications, described in the appendix, make the theory more compatible with the model characteristics. As shown in the appendix, these modifications have not produced any qualitative changes in the characteristics of the linear growth rates and updraft slopes, though some quantitative changes are produced.

This linear, Boussinesq theory assumes that CSI consists of a sloping saturated updraft bracketed by two sloping, unsaturated downdrafts. The updraft is assumed to be entirely saturated and the downdraft is assumed to be entirely unsaturated, so no unsaturated regions with upward motion are allowed to exist. Therefore, the width of this saturated updraft, $l_{up}$, is the same as the width of the updraft region, $l_u$, and is assumed to be constant over the entire depth of the circulation (Fig. 12a). This depth is the height scale $H$. The updraft slope angle, $\phi_u$, is the angle the sloping updraft makes with the horizontal. Both the saturated updraft and the unsaturated downdrafts are assumed to
Fig. 11. The mass-weighted changes (a) $\Delta(p^*\theta)$ (cb K) and (b) $\Delta(p^*u)$ (cb m s$^{-1}$), where $\Delta$ indicates the difference between the values at 48 h and those at 0 h for CTRL. Dashed isothes indicate negative changes. Division by $p^*$ ($\approx 85$ cb) will give the changes in more familiar units. Horizontally (vertically) averaged vertical (horizontal) change profiles are also shown to the right (below) of each panel, where the division by $p^*$ has been done.
have environmental conditions that are spatially invariant within each area, and both downdraft areas are assumed to have the same conditions. Of course, all conditions are assumed to be invariant with time. The basic-state transverse potential temperature gradient is assumed to be in thermal wind balance with the basic-state longitudinal flow (i.e., the flow perpendicular to the transverse cross section). The unstable area is assumed to have a height scale \( H \), and viscosity is allowed to modulate the growth of the CSI, as described in the appendix.

The useful feature of this theory is that it provides theoretical growth rates and updraft slopes as a function of updraft width given the environmental conditions (e.g., buoyancy frequency \( N^2 \), \( q_0 \), etc.), the instability height scale \( H \), and the viscosity (diffusion). The difficulties in applying this theory come in determining the parameters describing the environmental conditions and in determining the updraft width and slope characteristics.

\( b. \) Determination of parameters and characteristics from model output

Figure 13a shows the vertical velocities at one sigma level (\( \bar{\rho} = 736 \text{ mb} \)) during the growth stage of CTRL.
The magnitude of each downdraft decays exponentially with distance from the centerline of the updraft, in good agreement with the structure assumed by the linear theory. However, in other ways, the structure of the CSI circulations in the numerical simulations differs somewhat from that assumed in the linear theory. Unsaturated regions of upward motion, which are not present in the linear theory, do exist in the nonlinear simulations, making the definition of the updraft more uncertain. In this study, two definitions of the updraft are examined, giving upper and lower bounds for the updraft width. Defining the updraft as existing wherever \( w > 0 \) yields a broad updraft, while a narrow updraft is obtained by defining the updraft as existing wherever \( w > 0 \) and the relative humidity, RH, is greater than 99.9%. In Fig. 12b, the former is denoted by \( L_u \) and the latter by \( L_{wu} \). The estimates of \( L_{wu} \) and \( L_u \) represent the average of the updraft widths determined at the three adjacent model levels that have the strongest \( v \) and \( w \) components. These levels are located at approximately the position indicated by the horizontal dashed double-headed arrows in Fig. 12b relative to the other updraft features. In addition, \( H \) in the numerical simulations is defined as the depth of the saturated region within the area of negative \( q_e \). The impact of the method of determining this important parameter is discussed further in section 5d.

To make quantitative estimates of updraft slopes from the model output, as well as of the slopes of the \( \theta_a \) and \( m \) surfaces, an updraft core must be defined that is representative of the updraft interior. For these purposes, an updraft core region is defined in the horizontal as the two grid points of greatest vertical velocity at each sigma level; in the vertical it is defined by the sequential levels at which the two points of maximum \( w \) correspond to the two points of maximum \( v \). This method of defining the updraft core eliminates points along the edge of the updraft or near the lower inflow and upper outflow regions of the updraft, where the updraft slopes are unrepresentative of the slopes in the updraft interior. The updraft core is then the sloping region in the center of the updraft, and its vertical extent relative to the other updraft features is given by \( h_i \) in Fig. 12b. The two grid points in the updraft core on each level are always saturated. The two central points in Fig. 13b, which is an exploded view of the updraft region in Fig. 13a, are defined to be the updraft core at this sigma level and time. This cross-sectional profile of vertical and horizontal velocities is typical for all cross-sectional profiles across the updraft core of CTRL at 18 hours.

