Collision Interactions of Solitons in a Baroclinic Atmosphere

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ABSTRACT

In this paper the interactions between two marginally unstable baroclinic wave packets in the two-layer Phillips model are investigated by using the multiple-scale method. It is shown that the interactions can be described by a set of two coupled nonlinear Schrödinger equations. Except for two special cases the equations have only four invariants of motion and cannot be solved by the inverse scattering method.

The equations are solved numerically to study the collision interactions between two solitons. It is found that though the coefficients in the equations are fixed, the behavior of the two solitons may be quite different, which is closely related to the initial states of the two solitons (the speeds and the amplitudes of the solitons well before the interactions). For some initial conditions the collision interactions may be soliton-like in that the properties of the two solitons change very little, while for other initial conditions some "inelastic" phenomena are observed: one soliton may be destroyed by the other, or two solitons may change their speeds and directions of propagation and fuse into a new bound state.

1. Introduction

The important role played by baroclinic waves in the large-scale dynamics of both the atmosphere and the ocean has been accepted since the early investigations by Charney (1947) and Eady (1949) of the linear stability of baroclinic flows. In the last two decades or more, considerable efforts have been made to obtain a theoretical understanding of the behavior of finite-amplitude baroclinic wave trains and wave packets. Pedlosky (1970, 1971, 1972a, 1979), Drazin (1970, 1972), Pedlosky and Frenzen (1980), Brindley and Moroz (1980), and Moroz and Brindley (1984) studied the time evolution of a single baroclinic wave train. They found that the amplitude oscillates periodically in time in the inviscid limit and approaches a steady solution in the strong viscous limit. In the intermediate case the behavior of the amplitude may be steady, oscillatory, or chaotic.

The resonant interactions of baroclinic waves were studied by Loesch (1974a,b) and Pedlosky (1975) for cases where one of the three waves is marginally unstable and the other two waves are neutral. They found oscillatory solutions analytically or numerically. The interaction of two unstable baroclinic waves were studied by Pedlosky and Polvani (1987). They obtained steady, periodic, or chaotic solutions for different parameter ranges.

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The time and space evolution of a baroclinic wave packet was studied by Pedlosky (1972b) by use of a two-layer model and by Moroz and Brindley (1981) by use of a continuously stratified model. They obtained the AB equations (where A is the amplitude of the wave packet and B is a measure of the modification of the mean flow induced by the wave packet). Gibbon et al. (1979) showed that the AB equations can be transformed either into the sine–Gordon equation or the so-called self-induced transparency equations, which can be solved by means of the inverse scattering method and possess a variety of soliton solutions.

In the work of Pedlosky (1972b) the theory of the finite-amplitude baroclinic wave packets was developed for a flow near the minimal critical shear. In this case there exists only one wave packet that is marginally unstable. The interactions between unstable wave packets belonging to different wave modes may not exist. As the vertical shear of the flow exceeds the minimal critical shear significantly, however, there are more and more wave packets that become unstable, and these wave packets may interact with each other. How do they behave when two or more wave packets meet? This problem is of course of theoretical and synoptic importance and little attention has been paid to it. In the present paper an attempt is made to explore this problem. As a first step, we will study the interactions between two baroclinic wave packets only. Within the framework of the weakly nonlinear theory we can treat only the marginally unstable waves, so the two wave packets considered hereinafter will be chosen to be marginally unstable ones.

2. Derivation of the evolution equations

The model used is the conventional two-layer model (e.g., see Pedlosky 1987). The model consists of two layers of incompressible, homogeneous fluids of slightly different densities in a gravitationally stable configuration (light fluid on top), rotating with angular speed Ω about the vertical axis. The fluid is contained in a channel of width L. The nondimensional quasi-geostrophic governing equations without dissipation and topographic forcing are

$$\left(\frac{\partial}{\partial t} + U_n \frac{\partial}{\partial x}\right)[\nabla^2\phi_n + F(\phi_1 - \phi_2)(-1)^n] + [\beta - F(U_1 - U_2)(-1)^n]\frac{\partial \phi_n}{\partial x} = -\mathbf{J}[\phi_n, \nabla^2\phi_n + F(\phi_1 - \phi_2)(-1)^n], \quad n = 1, 2. \tag{2.1}$$

The basic shear flow is U_n, and the parameter F is the square of the ratio of the channel width L to the deformation radius $(g'D)^{1/2}/f$, where D is the undisturbed depth of each layer. The parameter β is $\beta_* L^2/U$, where β_* is dimensional planetary vorticity gradient and U is the scale for the horizontal velocity field.

For the linear problem, instability occurs when

$$\beta < F(U_1 - U_2) \equiv \beta_M. \tag{2.2}$$

The critical curve is given by

$$\beta_l = \frac{U_1 - U_2}{2F} K^2(4F^2 - K^4)^{1/2}, \tag{2.3}$$

where K^2 is square of the total wavenumber, that is, $k^2 + l^2$. Thus, for each $\beta < \beta_M$ there are two values of K^2 for which a wavelike mode is just marginally stable—namely,

$$K^2 = \sqrt{2}F\left\{1 \pm \left[1 - \frac{\beta^2}{F^2(U_1 - U_2)^2}\right]^{1/2}\right\}^{1/2}. \tag{2.4}$$

The minus sign yields the long-wave solution K_1, while the plus sign yields the short marginal wave K_2 at the same value of β.

For each (β, K^2) pair, the phase speed of the marginal wave is given by

$$c = \frac{U_1 + U_2}{2} - \frac{\beta(K^2 + F)}{K^2(K^2 + 2F)}. \tag{2.5}$$

It can be shown that the long wave always travels with a phase speed less than U_2, while the short wave always travels with a speed greater than U_2.

We will examine the evolution of the wave packet pair for values of β slightly less than the critical value for a given shear; that is,

$$\beta = \beta_l - \Delta, \tag{2.6}$$

where β_l is given by (2.3). In the neighborhood of the critical points, ϕ_n will be a function of x, y, t and the long space scales and timescales:

$$T_1 = |\Delta|^{1/2}t, \quad T_2 = |\Delta|t, \quad X_1 = |\Delta|^{1/2}x, \quad X_2 = |\Delta|x. \tag{2.7}$$

Thus, in (2.1),

$$\begin{cases} \dfrac{\partial}{\partial t} \to \dfrac{\partial}{\partial t} + |\Delta|^{1/2}\dfrac{\partial}{\partial T_1} + |\Delta|\dfrac{\partial}{\partial T_2}, \\ \dfrac{\partial}{\partial x} \to \dfrac{\partial}{\partial x} + |\Delta|^{1/2}\dfrac{\partial}{\partial X_1} + |\Delta|\dfrac{\partial}{\partial X_2}. \end{cases} \tag{2.8}$$

The solution to (2.1) is sought in the form of an asymptotic series

$$\phi_n = |\Delta|^{1/2}\phi_n^{(1)} + |\Delta|\phi_n^{(2)} + |\Delta|^{3/2}\phi_n^{(3)} + \cdots. \tag{2.9}$$

The insertion of (2.6), (2.8), and (2.9) into (2.1) yields a sequence of linear problems for the $\phi_n^{(j)}$ after terms of like order in $|\Delta|^{1/2}$ are balanced.

The lowest-order solution at $O(|\Delta|^{1/2})$ yields

$$\phi_1^{(1)} = (A_1 e^{i\theta_1} + A_2 e^{i\theta_2}) \sin ly, \quad \phi_2^{(1)} = (\gamma_1 A_1 e^{i\theta_1} + \gamma_2 A_2 e^{i\theta_2}) \sin ly, \tag{2.10}$$

where

$$\theta_n = k_n(x - c_n t), \quad \gamma_n = \frac{K_n^2 - F}{F} - \frac{\beta_l + F(U_1 - U_2)}{F(U_1 - c_n)}. \tag{2.11}$$

In (2.11), the subscript n labels each of the two marginal wave packets, that is, solutions that satisfy (2.4) and (2.5). For each marginal wave packet, γ_n is the ratio between the amplitudes in the lower and upper layers. The asterisks in (2.10) denote the complex conjugate of the preceding term. The amplitudes A_n are functions of (X_1, X_2, T_1, T_2) and are still undetermined.

