Nonnormality Increases Variance

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ABSTRACT

Recently, a new theoretical and conceptual model of quasigeostrophic turbulence has been advanced in which eddy variance is regarded as being maintained by transient growth of perturbations arising from sources including the nonlinear interactions among the eddies, but crucially without a direct contribution of unstable modal growth to the maintenance of variance. This theory is based on the finding that stochastic forcing of the subcritical atmospheric flow supports variance arising from induced transfer of energy from the background flow to the disturbance field that substantially exceeds the variance expected from the decay rate of the associated normal modes in an equivalent normal system. Herein the authors prove that such amplification of variance is a general property of the stochastic dynamics of systems governed by nonnormal evolution operators and that consequently the response of the atmosphere to unbiased forcing is always underestimated when consideration is limited to the response of the system's individual normal modes to stochastic excitation.

1. Introduction

A subset of perturbations in a baroclinic, barotropic, or mixed jet can exhibit substantial transient growth even when all perturbations decay with time in the asymptotic limit. The meteorological applications of this phenomenon have included development of a theory for the formation of cyclones (Farrell 1984, 1989; Montgomery and Farrell 1992) and for the growth of errors in numerical simulations (Lacarra and Talagrand 1988; Farrell 1990; Molteni and Palmer 1993; Mureau et al. 1993; see also Trefethen 1992).

It has sometimes been argued that these optimally growing perturbations are exceptional in the sense that in the absence of asymptotically growing modes transient growth could not sustain variance levels characteristic of the midlatitude atmosphere unless the initial conditions were biased to favor the optimally growing subset. The well-known result of constancy of perturbation energy for an initially isotropic set of perturbations imposed on unbounded constant shear flow is often cited as supporting this argument (Kraichnan 1976; Shepherd 1985; Farrell and Ioannou 1993a). It is plausible to conclude from this example that variance levels characteristic of the midlatitude atmosphere or of turbulent laboratory flows require at least the sporadic intercession of exponential instabilities, which through equilibration and nonlinear spectral scattering maintain the observed levels of variance (note that the mean level of perturbation kinetic energy in the midlatitude atmosphere is $\sim 10\%$ of the mean kinetic energy, while in turbulent laboratory flows the corresponding variance is $\sim 1\%$).

However, it has been recently found in physically motivated model studies of the midlatitude jet and laboratory channel flows (Farrell and Ioannou 1993b,c, 1994a,b) that unbiased stochastic forcing quite generally leads to maintained levels of variance that greatly exceed the variance levels resulting from the balance between energy accumulated from stochastic forcing and energy dissipated by the normal modes in isolation as would be anticipated from classical stochastic theory of normal dynamical systems (Wang and Uhlenbeck 1945). These examples reveal that extraction of energy from the background shear is a robust means of amplifying variance and strongly suggest that transient growth, which is a necessary consequence of the nonnormal evolution operator of the linearized dynamics, significantly contributes to maintaining the atmospheric variance in apparent contradiction to the earlier results obtained for unbounded constant shear flow.

Here we provide a resolution of this apparent contradiction. It is first shown that in nonnormal dynamical systems like the midlatitude atmosphere the maximum perturbation energy amplification over a given time interval results from a perturbation that is not of modal form and that this amplification exceeds that of the maximally growing mode over the same time interval. We also show that the minimum perturbation energy amplification is smaller than the amplification due to the most damped normal mode. From these two results alone it is not immediately obvious that any net energy gain can be achieved when all perturbations are excited

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in an unbiased fashion. However, for nonnormal dynamical systems this turns out to be necessarily the case, and a proof is provided that the mean perturbation energy growth at any time always exceeds the mean of the energy growth of the modes taken in isolation. Moreover, in the case of a stable modal spectrum stationary statistics exist, and it can be further shown that the maintained variance in nonnormal dynamical systems always exceeds the variance accumulated from the forcing by the individual modes in isolation, provided that the physical system can be approximated by a finite dimensional evolution operator. In addition, it is demonstrated that these are general properties of the linearized dynamics of systems governed by nonnormal operators.