The updraft slope angle (\( \phi_u \)) is calculated from the model output \( v \) and \( w \) components as the average angle within this updraft core, and the slope angles of the \( m \) and \( \theta_a \) surfaces represent averages over the depth of the updraft core. All angles are taken with respect to constant height surfaces.

c. Growth rate and updraft slope comparisons

Figure 14a compares the nonlinear \( GR_n \) calculated from the model output at three different times in simulation CTRL, with the applicable theoretical linear growth rate curves (curves 1 and 2) calculated from the theory in the appendix using the environmental characteristics given in Table 1. The updraft widths \( L_{wu} \) and \( L_u \) are also estimated from the model output to position the three nonlinear \( GR \) on the plot. Curve 1 represents linear growth rates for the environmental conditions and diffusion-parameter values at the center of the initial perturbation. Curve 2 represents the linear growth rates for the initial environmental conditions at the point at which the 12-h domain-maximum \( v \) and domain-maximum \( w \) occur, using the scale height and the diffusion characteristics present at this point at 12 hours. Hence, curve 2 gives a theoretical estimate of the linear growth rate characteristics at 12 hours, which is the time of max \( GR \) for CTRL. We recognize that 1) linear theory is, strictly speaking, only applicable
would yield a linear growth-rate curve that would pass through the simulation $GR$. The environmental conditions for curve 2 are less unstable than those for curve 1, the diffusion is greater for curve 2 than curve 1, and the height scale for the circulation at 12 hours (used for curve 2) has increased from 1.6 to 3.5 km (Table 1). Since the first two changes would decrease the growth rate, the expected 12-h linear growth rate is greater than that expected from the 0-h linear growth-rate curve for modes with $L_w > 32$ km because the height scale increases as the saturated region becomes deeper. This suggests that the increase in the depth of the saturated region is important for the max $GR$ in the simulation to occur at 12 hours rather than earlier. The effect of the increase in the height scale more than compensates for the effects of the parcel movement into a less unstable environment and the increased diffusion.

Though the nonlinear $GR$, positioned with $L_w$ are marked on Fig. 14, other tests (see Persson and Warner 1993) show that $L_w$ has no apparent relationship to max $GR$, whereas a clear relationship between $L_w$ and max $GR$ is evident. Also, for classical upright convection, the growth rate has been shown to be a function of the updraft width, defined as the area of $w > 0$ (Bjerknes 1938). Hence, we used the $GR$, positioned with $L_w$ rather than $L_w$ in the above discussion. Also, probably no significance should be ascribed to the apparent correspondence of the CTRL $GR$ at 18 and 24 hours with curve 2, which represents the environmental conditions at 12 h. Their correspondence is shown here as just a curiosity. It is even more difficult to justify using linear theory at these later times since substantial changes in the environmental conditions have occurred and the model atmosphere is even further from thermal wind balance.