The $O(|\Delta|)$ problem for $\phi_n^{(2)}$ is

$$\left(\frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x}\right)[\nabla^2\phi_1^{(2)} - F(\phi_1^{(2)} - \phi_2^{(2)})] + \frac{\partial q_1}{\partial y}\frac{\partial \phi_1^{(2)}}{\partial x} = \sum_{N=1}^{2}\frac{\partial q_1/\partial y}{U_1 - c_n}\left\{\frac{\partial}{\partial T_1} + \left[U_1 - \frac{\partial q_1/\partial y - 2k_n^2(U_1 - c_n)}{(\partial q_1/\partial y)/(U_1 - c_n)}\right]\frac{\partial}{\partial X_1}\right\} \times A_n e^{i\theta_n}\sin ly + i\frac{l(\partial q_1/\partial y)(c_2 - c_1)}{2(U_1 - c_1)(U_1 - c_2)} \times [(k_1 + k_2)A_1A_2e^{i(\theta_1+\theta_2)} + (k_1 + k_2)A_1A_2^* e^{i(\theta_1 - \theta_2)}]\sin 2ly, \tag{2.12a}$$

$$\left(\frac{\partial}{\partial t} + U_2\frac{\partial}{\partial x}\right)[\nabla^2\phi_2^{(2)} - F(\phi_2^{(2)} - \phi_1^{(2)})]$$
$$+ \frac{\partial q_2}{\partial y}\frac{\partial \phi_2^{(2)}}{\partial x} = \sum_{n=1}^{2}\frac{(\partial q_1/\partial y)\gamma_n}{U_2 - c_n}\left\{\frac{\partial}{\partial T_1}\right.$$
$$\left.+ \left[U_2 - \frac{\partial q_2/\partial y - 2k_n^2(U_2 - c_n)}{(\partial q_2/\partial y)/(U_2 - c_n)}\right]\frac{\partial}{\partial X_1}\right\}$$
$$\times A_n e^{i\theta_n}\sin ly + i\frac{l(\partial q_2/\partial y)(c_2 - c_1)\gamma_1\gamma_2}{2(U_2 - c_1)(U_2 - c_2)}$$
$$\times [(k_1 - k_2)A_1A_2e^{i(\theta_1+\theta_2)}$$
$$+ (k_1 + k_2)A_1A_2^*e^{i(\theta_1-\theta_2)}]\sin 2ly, \quad (2.12b)$$

where

$$\frac{\partial q_n}{\partial y} = \beta_t - (-1)^n F(U_1 - U_2). \quad (2.13)$$

The inhomogeneities in Eqs. (2.12a,b) consist of two parts. One part is produced by the interaction between the two marginal waves, which gives rise to the "forced" solutions

$$\phi_{1f}^{(2)} = R_1A_1A_2e^{i(\theta_1+\theta_2)}\sin 2ly + Q_1A_1A_2^*e^{i(\theta_1-\theta_2)}\sin 2ly,$$
$$\phi_{2f}^{(2)} = R_2A_1A_2e^{i(\theta_1+\theta_2)}\sin 2ly$$
$$+ Q_2A_1A_2^*e^{i(\theta_1-\theta_2)}\sin 2ly, \quad (2.14)$$

where R_1, R_2, Q_1, and Q_2 are real constants given in appendix A. Another part is proportional to $\exp(ik_nx - i\omega_nt)\sin ly$. We attempt to find the particular solutions corresponding to this part in the form

$$\phi_{1p}^{(2)} = \sum_{n=1}^{2} A_n^{(2)}e^{i\theta_n}\sin ly,$$
$$\phi_{2p}^{(2)} = \sum_{n=1}^{2} B_n^{(2)}e^{i\theta_n}\sin ly. \quad (2.15)$$

Substitution into (2.12a,b) yields, after some algebra,

$$-\gamma_nA_n^{(2)} + B_n^{(2)} = \frac{\partial q_1/\partial y}{iFk_n(U_1 - c_n)^2}\left\{\frac{\partial}{\partial T_1} + \left[c_n + \frac{2k_n^2(U_1 - c_n)^2}{\partial q_1/\partial y}\right]\frac{\partial}{\partial X_1}\right\}A_n,$$
$$-\gamma_nA_n^{(2)} + B_n^{(2)} = -\frac{(\partial q_2/\partial y)\gamma_n^2}{iFk_n(U_2 - c_n)^2}\left\{\frac{\partial}{\partial T_1} + \left[c_n + \frac{2k_n^2(U_2 - c_n)^2}{\partial q_2/\partial y}\right]\frac{\partial}{\partial X_1}\right\}A_n. \quad (2.16)$$

Since the determinant of the coefficients of $A_n^{(2)}$ and $B_n^{(2)}$ vanishes, a solution is possible only if

$$\frac{2k_n^2}{iFk_n}(1 + \gamma_n^2)\frac{\partial A_n}{\partial X_1} = 0. \quad (2.17)$$

To obtain (2.17), we have used the relation

$$\frac{\partial q_1/\partial y}{(U_1 - c_n)^2} + \gamma_n^2\frac{\partial q_2/\partial y}{(U_2 - c_n)^2} = 0. \quad (2.18)$$

As γ_n is real and k_n is not equal to zero, (2.17) requires that

$$\frac{\partial A_n}{\partial X_1} = 0. \quad (2.19)$$

This condition does not imply that A_n is space independent; rather, it is a function of slower space scale X_2.

Equations (2.16) are hence redundant, and solving for $B_n^{(2)}$ yields

$$B_2^{(2)} = \gamma_nA_n^{(2)} + \frac{\partial q_1/\partial y}{iFk_n(U_1 - c_n)^2}\frac{\partial A_n}{\partial T_1}. \quad (2.20)$$

Without a loss in generality, we may set $A_n^{(2)}$ equal to zero.

To this order, the solutions can be written as

$$\phi_1^{(2)} = R_1A_1A_2e^{i(\theta_1+\theta_2)}\sin 2ly + Q_1A_1A_2^*e^{i(\theta_1-\theta_2)} \times \sin 2ly + \Phi_1(y, T_1, T_2, X_2),$$
$$\phi_2^{(2)} = R_2A_1A_2e^{i(\theta_1+\theta_2)}\sin 2ly + Q_2A_1A_2^*e^{i(\theta_1-\theta_2)} \times \sin 2ly + \sum_{n=1}^{2}\frac{\partial q_1/\partial y}{iFk_n(U_1 - c_n)^2}\frac{\partial A_n}{\partial T_1}e^{i\theta_n} \times \sin ly + \Phi_2(y, T_1, T_2, X_2). \quad (2.21a,b)$$

Here we have added the homogeneous solutions Φ_1 and Φ_2 of (2.12a,b), which present corrections to the zonal flow.