2. Formulation and proof of the main result

A stochastically forced linear dynamical system can be represented in the general form

$$\frac{dx_i}{dt} = \mathcal{A}_{ij}x_j + \mathcal{F}_{ij}\epsilon_j, \qquad (1)$$

in which x is a complex state vector; $\mathcal{A}$ is the generally nonnormal operator of the linearized dynamical system; and ϵ is a Gaussian white-noise forcing, which is spatially δ correlated, with the spatial distribution of the forcings imposed by a unitary $\mathcal{F}$, by which all available spatial scales are equally excited. Note that with unitary forcing the specific distribution of the forcings, given by the columns of $\mathcal{F}$, does not affect the resulting statistics (Farrell and Ioannou 1993c). Generalized velocity coordinates have been chosen to represent the state vector so that the steady-state energy takes the simple form of the Euclidean norm of the state vector: $\langle E^\infty \rangle = \lim_{t\to\infty}\langle x^\dagger(t)x(t)\rangle$, which exists for asymptotically stable systems $\mathcal{A}$, for which necessarily all eigenvalues have negative real parts [we denote ensemble averaging by $\langle\ \rangle$, complex conjugation by $*$, and hermitian conjugation by †]. When the differential operator in (1) is approximated, as in all numerical simulations of the atmosphere, $\mathcal{A}$ is represented by a finite dimensional matrix. We note that for shear flows the governing linear operator $\mathcal{A}$ is nonnormal (i.e., $\mathcal{A}^\dagger\mathcal{A} \neq \mathcal{A}\mathcal{A}^\dagger$) with the consequence that the eigenvectors of the operator that correspond to distinct eigenvalues are not orthogonal.

It can be shown that for an asymptotically stable $\mathcal{A}$ the maintained variance is given by (Farrell and Ioannou 1993b)

$$\langle E^\infty \rangle = \mathrm{trace}\left(\int_0^\infty e^{\mathcal{A}t}\mathcal{F}\mathcal{F}^\dagger e^{\mathcal{A}^\dagger t}\,dt\right)$$
$$= \mathrm{trace}\left(\int_0^\infty e^{\mathcal{A}t}e^{\mathcal{A}^\dagger t}dt\right), \qquad (2)$$

which is independent of the spatial forcing distribution since $\mathcal{F}\mathcal{F}^\dagger = I$, where I is the identity matrix. Note that $E^t = x^\dagger e^{\mathcal{A}^\dagger t}e^{\mathcal{A}t}x$ gives the energy growth at time t for an initial perturbation x with unit energy, and trace (E^t) gives the mean growth at time t when all perturbations are equally excited.

We will prove that under the above conditions,

$$\langle E^\infty \rangle \geq \sum_{i=1}^{N} \frac{1}{-(\lambda_i + \lambda_i^*)}, \qquad (3)$$

where λ_i are the eigenvalues of $\mathcal{A}$, and N is the dimension of the system. The sum on the rhs of (3) is the variance that would result if the operator $\mathcal{A}$ were a normal operator with the same spectrum as $\mathcal{A}$, in which case the maintained variance is well known to be the sum of the inverse of twice the decay rates of the individual modes (Wang and Uhlenbeck 1945).

We first establish that the maximum energy increase that can be attained by any perturbation over a given time t is greater than that attained by the fastest growing mode and that the minimum energy attained by any perturbation is less than that attained by the least growing mode; that is,

$$\min\left(\frac{x^\dagger e^{\mathcal{A}^\dagger t}e^{\mathcal{A}t}x}{x^\dagger x}\right) \leq e^{(\lambda_i+\lambda_i^*)t} \leq \max\left(\frac{x^\dagger e^{\mathcal{A}^\dagger t}e^{\mathcal{A}t}x}{x^\dagger x}\right), \qquad (4)$$

where λ_i is any eigenvalue of $\mathcal{A}$ and the extrema are taken over all x. Consider the singular value decomposition $e^{\mathcal{A}t} = USV^\dagger$, where U and V are unitary and S diagonal with real nonnegative entries called singular values. Define $\mathcal{B} = V^\dagger e^{\mathcal{A}t}V$ so that $\mathcal{B}^\dagger\mathcal{B} = S^2$. The maximum energy attained by any perturbation is the greatest singular value of S^2, while the minimum energy attained is given by the least singular value of S^2. Note also that $\mathcal{B}$ is similar to $e^{\mathcal{A}t}$ and therefore $\mathcal{B}$ has the same eigenvalues as $e^{\mathcal{A}t}$. Consider the Rayleigh quotient,