Figure 13c shows the updraft slopes calculated from the horizontal and vertical velocities from the points shown in Fig. 13b, where the two central points define the updraft core. Within the updraft core at this level, the updraft slope averages $0.84^\circ$, while on the flanks the slope is greater or less than this. The slopes of the $m$ and $\theta_w$ surfaces through these grid points at 0 and 18 hours are also shown for comparison. Certainly, within the updraft core, the updraft slope is between the slopes of the $m$ surfaces and $\theta_w$ surfaces at either time. Linear theory would predict an updraft slope between the $m$ and $\theta_w$ slopes at 0 hours, and Fig. 13c shows that this expectation also holds true for nonlinear CSI. We can also note the large changes in the $m$ and $\theta_w$ slopes at 18 hours across the updraft, resulting from the buoyant destabilization along the top of the updraft and the inertial destabilization along the underside of the updraft, as discussed in section 4a. The updraft slope averaged over all 11 levels of the updraft core at 18 hours is $0.84^\circ$, while the average $m$ and $\theta_w$ slopes are $0.62^\circ$ and $1.26^\circ$ at 0 hours, and $0.43^\circ$ and $47.1^\circ$ at 18 hours, respectively. Hence, the nonlinear updraft slope during the growth stage in CTRL is between the slopes of the
Table 1. Parameter values used in all growth rate curves calculated from the linear theory of Xu, modified as in the appendix. The curve numbers are referenced in the text and are used to identify the parameter value combinations. All values are obtained at the given time and location, indicated by the average \( \sigma \) level pressure \((\bar{p})\) and the horizontal distance \((y)\). The parameter values for curves 3–7 represent those at the points shown in Fig. 7d. The parameters \( F, S, S_w, N_0^2, \) and \( R_w \) are the inertial frequency, the dry baroclinic frequency, the moist baroclinic frequency, a moist pseudo buoyancy frequency (it is not the actual buoyancy frequency; see Emanuel (1983)), and the modified Richardson number, respectively. They are further defined in the appendix. The other parameters are the dry buoyancy frequency \((N_0^2)\), the instability height scale \((H)\), the vertical diffusion coefficient \((K_v)\), and the horizontal diffusion coefficient \((K_h)\).

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<td>(61 \times 10^{10})</td>
<td>(29 \times 10^{10})</td>
<td>(45 \times 10^{10})</td>
<td>(45 \times 10^{10})</td>
<td>(22.5 \times 10^{10})</td>
</tr>
</tbody>
</table>

Initial \( m \) and \( \theta_w \) surfaces, as well as between the slopes of the nonlinearly modified \( m \) and \( \theta_w \) surfaces. The updraft slope in this simulation appears to be closer to the slope of the \( m \) surfaces, especially if the nonlinearly modified \( m \) and \( \theta_w \) surface slopes are used for the comparison.

Although Fig. 13c shows that the nonlinear updraft slope is qualitatively in agreement with the linear theory, Fig. 14b shows that the nonlinear updraft slope for CTRL at 12 hours is about 0.14° (15%) larger than that expected from linear theory. The decrease in stability implied by the nonlinear steepening of the \( \theta_w \) surfaces (Fig. 13c) may explain why the updraft slope is slightly greater than expected; that is, the upright stability is locally reduced during the course of the nonlinear CSI evolution, thereby allowing an enhancement of the vertical velocity, which produces a steeper updraft slope.

d. Linear stability of final state

When the CSI circulation in CTRL has ceased (e.g., at 48 hours), the former updraft region is left in a state in which \( q_v \) is still negative and in which a narrow, broken, saturated region still exists (Fig. 7d). In section 4c, it is suggested that a possible reason for the decay of the CSI circulation, and its eventual cessation is the depletion of the water vapor supply by the precipitation. Ideally, to diagnose the reason for the decay, some theory should be applied near the transition from the growth stage to the decay stage to assess what conditions lead to the decay of the circulation. Unfortunately, the application of the linear theory of Xu (1986) at this time would not be justified because 1) the assumption of geostrophic zonal flow does not hold and 2) a large-amplitude transverse CSI circulation is present, which is inconsistent with the linear-theory assumption of weak perturbation motions. However, applying this linear theory at 48 hours appears justified because the \( u_v \) components in the model atmosphere are once again quite small and the transverse motions are weak. By obtaining linear growth-rate curves at this later time, we can assess which changes occur during the course of the CSI life cycle, if any, that modify the linear CSI growth rate curves in a manner that can help explain why a CSI circulation is not initiated by the final conditions at 48 hours. Note that this does not directly tell us why the initial circulation decays, though it would be suggestive.

To apply the linear theory at 48 hours, the linear growth-rate curves are calculated for conditions found at five points along the sloping decayed updraft at 48 hours (Table 1; also see Fig. 7d for the locations of these points). However, applying the linear theory requires the estimation of \( H \). It is uncertain exactly what values should be used for this parameter since its value depends on whether a very narrow saturated region can be considered sufficient to support a circulation. Each \( H \) in Table 1 is estimated from the approximate depth of the saturated areas that are at least two grid points wide in the vicinity of each point in Fig. 7d. This manner of defining \( H \) is rather arbitrary, but the results suggest that some effects other than just the depletion of the water vapor supply may have led to the cessation of the CSI circulation. If a scale height encompassing the entire troposphere were assumed, the linear theory would predict that these final conditions should lead to the development of a new CSI circulation, which, of course, does not happen. This suggests that the breaks, or near breaks, in the saturated region cannot support a CSI circulation.