At this order both the mean flow corrections and the amplitudes of the wave packets are still undetermined. To determine them, the $O(|\Delta|^{3/2})$ problem for $\phi_n^{(3)}$ must be considered:

$$\left(\frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x}\right)[\nabla^2\phi_1^{(3)} + F(\phi_2^{(3)} - \phi_1^{(3)})]$$
$$+ \frac{\partial q_1}{\partial y}\frac{\partial \phi_1^{(3)}}{\partial x} = -\left(\frac{\partial}{\partial T_2} + U_1\frac{\partial}{\partial X_2}\right)[\nabla^2\phi_1^{(1)}$$
$$+ F(\phi_2^{(1)} - \phi_1^{(1)})] - 2\left(\frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x}\right)\frac{\partial^2\phi_1^{(1)}}{\partial x\partial X_2}$$
$$- \frac{\partial q_1}{\partial y}\frac{\partial \phi_1^{(1)}}{\partial X_2} + \frac{\Delta}{|\Delta|}\frac{\partial \phi_1^{(1)}}{\partial x} - \frac{\partial}{\partial T_1}[\nabla^2\phi_1^{(2)} + F(\phi_2^{(2)}$$
$$- \phi_1^{(2)})] - \mathbf{J}[\phi_1^{(1)}, \nabla^2\phi_1^{(2)} + F(\phi_2^{(2)} - \phi_1^{(2)})]$$
$$- \mathbf{J}[\phi_1^{(2)}, \nabla^2\phi_1^{(1)} + F(\phi_2^{(1)} - \phi_1^{(1)})], \quad (2.22a)$$

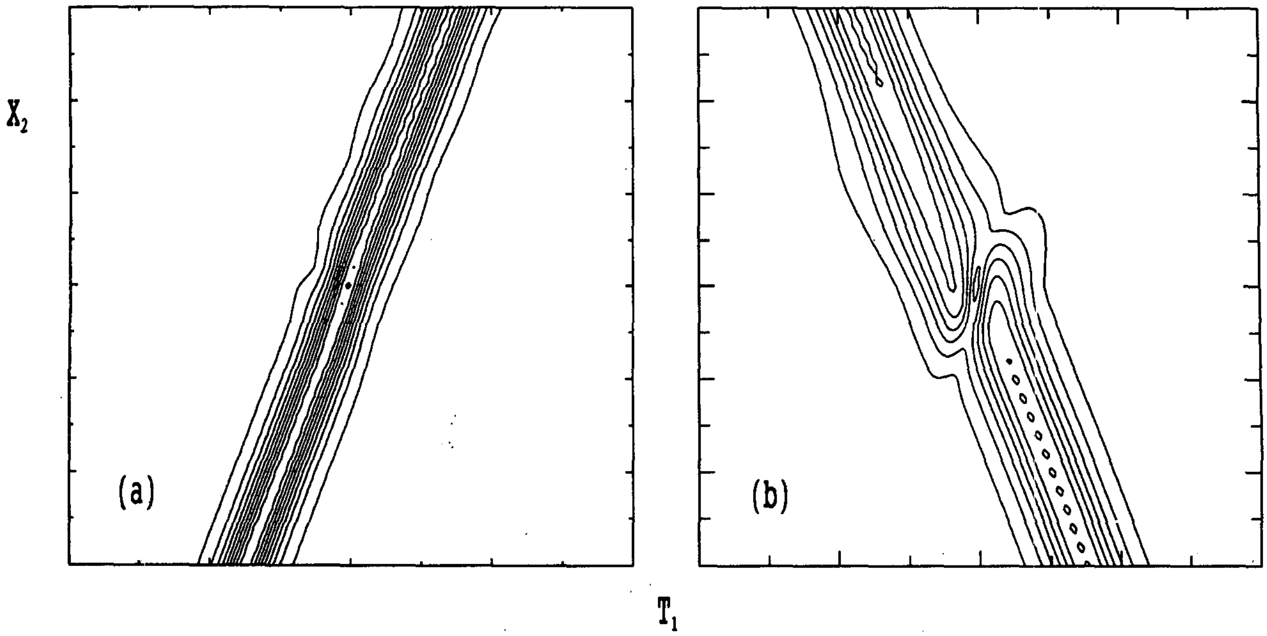

FIG. 1. Collision of a larger soliton with a small soliton: $\eta_1 = 1.0$, $\eta_2 = 0.6$, $T_0 = 15.0$, and $\Delta\xi = 1.0$. (a) Evolution of soliton 1; (b) evolution of soliton 2.

$$\left(\frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial x}\right)[\nabla^2\phi_2^{(3)} + F(\phi_1^{(3)} - \phi_2^{(3)})] + \frac{\partial q_2}{\partial y}\frac{\partial \phi_2^{(3)}}{\partial x} = -\left(\frac{\partial}{\partial T_2} + U_2 \frac{\partial}{\partial X_2}\right)[\nabla^2\phi_2^{(1)} + F(\phi_1^{(1)} - \phi_2^{(1)})] - 2\left(\frac{\partial}{\partial t} + U_2\frac{\partial}{\partial x}\right)\frac{\partial^2\phi_2^{(1)}}{\partial x \partial X_2} - \frac{\partial q_2}{\partial y}\frac{\partial\phi_2^{(1)}}{\partial X_2} + \frac{\Delta}{|\Delta|}\frac{\partial \phi_2^{(1)}}{\partial x} - \frac{\partial}{\partial T_1}[\nabla^2\phi_2^{(2)} + F(\phi_1^{(2)} - \phi_2^{(2)})] - \mathbf{J}[\phi_2^{(1)}, \nabla^2\phi_2^{(2)} + F(\phi_1^{(2)} - \phi_2^{(2)})] - \mathbf{J}[\phi_2^{(2)}, \nabla^2\phi_2^{(1)} + F(\phi_1^{(1)} - \phi_2^{(1)})]. \quad (2.22b)$$

The solutions given by (2.21a,b) then combine with the two marginal wave packets to provide inhomogeneities in (2.22a,b). Among them there are inhomogeneous terms independent of (x, t). Considering the form of the linear operators on the left sides of (2.22a,b), it is clear that these inhomogeneities are secular terms and must be removed, leading to conditions that relate the $O(|\Delta|)$ corrections to the mean flow to the wave amplitudes; that is,

$$\frac{\partial}{\partial T_1}\left[\frac{\partial^2\Phi_1}{\partial y^2} + F(\Phi_2 - \Phi_1)\right] = \sum_{n=1}^{2}\frac{l\partial q_1/\partial y}{(U_1 - c_n)^2}\frac{\partial}{\partial T_1}|A_n|^2 \sin 2ly,$$

$$\frac{\partial}{\partial T_1}\left[\frac{\partial^2\Phi_2}{\partial y^2} + F(\Phi_1 - \Phi_2)\right] = -\sum_{n=1}^{2}\frac{l\partial q_1/\partial y}{(U_1 - c_n)^2}\frac{\partial}{\partial T_1}|A_n|^2 \sin 2ly, \quad (2.23a,b)$$

with

$$\frac{\partial \Phi_n}{\partial y} = 0 \quad \text{at} \quad y = 0, \pi. \quad (2.24)$$

The solutions of (2.23a,b) are

$$\Phi_1 = -\Phi_2 = \Phi = -\frac{1}{4l^2 + 2F}\sum_{n=1}^{2}\frac{l\partial q_1/\partial y}{(U_1 - c_n)^2} \times |A_n|^2\left(\sin 2ly - \frac{\sqrt{2}l \sinh\sqrt{2F}(y - \pi/2)}{\sqrt{F}\cosh\sqrt{F/2}\pi}\right). \quad (2.25)$$

There are inhomogeneous terms in (2.22a,b) that are proportional to $\exp(i\theta_n)\sin ly$. To obtain nonzero solutions, as is done in the $O(|\Delta|)$ problem, the following conditions must be satisfied:

$$i\chi_1\frac{\partial A_1}{\partial X_2} + \alpha_1\frac{\partial^2 A_1}{\partial T_1^2} + (\beta_1|A_1|^2 + \lambda_1|A_2|^2)A_1 + \sigma_1 A_1 = 0,$$

$$i\chi_2\frac{\partial A_2}{\partial X_2} + \alpha_2\frac{\partial^2 A_2}{\partial T_1^2} + (\beta_2|A_2|^2 + \lambda_2|A_1|^2)A_2 + \sigma_2 A_2 = 0, \quad (2.26a,b)$$