$$\mathcal{Q}(x) = \frac{x^\dagger\mathcal{B}^\dagger\mathcal{B}x}{x^\dagger x}, \qquad (5)$$

which satisfies

$$\min(S^2) \leq \mathcal{Q}(x) \leq \max(S^2). \qquad (6)$$

Clearly, for any eigenvector x_i of $\mathcal{B}$ with eigenvalue λ_i we have $\mathcal{Q}(x_i) = e^{(\lambda_i+\lambda_i^*)t}$, and consequently,

$$\min\left(\frac{x^\dagger e^{\mathcal{A}^\dagger t}e^{\mathcal{A}t}x}{x^\dagger x}\right) \leq e^{(\lambda_i+\lambda_i^*)t} \leq \max\left(\frac{x^\dagger e^{\mathcal{A}^\dagger t}e^{\mathcal{A}t}x}{x^\dagger x}\right), \qquad (7)$$

which establishes the inequality.

Remarkably, in addition to (7) a further constraint is always true: at each instant in time the mean energy growth, when all perturbations are equally excited, exceeds the corresponding mean growth of the normal modes of $\mathcal{A}$ taken in isolation. Specifically, we will prove that for any finite dimensional nonnormal operator $\mathcal{A}$,

$$\text{trace}(e^{\mathcal{A}t}e^{\mathcal{A}^\dagger t}) \geq \sum_{i=1}^{N} e^{(\lambda_i+\lambda_i^*)t}, \quad (8)$$

and equality holds if and only if the operator is normal, which would require the absence of barotropic and baroclinic shear for a zonal atmospheric jet.

In order to prove (8) consider the Schur decomposition of $e^{\mathcal{A}t}$ (Golub and Van Loan 1989):

$$e^{\mathcal{A}t} = U(D + T)U^\dagger, \quad (9)$$

where U is unitary and D is diagonal with entries $e^{\lambda_i t}$, with λ_i the eigenvalues of $\mathcal{A}$, and T is an upper triangular matrix with zero elements along the diagonal. Since trace $(DT^\dagger) = 0$, we obtain

$$\text{trace}(e^{\mathcal{A}t}e^{\mathcal{A}^\dagger t}) = \text{trace}(DD^\dagger + TT^\dagger) \geq \text{trace}(DD^\dagger) = \sum_{i=1}^{N} e^{(\lambda_i+\lambda_i^*)t}, \quad (10)$$

which establishes inequality (8). Only when $\mathcal{A}$ is normal does $T = 0$ and equality hold.

Integrating (10) over time yields the desired inequality for the maintained variance:

$$\text{trace}\left(\int_0^\infty e^{\mathcal{A}t}e^{\mathcal{A}^\dagger t}dt\right) \geq \sum_{i=1}^{N} \frac{1}{-(\lambda_i + \lambda_i^*)}, \quad (11)$$

with equality only when $\mathcal{A}$ is normal. This result is a property of the nonnormality of the operator, and it holds regardless of whether there are perturbations that lead to initial growth: it holds even when all perturbations initially decay, that is, when all the eigenvalues of $\mathcal{A} + \mathcal{A}^\dagger$ are negative.

Alternatively, we can proceed in the frequency domain (Farrell and Ioannou 1993b, 1994a) using the fact that the ensemble average energy is given as an integral over real frequencies by

$$\langle E^\infty \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)d\omega, \quad (12)$$

with the individual frequency response

$$F(\omega) = \text{trace}[\mathcal{R}^\dagger(\omega)\mathcal{R}(\omega)] \quad (13)$$

expressed in terms of the resolvent of the operator $\mathcal{A}$:

$$\mathcal{R}(\omega) = (i\omega\mathcal{I} - \mathcal{A})^{-1}, \quad (14)$$

with $\mathcal{I}$ the identity. The eigenvalues of the resolvent are $(i\omega - \lambda_i)^{-1}$, and arguments similar to those that led to (8) show that