Although the proper definition for \( H \) may be different than that used here, our current definition of \( H \) (i.e., the depth of the region that is both saturated and has a negative \( q_v \)) seems reasonable. No matter what the def-
inition, the importance of $H$ for the application of the linear theory is emphasized. A further demonstration of the significance of $H$ is given in Fig. 14, which shows the linear growth rate curve for the initial conditions of SMLH (see Table 1). The initial conditions for this simulation are designed so the value of $H$ is fairly small (1.0 km), thereby leading to unstable modes with updraft widths narrower than those for CTRL. No CSI circulation develops from these initial conditions, illustrating the effect of $H$.

The growth rate curves shown in Fig. 15 are calculated from the linear theory using the environmental conditions at the five points along the decayed updraft of CTRL in Fig. 7d. These linear growth rate curves can easily be separated into two categories. The curves corresponding to the three lower-tropospheric points show unstable modes only for updraft widths of less than about 50 km. This is to be compared with curves 1 and 2 in Fig. 14a, representative of the growth stage, for which unstable modes with updraft widths of up to 87 and 146 km exist, respectively. Hence, it appears that stabilization of the lowest part of the updraft has occurred for the modes with the widest updrafts. Presumably, the resolution of the model is insufficient to resolve the narrow updrafts of the unstable modes that remain. The unstable modes with resolvable updraft widths in the lower part of the updraft have been removed by a change in the environmental characteristics and by the removal of the water vapor, thereby reducing $H$. Larger values of the linear theory parameters $F/ N_d$ and $FN_d$ at these three lower points in Table 1, compared to the corresponding values for curve 1, can be seen. This increase is apparently due to the increase of the low-level cyclonic horizontal shear by the end of the simulation. No other consistent changes of the environmental parameters for these three points compared to those for curve 1 can be seen.

The environmental conditions for the point at 829 mb at 48 hours (curve 4, Table 1) will produce a marginally unstable mode with an updraft width of about the same width as that for curve 1 if a value of $H = 1.6$ km is used rather than the value of 1.0 km estimated at this point. Hence, the removal of the resolvable unstable modes can be argued to be principally due to the reduction of $H$ for this point. The change in the environmental conditions (inertial stabilization) for the other two points is sufficient to substantially decrease the updraft width of the marginally unstable modes for these points, even if a value of $H = 1.6$ km were used in the linear theory. Hence, the conclusion can be made that the nonlinear changes in the environmental conditions and in the height-scale $H$ explains the removal of the unstable modes with resolvable updraft widths in the lower troposphere. The changes in $H$ are due to the depletion of the water vapor supply since $q_w$ remains negative at all points. The removal of the modes with resolvable updraft widths suggests that this led to the cessation of the CSI circulation in the lower troposphere.

The upper-tropospheric points (those at 421 and 523 mb) indicate that unstable modes with very large updraft widths can be present in a region that has been inertially destabilized. The linear growth rates are not particularly large, but the cessation of CSI at the top of the updraft most likely has been due to the depletion of the moisture supply. If any saturated region remains in this area, the updraft of an unstable mode will certainly be resolved. This conclusion for these two points is not as sensitive to the estimate of $H$ since a reduction of $H$ from 3 to 1 km for these points still allows for weakly unstable modes with updraft widths of 69 km (at 523 mb) or more (at 421 mb). A reduction of $H$ to 0.5 km removes all unstable modes for the points at 931, 678, and 523 mb. In this case, the widest updraft width of an unstable mode is about 22 km at 829 mb and about 50 km at 421 mb. The apparent stability of the upper-tropospheric region of negative $q$ (Fig. 7b) may also be due to the inability of the model to resolve the unstable SI modes. However, the development of a theoretical SI model, similar to the one used for CSI, to verify this conjecture is beyond the scope of this paper on CSI.