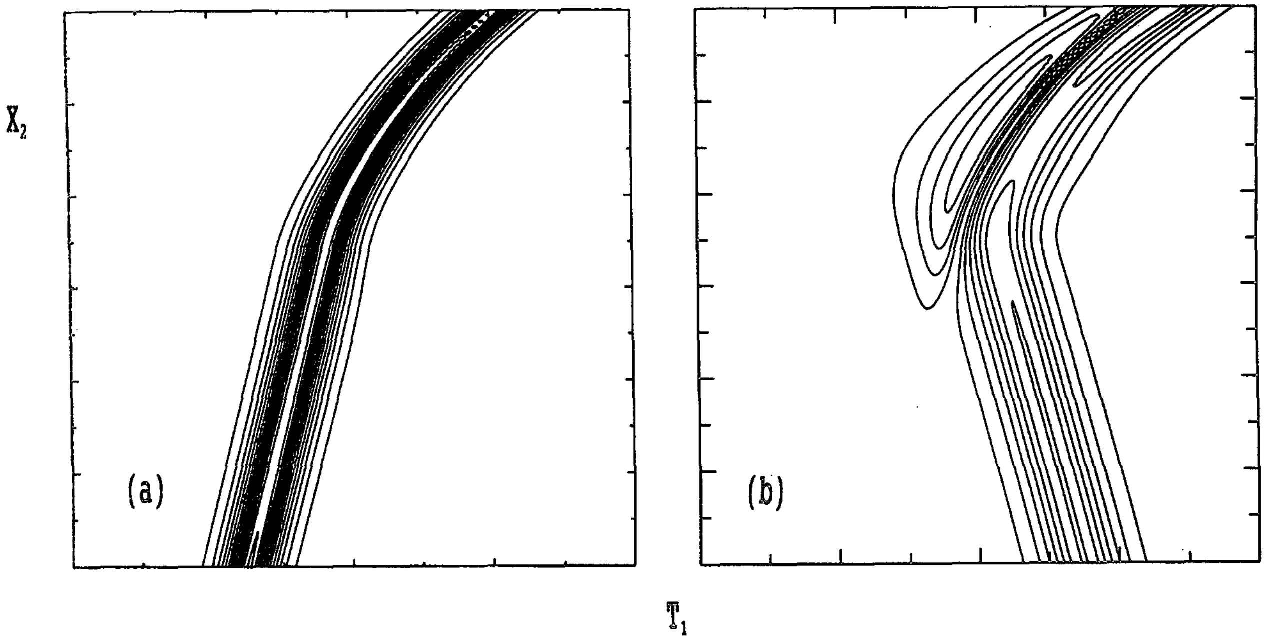

FIG. 2. As in Fig. 1 except $\Delta\xi = 0.2$.

where χ_n, α_n, β_n, λ_n, and σ_n are constants given in appendix B. Equations (2.26a,b) are the fundamental equations governing the evolution of the two interacting marginal baroclinic wave packets. In the following sections we will examine the properties of their solutions.

3. Soliton solution of the unstable nonlinear Schrödinger equation

If only one marginal wave packet (see A_1) is considered, it is clear from Eqs. (2.26a,b) that the evolution equation is

$$i\chi_1 \frac{\partial A_1}{\partial X_2} + \alpha_1 \frac{\partial^2 A_1}{\partial T_1^2} + \beta_1 |A_1|^2 A_1 + \sigma_1 A_1 = 0. \quad (3.1)$$

Equation (3.1) is the unstable nonlinear Schrödinger (UNS) equation. It becomes the conventional nonlinear Schrödinger equation under interchange of X_2 and T_1. It is also a special case of what is now often called the Ginzburg–Landau equation when the coefficients are complex. If we define

$$A_1(X_2, T_1) = e^{(i\sigma_1/\chi_1)X_2} B(X_2, T_1), \quad (3.2)$$

we can rewrite (3.1) in the following standard form of unstable nonlinear Schrödinger equation:

$$\chi_1 \frac{\partial B}{\partial X_2} + \alpha_1 \frac{\partial^2 B}{\partial T_1^2} + \beta_1 |B|^2 B = 0. \quad (3.3)$$

It is very interesting to note that Pedlosky (1972b) showed that the marginal wave packet for the minimum critical shear satisfies the *AB* equations. We show here that the marginal wave packet satisfies the UNS equation when the vertical shear is above the value of minimum critical shear. It is also interesting to note that when the β effect is excluded in the two-layer model, the marginal wave packet also satisfies the UNS equation (Gibbon and McGuinness 1981).

The unstable nonlinear Schrödinger equation is typical in unstable media (e.g., Yajima and Tanaka 1988), and it can be solved by the inverse scattering method (Yajima and Wadati 1990). The soliton consists of radiation plus a number of solitons. A single soliton solution to the UNS equation (3.1) can be obtained in the following way. We allow

$$A_1 = e^{i(\sigma_1/\chi_1 + r)X_2 - isT_1}\, \psi(\theta), \quad \theta = X_2 - UT_1, \quad (3.4)$$

where r and s are constants. On substitution, the ordinary differential equation for ψ is

$$\alpha_1 U^2 \psi'' + i(\chi_1 + 2\alpha_1 sU)\psi' - (r\chi_1 + \alpha_1 s^2)\psi + \beta_1 \psi^3 = 0. \quad (3.5)$$

We now choose

$$sU = -\frac{\chi_1}{2\alpha_1}, \quad r\chi_1 + \alpha_1 s^2 = \alpha_1 \eta^2. \quad (3.6)$$

Then ψ may be taken to be real and

$$\psi'' - \frac{\eta^2}{U^2}\psi + \frac{\beta_1}{\alpha_1 U^2}\psi^3 = 0. \quad (3.7)$$

It may be integrated once to

$$\psi' = \left[\frac{\beta_1}{2\alpha_1 U^2}\psi^2\left(\frac{2\alpha_1}{\beta_1}\eta^2 - \psi^2\right)\right]^{1/2}, \quad (3.8)$$

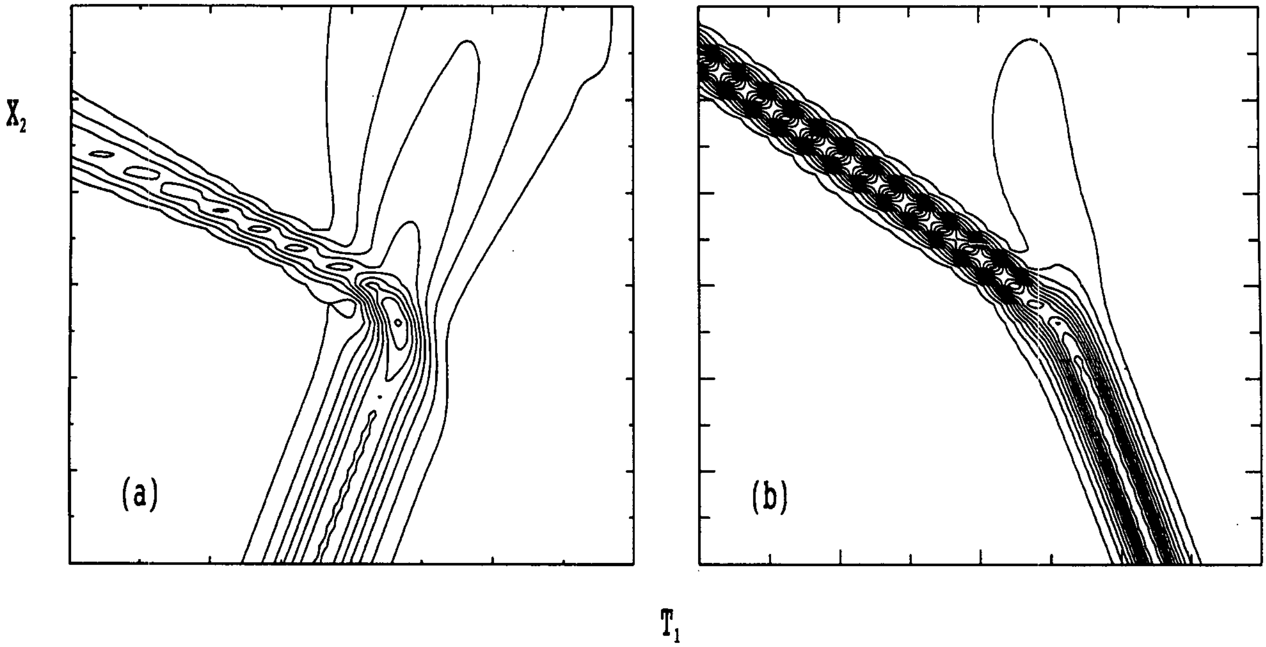

FIG. 3. Collision of a larger soliton with a small soliton: $\eta_1 = 0.5$, $\eta_2 = 1.0$, $T_0 = 15.0$, and $\Delta\xi = 1.0$. (a) Evolution of soliton 1; (b) evolution of soliton 2.