$$\text{trace}[\mathcal{R}^\dagger(\omega)\mathcal{R}(\omega)] \geq \sum_{i=1}^{N} \frac{1}{|i\omega - \lambda_i|^2}, \quad (15)$$

with equality only when $\mathcal{A}$ is normal. This inequality has the important implication that the frequency response of a nonnormal operator would always be underestimated if it were simply characterized as a summation of the contributions from the poles of the resolvent as would be the case if the operator were normal [for an illuminating discussion cf. Trefethen et al. (1993) and Reddy and Henningson (1993)]. In the atmosphere this implies, for instance, that the excitation of quasi-stationary perturbations is inevitably underestimated in atmospheric models in which the sources of nonnormality of the operator are ignored (i.e., models that do not include the baroclinic or barotropic shear or the departure from zonality of the background flow). Integration of (15) over all frequencies again gives inequality (11).

3. Conclusions

Because of the nonnormality of the linear dynamical operator governing the evolution of perturbations on a spatially varying atmospheric basic state, the maximum perturbation growth at any finite time exceeds the growth due to the maximally growing normal mode. In addition, the mean perturbation growth when all perturbations are equally excited always exceeds the corresponding mean growth of the modes taken in isolation. It follows that studies restricted to the growth of individual normal modes of the atmosphere lead necessarily to an underestimate of the instability of the atmosphere. Further, the maintained variance in the necessarily nonnormal atmospheric flow will always be underestimated by the variance maintained in an otherwise equivalent normal system.

We have shown this result for the case of finite dimensional operators, which covers all practical representations of atmospheric dynamics. Although the spectrum of the analytic operator governing the linear evolution of diffusively damped perturbations on a background flow in a bounded domain can be shown to consist of discrete points (DiPrima and Habetler 1969; Herron 1980), it is customary to idealize the dissipation of geophysical flows and disregard the zonal asymmetries of the background flow with the result that part of the spectrum is represented by a continuum. For example, the case of unbounded constant shear flow, for which analytic solutions exist, has often been investigated. In this flow there is no discrete spectrum, and isotropic initial perturbations in the inviscid case and the case of Rayleigh damping lead to no net energy amplification in the presence of shear compared to that found without shear (in which case the operator is normal), while higher-order dissipations lead only to modest increase with shear of the variance over that which would be maintained in the absence of shear (Kraichnan 1976; Shepherd 1985; Farrell and Ioannou 1993a). In the last of these references it was shown that in order to achieve a substantial increase of variance with shear when the spectrum is a continuum, it is necessary to model the role of modes as persistent structures. When this depository role was modeled as an occlusion a secular increase of variance with shear was found. While no natural persistent structures exist in two-dimensional unbounded constant shear flows there are persistent perturbations in shear flows in the three-dimensional planetary boundary layer, and for that case it can be shown that a subset of initial disturbances will evolve into slowly dissipating streamwise rolls, allowing a large accumulation of maintained variance compared to that maintained in the case with no shear (Butler and Farrell 1992; Reddy and Henningson 1993; Trefethen et al. 1993; Farrell and Ioannou 1993d,e, 1994a). Analogously, in idealized atmospheric flows a small number of modal solutions, albeit decaying, act as depository of the nonnormal growth of the continuum spectrum leading to large variances under stochastic excitation (Farrell and Ioannou 1993b).

The dominance of nonnormality of the linearized evolution operator in atmospheric dynamics makes it possible to accurately model the observed atmospheric variance and obtain the atmospheric eddy statistics by stochastic forcing of the linearized dynamical operator associated with the atmospheric flow (Farrell and Ioannou 1994b,c). These results suggest a model for turbulence in strongly maintained shear flows like the midlatitude atmospheric jet, or pressure-driven laboratory channel flows, in which the turbulent state is maintained by stochastic forcing of the background flow arising from nonlinear scattering of eddy energy perhaps augmented by other perturbation sources but in any case not dependent upon continual excitation from exponential instabilities. For this theory to be valid the nonmodal growth must be sufficiently large to allow possibly inefficient nonlinear scattering to provide adequate excitation for self-maintenance of the stochastic forcing (Farrell and Ioannou 1993c).

In this work it has been shown that an amplification inevitably results from nonnormal dynamics. Whether this amplification self-consistently supports the observed variance is the subject of continued study.

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