6. Summary and discussion

This work provides detailed analyses of a simulation of the complete life cycle of nonlinear CSI in terms of theoretically and observationally significant parameters and uses quantitative comparisons with linear theory as an aid in interpreting the results of the simulation. During the growth stage (0–24 hours), buckling of $\theta_w$, $m$, and $m_s$ surfaces occurs, indicating the development of localized areas of upright potential instabil-
ity, negative absolute vorticity, and possible inertial instability. In addition, subgeostrophic zonal flow is coincident with the sloping updraft during the growth stage, and supergeostrophic zonal flow occurs near the tops of the two downdrafts. This ageostrophy produces northward and southward inertial accelerations, respectively, and implies that \( m \) and \( m \) surfaces are not coincident within and near the updraft. The strongest precipitation rates (condensation rates) are small. The tropospherically averaged maximum nonlinear growth rate is substantially less than that which would be expected from inviscid linear theory but is in good agreement with the growth rate expected from viscous linear theory for a mode with an updraft width equal to that present in the model output and with the estimated instability height scale. As for linear theory, the nonlinear updraft slope angle is substantially smaller than the slope angle of the \( \theta \) surfaces but somewhat greater than the slope angle of the \( m \) surfaces at the initial time. The discrepancy between the life cycle length of \( \sim 1-2 \) days and the apparently shorter observed timescales of symmetric neutralization (Reuter and Yau 1990) and rainband life cycles (Parsons and Hobbs 1983) is as yet not understood.

During the decay stage (24–48 hours), the mostly saturated sloping updraft and the two sloping downdrafts are still present, though the meridional and vertical velocities associated with them decrease throughout this period. The northward inertial acceleration within the CSI updraft ceases, as indicated by the nearly ageostrophic zonal flow within the updraft region during this stage. Areas of superageostrophic flow still remain above and below the updraft near the tops of the downdrafts. The presence of subgeostrophic flow coincident with the updraft appears to be a possible indicator of whether CSI is in its growth or decay stage.

By the end of the decay stage, significant changes in both the mass and momentum fields have been produced, leading to localized changes in the buoyant and inertial instabilities. Although \( q_{w} \) decreases within the area occupied by the updraft during the life cycle of CSI and the initial area of negative \( q_{w} \) expands through advection into a region that initially has positive \( q_{w} \), surface “soundings” within the updraft show that thermodynamic stabilization occurs during the course of the simulation. The dominance of horizontally differential latent heating, principally during the growth stage, produces an area of negative dry potential vorticity (\( q \) along the underside and near the top of the sloping updraft, as well as an area of increased \( q \) in the lower troposphere along the warm side of the updraft. These changes in the \( q \) field remain at the end of the life cycle.

The lack of removal of the area of negative \( q_{w} \) indicates that some mechanism other than the neutralization of the initial instability is necessary to explain the cessation of CSI. It appears as if two reasons are possible: the depletion of moisture and other nonlinear environmental changes. The removal of all condensate as precipitation and the lack of moisture sources lead to the eventual breakup of the cloud into smaller saturated areas. In effect, the depletion of moisture reduces the height scale of the unstable region, which, as is seen from the linear theory of Xu (1986), will tend to reduce the updraft widths of unstable CSI modes. Similarly, the increased updraft and inertial stability in the lower troposphere have changed the environmental conditions so that, for a constant height scale, the updraft widths of the unstable modes have become smaller. Hence, due to both of these nonlinear changes, the finite grid spacing of the numerical model is not capable of resolving the updrafts of the remaining unstable modes. The removal of resolvable updraft widths due to changes in the environmental conditions can only be an argument for the stabilization of the lower troposphere. In the upper troposphere, the depletion of moisture appears to be the only possible reason for the cessation of CSI, since the updraft widths of the remaining unstable modes are easily resolved by the model grid. Since the sloping CSI updraft from the lower troposphere is the source of moisture for the upper troposphere, this hypothesis seems reasonable though speculative.