which can be solved in elliptic functions. The limiting case of the solitary wave is possible when $\beta_1/\alpha_1 > 0$. The solution is

$$\psi = \sqrt{\frac{2\alpha_1}{\beta_1}}\,\eta\,\mathrm{sech}\left[-\frac{\eta}{U}(X_2 - UT_1)\right]. \qquad (3.9)$$

On substitution into (3.4), we obtain

$$A_1 = \sqrt{\frac{2\alpha_1}{\beta_1}}\,\eta\,\mathrm{sech}\left[-\frac{\eta}{U}(X_2 - UT_1)\right] \times \exp\left[i\frac{\alpha_1}{\chi_1}\left(\eta^2 + \frac{\sigma_1}{\alpha_1} - s^2\right)X_2 - isT_1\right], \qquad (3.10)$$

where the parameters η and U are independent parameters and are determined by the initial state of A_1, and s is given by (2.6). The solution (3.10) represents in envelope solitary wave. Clearly, the velocity of the envelope soliton is independent of its amplitude, while the velocity of the carrier wave is amplitude dependent. It is also clear from (3.10) that the width of the soliton is directly proportional to its velocity, inversely proportional to its amplitude.

4. Collision interactions of solitons

The interactions between two marginal waves are described by the coupled equations (2.26). These equations are usually called the coupled nonlinear Schrödinger equations, which also appear in other fields of physics (see, e.g., Manakov 1973; Crosignani et al. 1982). Zakharov and Schulman (1982) have shown that in the case

$$\frac{\alpha_1}{\chi_1} = \frac{\alpha_2}{\chi_2}, \quad \frac{\beta_1}{\chi_1} = \frac{\beta_2}{\chi_2} = \frac{\lambda_1}{\chi_1} = \frac{\lambda_2}{\chi_2}, \qquad (4.1)$$

or

$$\frac{\alpha_1}{\chi_1} = -\frac{\alpha_2}{\chi_2}, \quad \frac{\beta_1}{\chi_1} = \frac{\beta_2}{\chi_2} = -\frac{\lambda_1}{\chi_1} = -\frac{\lambda_2}{\chi_2}, \qquad (4.2)$$

the system (2.26) has a infinite number of conservation laws and can be solved by the inverse scattering method. Except for the above two cases the system (2.26) has only four conservation laws and is not integrable; that is, it cannot be solved by the inverse scattering method. The four conservation laws can be easily obtained from (2.26); they are

$$\frac{d}{dX_2}\int_{-\infty}^{\infty} |A_1|^2 dT_1 = 0, \qquad (4.3)$$

$$\frac{d}{dX_2}\int_{-\infty}^{\infty} |A_2|^2 dT_1 = 0, \qquad (4.4)$$

$$\frac{d}{dX_2}\int_{-\infty}^{\infty}\left(A_1\frac{\partial A_1^*}{\partial T_1} + \frac{\lambda_1}{\lambda_2}A_2\frac{\partial A_2^*}{\partial T_1}\right)dT_1 = 0, \qquad (4.5)$$

$$\frac{d}{dX_2}\int_{-\infty}^{\infty}\left(\alpha_1\left|\frac{\partial A_1}{\partial T_1}\right|^2 + \alpha_2\frac{\lambda_1}{\lambda_2}\left|\frac{\partial A_2}{\partial T_1}\right|^2 - \frac{\beta_1}{2}|A_1|^4 - \frac{\beta_2}{2}\frac{\lambda_1}{\lambda_2}|A_2|^4 - \lambda_1|A_1|^2|A_2|^2\right)dT_1 = 0. \qquad (4.6)$$

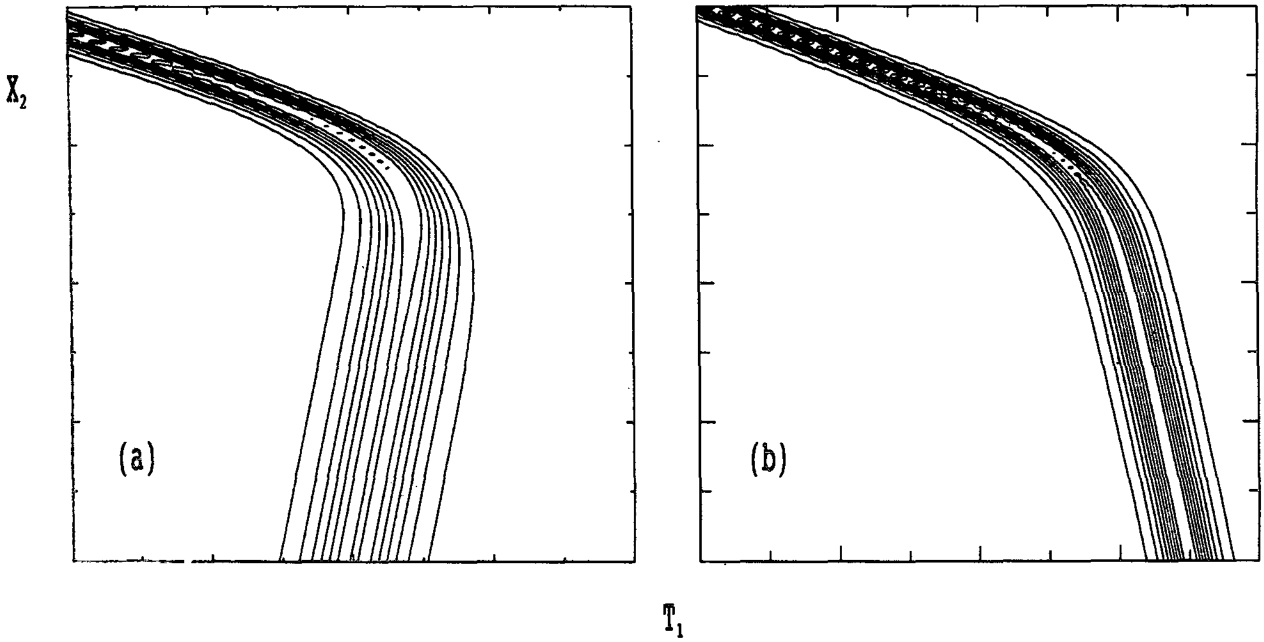

Fig. 4. As in Fig. 3 except $\Delta\xi = 0.2$.

It is clear from (4.3) and (4.4) that each wave conserves its energy. In other words there is no interchange of energy between the two wave packets.

In the atmosphere the condition (4.1) or (4.2) cannot be satisfied in general, so the system (2.26) is not integrable. In the following we will study the collisions between two solitons numerically. We take $U_1 - U_2 = 1.5$, $F = 3.0$, $l = 1.0$, and $\beta = 3.0$. In this case we have $\chi_1 = -1.2$, $\alpha_1 = 3.1$, $\beta_1 = 3.1$, $\lambda_1 = -2.7$; $\chi_2 = -1.0$, $\alpha_2 = -2.4$, $\beta_2 = -3.5$; and $\lambda_2 = 2.5$. We may set the initial conditions as[1]

$$A_1(X_2 = 0, T_1) = \sqrt{\frac{2\alpha_1}{\beta_1}}\,\eta_1 \operatorname{sech}[\eta_1(T_1 + T_0)] \times \exp[-is_1(T_1 + T_0)], \quad (4.7)$$

$$A_2(X_2 = 0, T_1) = \sqrt{\frac{2\alpha_2}{\beta_2}}\,\eta_2 \operatorname{sech}(\eta_2 T_1) \exp(-is_2 T_1). \quad (4.8)$$

Well before the collision, the evolution of the two soliton is

$$A_1 = \sqrt{\frac{2\alpha_1}{\beta_1}}\,\eta_1 \operatorname{sech}[\eta_1(T_1 + T_0 - \xi_1 X_2)] \times \exp\left[i\frac{\alpha_1}{\chi_1}\left(\frac{\sigma_1}{\alpha_1} + \eta_1^2 - s_1^2\right)X_2 - is_1(T_1 + T_0)\right], \quad (4.9)$$

$$A_2 = \sqrt{\frac{2\alpha_2}{\beta_2}}\,\eta_2 \operatorname{sech}[\eta_2(T_1 - \xi_2 X_2)] \times \exp\left[i\frac{\alpha_2}{\chi_2}\left(\frac{\sigma_2}{\alpha_2} + \eta_2^2 - s_2^2\right)X_2 - is_2 T_1\right], \quad (4.10)$$

where $\xi_n = 1/U_n$ and the relation between U_n and S_n is given by (3.4).