The major differences in the depicted CSI life cycle from that of dry symmetric instability (DSI), shown by Thorpe and Rotunno (1989, referred to as TR) and Ducrocq (1993, referred to as DQ), are likely due to the necessity of saturation for CSI, resulting in only one updraft rather than the multiple updrafts seen for DSI. The entire patch of negative \( q \) is unstable for DSI, leading to the multiple cells, but only the updraft (and a small part of the lower downdraft in which \( q \) becomes negative) is unstable in this CSI case. The decay of this one updraft ends the transverse circulation, leaving a large but deformed remaining area of negative \( q_{w} \). Subpatches of negative \( q \) can remain at the end of the DSI life cycle as well, and, as stated by TR, these subpatches “are viscous stable to further DSI by virtue of their geometry.” Our results for CSI suggest that the important aspect of the geometry is the depth scale of the subpatches of negative \( q \) for DSI and the saturated areas of negative \( q_{w} \) in our case. However, DQ shows that the final state of DSI may have greater or lesser amounts of remaining subpatches of negative \( q \) depending on the diffusive characteristics of the simulation. DQ’s results suggest that in order to neutralize a large but limited region of initially negative \( q \), multiple DSI circulations may be necessary to create a jumble of smaller subpatches of large gradients of \( m \) and \( \theta \) that can then be removed by diffusion. As illustrated by DQ, this is best accomplished by using time- and space-dependent diffusion coefficients, allowing rapid initial growth of the DSI circulations and later dissipation of the large local gradients formed. Also, TR suggests that downgradient diffusion of \( q \) rather than \( m \) and \( \theta \) should be used, but no such diffusion scheme is introduced.
The simulations of CSI in this study use diffusion coefficients that are dependent on local gradients, but the important additional criterion of saturation for instability appears not to allow the multiple updrafts probably necessary to produce sufficient reduction of scale in order that local diffusion can neutralize the area of negative $q_w$. However, there is a reduction in the sizes of the saturated areas, and a tendency toward smaller scales for CSI is suggested by the removal of the unstable modes with larger updraft widths from the initial time to the end of the simulation.

To the extent that it is possible to compare, other aspects in the DSI life cycles appear to be similar to those depicted for CSI. The local minima in $q$ attained during the DSI life cycles could be in agreement with the local minima in $q_w$ in this study, though TR and DQ do not show where these minima are located with respect to the individual circulations. Suggestions of weak intrusions of negative $q$ into the region of initially positive $q$, possibly due to advection of negative $q$, are also present in Fig. 11c of TR and Figs. 6 and 9 of DQ. The multiple cells may limit the extent of development of any one individual cell and prevent them from protruding as far into the positive $q$ zone as the negative $q_w$ extended into the positive $q_w$ zone for CSI. No m-surface soundings are provided for estimating the extent of possible parcel movement, nor are individual cells examined in detail. Later in the simulation of TR, a large thermally direct circulation dominates and begins to entrain stable air from the surroundings, which DQ suggests may be due to inconsistent vertical and horizontal resolution. This is not a problem in the present CSI simulation. Note, however, that the life cycles of DSI and CSI documented in TR, DQ, and the present study may be only of academic interest once large-scale forcing is applied and a quasi balance between the instabilities and the forcing becomes possible.

This study has been able to document the life cycle of unforced CSI, extending our knowledge of CSI into the nonlinear realm, and is a step in tying together previous theoretical studies with future idealized modeling studies and observational or modeling studies of real-data cases. In real-data cases, various types of forcing, such as geostrophic deformational forcing or forcing from the planetary boundary layer, can occur at the same time as the release of CSI, and these need to be incorporated in future idealized studies, as do microphysical effects. The inclusion of rain evaporation has an impact on CSI (Innocentini and Caetano 1992) since substantial evaporation may occur below the sloping cloud, possibly leading to a reintroduction of the moisture into the CSI updraft, a change in the resulting dry potential vorticity field, and even the slantwise or upright destabilization of the model domain below the initial CSI updraft. Mixing moist updraft air with the dry environment above the updraft might also produce penetrative downdrafts that could disrupt the sloping updraft. A methodical progression of including the various physical mechanisms believed to occur in the real atmosphere is necessary in order to understand how the rainband observations relate to the CSI theory.

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APPENDIX

Modifications of Xu's Linear Theory

The linear theory of Xu (1984, 1986) was modified to make it more compatible with the numerical model equations. In the discussion below, the following variables are used:

$$F^2 = f(f - \partial U / \partial y),$$

$$S^2 = f \partial U / \partial z = -(g / \theta_0) (\partial \theta_0 / \partial y),$$

$$N^2_\theta = (g / \theta_0) \partial \theta_0 / \partial z,$$

$$N^2_\alpha = (g / \theta_0) \partial \theta_\alpha / \partial z,$$

and

$$S^2_{\alpha \theta} = -(g / \theta_0) \partial \theta_\alpha / \partial y.$$ 

The terms $U$, $\theta_0$, $\theta_\alpha$, and $\alpha$ are basic-state variables representing zonal velocity, potential temperature, wet-bulb potential temperature, and specific volume, respectively, and $u$, $v$, $w$, $\theta'$, $\theta_\alpha'$, and $p$ represent perturbation quantities with the standard meteorological meanings.