Numerical simulations show that the collision behavior of the two solitons may be quite different depending on the initial amplitudes and velocities of the solitons. The numerical results are given in Figs. 1–6. In these figures the horizontal axis is the slow time T_1, the vertical axis is the slow space X_2, and the curves are the contour lines of $|A_1|^2$ or $|A_2|^2$. Figures 1 and 2 show collisions of a larger soliton with a smaller soliton: here $\eta_1 = 1.0$, $\eta_2 = 0.6$, and $T_0 = 15.0$. Figure 1 shows the fast collision: $\Delta\xi = \xi_1 - \xi_2 = 1.0$. As can be seen, in this case the solitons maintain their shapes and velocities unchanged; they only suffer a phase shift. Figure 2 shows their slow collision: here $\Delta\xi = 0.2$. The collision makes the larger soliton propagate faster, while the smaller soliton is reflected back. Fi-

[1] Here we solved the nonstandard initial value problem, not the standard one. The reader is referred to appendix C for a discussion about this problem.

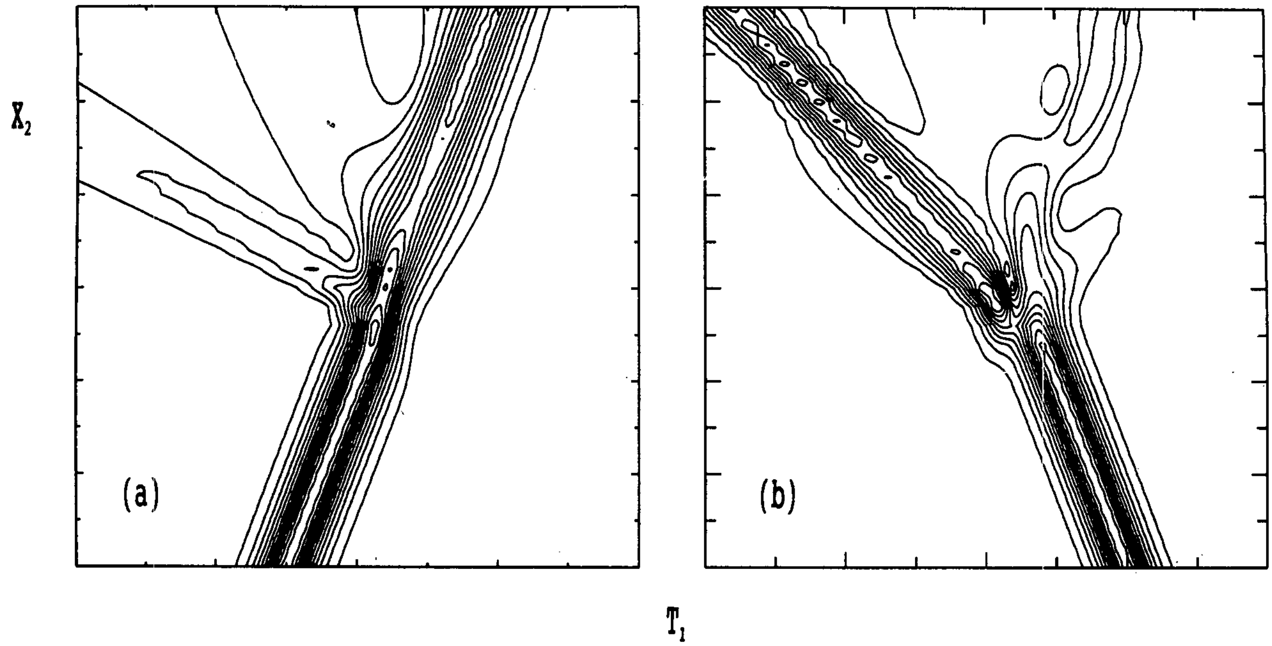

FIG. 5. Collision of two solitons of nearly equal strength: $\eta_1 = 1.0$, $\eta_2 = 1.0$, $T_0 = 10.0$, and $\Delta\xi = 1.0$. (a) Evolution of soliton 1; (b) evolution of soliton 2.

nally, they propagate together, making up a new bound state.

Figures 3 and 4 also show collisions of a larger soliton with a smaller soliton, but here $\eta_1 = 0.5$, $\eta_2 = 1.0$, and $T_0 = 15.0$. Figure 3 shows the fast collision: $\Delta\xi = 1.0$. As can be seen, the larger soliton penetrates the smaller one. After the collision, it propagates faster than before the collision and its amplitude oscillates. While the small soliton is broken into two parts, the energy in each part disperses. Figure 4 shows the slow collision: $\Delta\xi = 0.2$. In this case, the smaller soliton is reflected back, and the larger soliton accelerates. Fi-

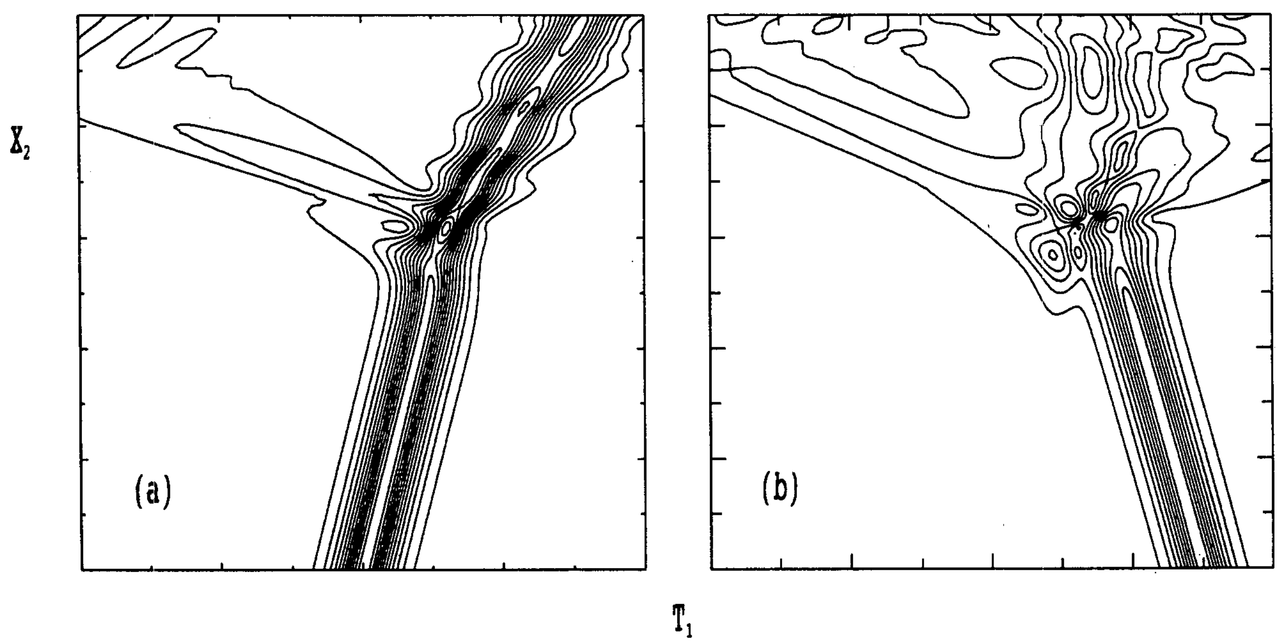

FIG. 6. As in Fig. 5 except $\Delta\xi = 0.2$.

nally, they fuse into a new bound state and their amplitudes oscillate. It is significant that the velocity of the bound state is about 7.5 times as large as the original velocity of the larger soliton.

Figures 5 and 6 show collisions of two solitons of nearly equal strength: here $\eta_1 = 1.0$, $\eta_2 = 1.0$, and $T_0 = 10.0$. Figure 5 shows the fast collision: $\Delta\xi = 1.0$. It can be seen from this figure that the two solitons penetrate each other and radiate a small amount of their energy. When the collision is slow, the situation is quite different: soliton 1 is transmitted, whereas soliton 2 is annihilated completely (see Fig. 6).

5. Concluding remarks

We have studied the baroclinic wave packets and their interactions. It is shown that away from the minimal vertical shear, a marginal baroclinic wave packet in the two-layer model satisfies the unstable nonlinear Schrödinger equation, which can be solved by the inverse scattering method and has envelope soliton solutions. It is also shown that two interacting marginal baroclinic wave packets can be described by a set of two coupled nonlinear Schrödinger equations. Except for two special cases these equations cannot be solved by the inverse scattering method. The collision interactions between two solitons have been studied numerically. The numerical simulations show that the collision behavior of the solitons may be quite different depending on the initial amplitudes and velocities of the solitons. For some initial conditions the collision may be soliton-like in the sense that the properties of the solitons change very little. For other initial conditions some "inelastic" phenomena are observed: one soliton may be destroyed completely by collision, or the solitons may change their speeds and directions of propagation and fuse into a new bound state.

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APPENDIX A

Parameters of the Forced Solutions

The constants appearing in the forced solutions in (2.14a,b) may be written

$$R_1 = \frac{\alpha_{12} l}{2k_{12}} \left[\frac{(c_2 - c_1)\partial q_1/\partial y}{(U_1 - c_1)(U_1 - c_2)(U_1 - \omega_{12}/k_{12})} \times \left(\frac{\partial q_2/\partial y}{U_2 - \omega_{12}/k_{12}} - (a_{12}^2 + F) \right) - \frac{F\partial q_2/\partial y(c_2 - c_2)\gamma_1\gamma_2}{(U_2 - c_1)(U_2 - c_2)(U_2 - \omega_{12}/k_{12})} \right] \times D^{-1}(k_{12}, 2l, \omega_{12}), \quad \text{(A1)}$$

$$R_2 = \frac{\alpha_{12} l}{2k_2} \left[\frac{(c_2 - c_1)\partial q_2/\partial y \gamma_1\gamma_2}{(U_2 - c_1)(U_2 - c_2)(U_2 - \omega_{12}/k_{12})} \times \left(\frac{\partial q_1/\partial y}{U_1 - \omega_{12}/k_{12}} - (a_{12}^2 + F) \right) - \frac{F\partial q_1/\partial y(c_2 - c_1)}{(U_1 - c_1)(U_1 - c_2)(U_1 - \omega_{12}/k_{12})} \right] \times D^{-1}(k_{12}, 2l, \omega_{12}), \quad \text{(A2)}$$

$$Q_1 = \frac{k_{12} l}{2\alpha_{12}} \left[\frac{(c_2 - c_1)\partial q_1/\partial y}{(U_2 - c_1)(U_2 - c_2)(U_2 - \sigma_{12}/\alpha_{12})} \times \left(\frac{\partial q_2/\partial y}{U_2 - \sigma_{12}/\alpha_{12}} - (d_{12}^2 + F) \right) - \frac{F\partial q_2/\partial y(c_2 - c_1)\gamma_1\gamma_2}{(U_2 - c_1)(U_2 - c_2)(U_2 - \sigma_{12}/\alpha_{12})} \right] \times D^{-1}(\alpha_{12}, 2l, \sigma_{12}), \quad \text{(A3)}$$

$$Q_2 = \frac{k_{12} l}{2\alpha_{12}} \left[\frac{(c_2 - c_1)\partial q_2/\partial y \gamma_1\gamma_2}{(U_2 - c_1)(U_2 - c_2)(U_2 - \sigma_{12}/\alpha_{12})} \times \left(\frac{\partial q_1/\partial y}{U_1 - \sigma_{12}/\alpha_{12}} - (d_{12}^2 + F) \right) - \frac{F\partial q_1/\partial y(c_2 - c_1)}{(U_1 - c_1)(U_1 - c_2)(U_1 - \sigma_{12}/\alpha_{12})} \right] \times D^{-1}(\alpha_{12}, 2l, \sigma_{12}), \quad \text{(A4)}$$

where

$$\alpha_{12} = k_1 - k_2, \quad k_{12} = k_1 + k_2, \quad \sigma_{12} = k_1 c_1 - k_2 c_2, \quad \omega_{12} = k_1 c_1 + k_2 c_2, \quad a_{12}^2 = (k_1 + k_2)^2 + 4l^2, \quad d_{12}^2 = (k_1 - k_2)^2 + 4l^2. \quad \text{(A5)}$$

The function D is defined as

$$D(k, l, \omega) \equiv \left[\frac{\partial q_1/\partial y}{U_1 - \omega/k} - (K^2 + F) \right] \times \left[\frac{\partial q_2/\partial y}{U_2 - \omega/k} - (K^2 + F) \right] - F^2. \quad \text{(A6)}$$

APPENDIX B

Constants in Evolution Equations

The coefficients appearing in (2.26a,b) may be written

$$\chi_n = -\frac{2k_n(1 + \gamma_n^2)}{F} \quad \text{(B1)}$$

$$\alpha_n = \frac{\partial q_1/\partial y(U_2 - U_1)K_n^4}{2F^3 k_n^2 (U_1 - c_n)^3 (U_2 - c_n)}, \quad \text{(B2)}$$

$$\beta_n = \frac{1}{(U_1 - c_n)^2} Y_n, \tag{B3}$$

$$\lambda_1 = \lambda_1^* + \frac{1}{(U_1 - c_2)^2} Y_1, \tag{B4}$$

$$\lambda_2 = \lambda_2^* + \frac{1}{(U_1 - c_2)^2} Y_2, \tag{B5}$$

$$\sigma_n = \frac{1}{F}\left[\frac{1}{U_1 - c_n} + \frac{\gamma_n^2}{U_1 - c_n}\right]\frac{\Delta}{|\Delta|}, \tag{B6}$$

where

$$Y_n = \frac{l^2 \partial q_1/\partial y}{F}\left(\frac{1}{U_1 - c_n} - \frac{\gamma_n^2}{U_2 - c_n}\right) - \frac{l^2 \partial q_1/\partial y}{4l^2 + 2F}\left(\frac{\partial q_1/\partial y}{F(U_1 - c_n)^2} - \frac{\partial q_2/\partial y \gamma_n^2}{F(U_2 - c_n)^2}\right) \times \left(1 + \frac{8l^2}{\sqrt{F/2}\,(4l^2 + 2F)} \tanh \sqrt{\frac{F}{2}}\pi\right), \tag{B7}$$

$$\lambda_1^* = \frac{1}{Fk_1}\left(\frac{m_{12}}{U_1 - c_1} + \frac{\gamma_1 n_{12}}{U_2 - c_1}\right), \tag{B8}$$

$$\lambda_2^* = \frac{1}{Fk_2}\left(\frac{m_{21}}{U_1 - c_2} + \frac{\gamma_2 n_{21}}{U_2 - c_2}\right). \tag{B9}$$