The three modifications made to Xu's theory are the following.

1) Hydrostatic, linearized, inviscid Boussinesq equations of motion are used rather than the nonhydrostatic ones.

2) For the part of the domain where $w > 0$, we make the assumptions

$$S^2_{\alpha \theta} = \frac{\theta'}{\theta_\alpha'} = R.$$  

(A1)
where $R$ is a constant that can differ from 1. Term $R$ is related to the ratio between the moist adiabatic lapse rate evaluated at a given temperature and pressure and the moist adiabatic lapse rate evaluated at $T = \theta_w$ and $p = 1000$ mb (Emanuel 1983). This ratio increases with height as the moist adiabatic lapse rate approaches the dry one and is typically in the range 1.0–1.5 in the lower and middle troposphere (Bennetts and Hoskins 1979). To solve (A2) below, Xu makes the assumption that $R = 1.0$, which is more restrictive than necessary.

3) Xu’s second-order bulk diffusion with constant coefficients is separated into second-order vertical diffusion and fourth-order horizontal diffusion with constant coefficients. No attempt is made to include variable coefficients in the linear theory.

Following Xu’s derivation using the inviscid Boussinesq equations, except for performing the derivation on a meridional plane rather than on a zonal plane and including modifications 1) and 2) above, the transverse circulation equations obtained are

\[
(\sigma^2 + F^2) \frac{\partial^2 \psi}{\partial z^2} + 2S^2 \frac{\partial^2 \psi}{\partial y \partial z} + N_0^2 \frac{\partial^2 \psi}{\partial y^2} = 0, \quad w \leq 0
\]  
(A2a)

\[
(\sigma^2 + F^2) \frac{\partial^2 \psi}{\partial z^2} + 2S^2 \frac{\partial^2 \psi}{\partial y \partial z} + RN_0^2 \frac{\partial^2 \psi}{\partial y^2} = 0, \quad w > 0, \quad \text{(A2b)}
\]

where $\psi = \psi(x, z)e^{-\sigma z}$ is a streamfunction such that $v = -\alpha_0 \partial \psi / \partial z$ and $w = \alpha_0 \partial \psi / \partial y$ and $\sigma$ is the growth rate. It is assumed that saturation occurs if and only if there is upward motion. The derivation then continues as in Xu (1986), with the hydrostatic assumption and the factor $R$ causing some differences in the relations obtained.

Xu includes a bulk formulation of viscosity by using a Rayleigh friction term in the equation of motion in the direction of the orientation of the two-dimensional plane. In our case, this is the meridional plane. The Rayleigh coefficient used by Xu, $K_x$, is

\[
K_x = \nu \nabla^2 = \nu \left[ \frac{1}{(l_u \sin \phi_w)^2} + \frac{1}{(l_d \sin \phi_w)^2} + \frac{1}{H^2} \right]
\]  
(A3)

where $\nu$ is the diffusion coefficient, $l_u$ and $l_d$ are the horizontal updraft and downdraft widths, respectively, and $H$ is the height scale of the instability. Therefore, $l_u \sin \phi_w$ is the updraft width in the direction perpendicular to the updraft flow. For our purposes, we separate $K$ into vertical and horizontal components,

\[
K = K_v + K_h,
\]  
(A4a)

where

\[
K_v = \nu_v \nabla^2 = \nu_v \left[ \frac{1}{(l_u \tan \phi_w)^2} + \frac{1}{H^2} \right]
\]  
(A4b)

and

\[
K_h = \nu_h \nabla^2 = \nu_h \frac{1}{l_v^2}.
\]  
(A4c)

Comparing (A3) and (A4), we see that the term involving the width of the downdraft has been ignored because we are concerned only with the isolated CSI mode, for which $l_d^2$ is small compared to the other terms. Because $\phi_w$ is generally quite small, we expect that the use of (A4b) should give similar results to that from (A3), and (A4c) will yield an additional dissipation that is very scale selective. Though the use of (A4) is not exactly equivalent to the use of (A3), it is more consistent with the numerical model since it allows $\nu_v$ and $\nu_h$ to be varied independently.

As done by Xu, this derivation of the linear theory is nondimensionalized using the timescale $(F N_0)^{1/2}$ and the height scale $H$, with the nondimensional variables being marked by asterisks. Therefore, using the updraft slope given by

\[
\tan \phi_w = (F^2 + \sigma^2)/S^2
\]  
(A5)

[the sign of the right-hand side differs between (A5) and Xu’s (20) because the derivation by Xu is done in the $x-z$ plane rather than the $y-z$ plane], we have

\[
K^* = \nu^*_v \left[ \frac{\text{Re}_m^{-1}}{l_v^* \left[ \frac{F}{(F N_0)^{1/2}} \right]^2 + 1} \right] + \nu^*_h l_v^* \frac{1}{l_v^*}
\]  
(A6)

where $K^* = K/(F N_0)^{1/2}$, $\nu^*_v = \nu_v/[H^2(F N_0)^{1/2}]$, $\nu^*_h = \nu_h/[H^2(F N_0)^{1/2}]$, $l_v^* = l_v/H$, and $\text{Re}_m = N_0^2 F^2/H^4$ [=$(\zeta_0 f) / \text{Re}$], which is a modified Richardson number of the basic-state flow, where $\zeta_0$ is the vertical component of the absolute vorticity. The nondimensional viscous growth rate, $\sigma^*_v$, is given by

\[
\sigma^*_v = (\sigma - K)/(F N_0)^{1/2}
\]  
(A7)

The method of calculating CSI susceptibility, growth rates, and updraft slopes is found in Xu (1986), with further details found in Xu (1984) and Persson (1991). In this (unmodified) linear theory, Xu solves (A2), obtaining a transcendental function, $\Psi$, which depends on $N_0^2/N_d^2$, $\text{Re}_m$ and $F N_d$, $\nu^*$, and $l_v^*$. For this modified linear theory, $\Psi$ is also a function of $R$ (or, equivalently, a function of $S_0^2$). Therefore, when assessing the stability of a given point from model output, values of $R$ (= $S^2/S_0^2$), $N_0^2/N_d^2$, $\text{Re}_m$, and $F N_d$ are determined at the point, while the values of $\nu$ and $H$ for the simulation are estimated, thereby determining $\nu^*$. This leaves $\Psi$
as a function of only $l_w^*$, and the stability growth rate—updraft slope analysis procedure continues as described by the references.

The effects of the modification of the theory on the predicted growth rates and updraft widths are seen in Fig. A1. Allowing $R$ to vary has a significant effect on the growth rates for all values of $l_w$. In fact, if $R > 1$, the modified theory will predict that a much more limited range of updraft widths will be unstable, that the growth rates of the unstable modes will be substantially less than that predicted by Xu’s unmodified theory, and that the instability envelope curve will show fewer unstable points. The first two effects are illustrated in Fig. A1. The significance of the value of $R$ for determining the growth rate can be understood by noting that $R > 1$ represents a reduction in baroclinicity from the case of $R = 1$; that is, the moist baroclinicity is less than the dry baroclinicity ($S^* < S^0$). Since CSI is dependent on the strength of the baroclinicity, a reduction in baroclinicity will lead to a reduction in the growth rate of the expected CSI or possibly even to a stabilization. The slope of the updraft is also predicted to be noticeably less for the modified theory than for Xu’s unmodified theory (Fig. A1b) since the growth rate is less [see (A5)]. Comparing the unmodified theory with the modified theory for $R = 1.0$, we conclude that the separation of the diffusion into vertical and horizontal components affects the growth rates only for the smallest updraft width, since the effect for such a shallow updraft is mainly due to the addition of the horizontal diffusion, which is strongly scale selective. It is doubtful that the hydrostatic assumption has made much difference between the linear theory results for this specific case, since the shallow updraft slope indicates that the hydrostatic assumption is valid. More details of the derivation of the modified theory and the comparisons with the unmodified theory are provided by Persson (1991).

REFERENCES