Here m_{12} and n_{12} are defined as

$$m_{12} = l\frac{\alpha_{12} + 2k_2}{2}\{[F(Q_2 - Q_1) - (\alpha_{12}^2 + 4l^2)Q_1] - [-(K_2^2 + F) + F\gamma_2]Q_1\} + l\frac{k_{12} - 2k_2}{2} \times \{[F(R_2 - R_1) - (k_{12}^2 + 4l^2)R_1] - [-(K_2^2 + F) + F\gamma_2]R_1\}, \tag{B10}$$

$$n_{12} = l\frac{\alpha_{12} + 2k_2}{2}\{\gamma_2[F(Q_1 - Q_2) - (\alpha_{12}^2 + 4l^2)Q_2] - [-\gamma_2(K_2^2 + F) + F]Q_2\} + l\frac{k_{12} - 2k_2}{2} \times \{\gamma_2[F(R_1 - R_2) - (k_{12}^2 + 4l^2)R_2] - [-\gamma_2(K_2^2 + F) + F]Q_2\}, \tag{B11}$$

where α_{mn}, k_{mn}, R_n, Q_n are defined in appendix A. The numbers m_{21} and n_{21} are given by the above expression, in which

$$\alpha_{12} \rightarrow -\alpha_{12},\ k_{12} \rightarrow k_{12},\ \gamma_2 \rightarrow \gamma_1,\ k_2 \rightarrow k_1. \tag{B12}$$

APPENDIX C

Discussion on the Use of the Nonstandard Initial Value Problem

Usually in the study of problems of wave propagation a wave motion is described in the way of the so-called standard initial value problem. First, the variation of some wave quantity with space, the so-called spatial wave, is specified at an initial time. Then, the evolution of this spatial wave with time is examined. For the coupled nonlinear Schrödinger equations we have not succeeded in solving the standard initial value problem. So when studying the collision interactions of two solitons of the coupled nonlinear Schrödinger equations in section 4 we used the nonstandard initial value problem instead. First, the packets are specified as functions of time at an initial position, which are called temporal waves; then, the evolution of these temporal waves with space is examined.

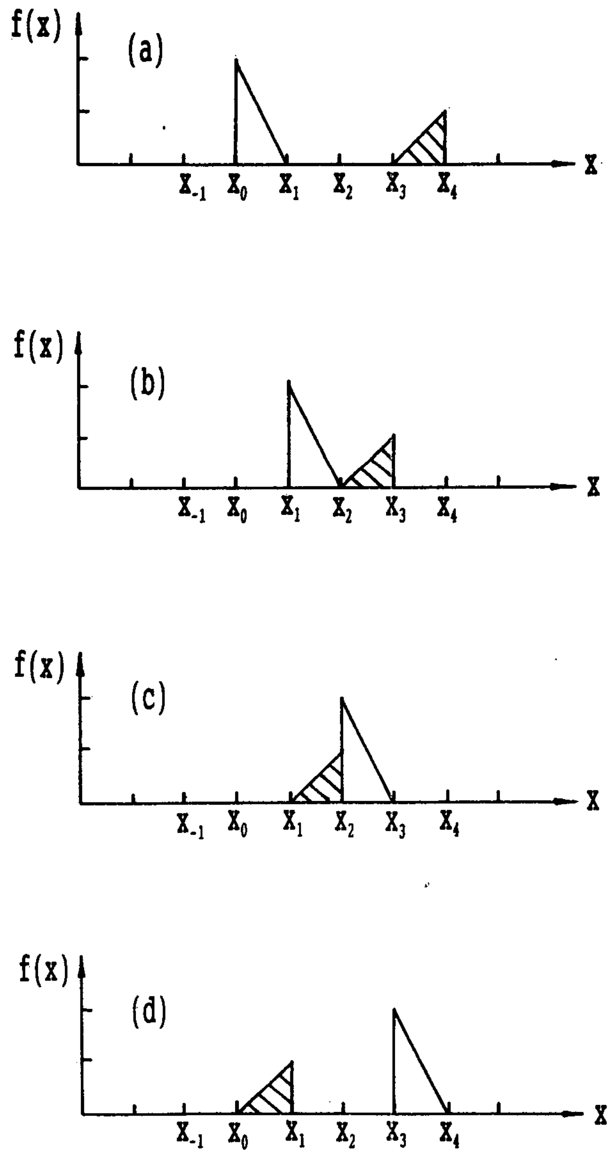

FIG. C1. Illustration of collision of two waves in the way of the standard initial value problem: (a) $t = t_0$, (b) $t = t_1$, (c) $t = t_2$, and (d) $t = t_3$.

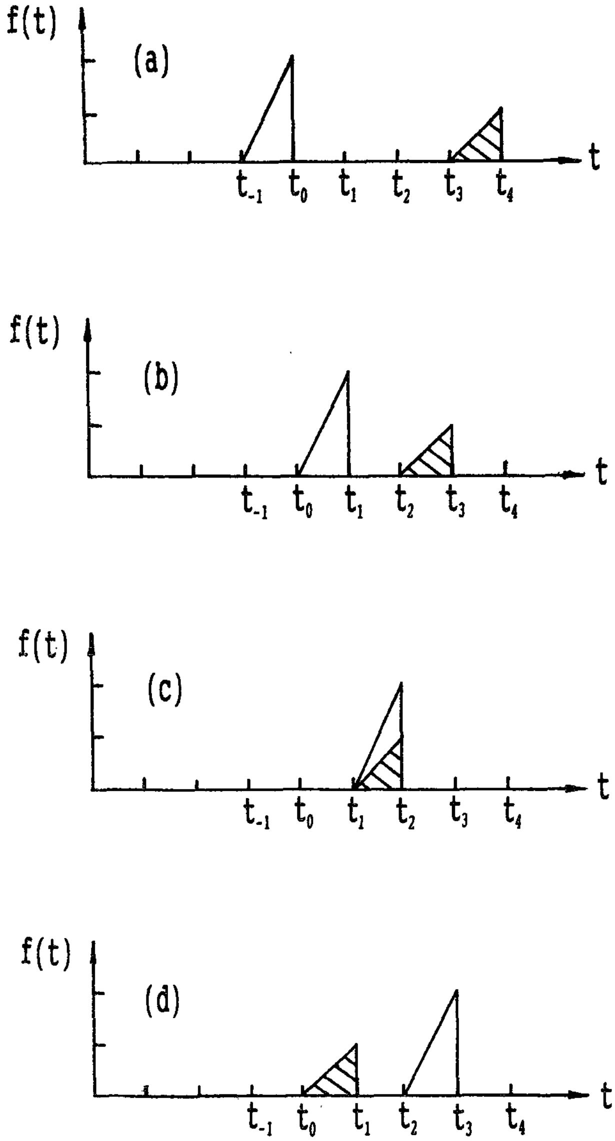

FIG. C2. Illustration of collision of two waves in the way of the nonstandard initial value problem: (a) $x = x_0$, (b) $x = x_1$, (c) $x = x_2$, and (d) $x = x_3$.

As the nonstandard initial value problem is not used as frequently as the standard one in the work of wave problems, the reader might not be familiar with it. In the following we will give an example of a head-on collision of two waves to see how the standard and the nonstandard initial value problems describe the same wave phenomenon. Figure C1 illustrates the collision process in the way of the standard initial value problem. Initially, that is, at t_0, there are two waves along the x axis. Wave 1 is located between x_0 and x_1, propagating right; wave 2 is located between x_3 and x_4, propagating left. Panels b, c, and d of Fig. C1 show their later evolution. These figures are self-explanatory; we do not give any further explanation about these figures, only point out that for simplicity and clearness we have assumed that the shapes and speeds of the two waves remain unchanged before, during, and after the collision.

In order to describe the above process in the way of the nonstandard initial value problem, let us examine some space points of the x axis in Fig. C1 to see how these two waves pass them. It can be seen from this figure that wave 1 passes through point x_0 during the time interval $t_1 - t_0$, while wave 2 passes through it during the time interval $t_3 - t_4$. Wave 1 passes through point x_1 during the time interval $t_0 - t_1$; wave 2 passes through it during the time interval $t_1 - t_2$. Wave 1 passes through point x_3 during the time interval $t_2 - t_3$; wave 2 passes through it during the time interval $t_0 - t_1$.

Using the above analysis, we are able to describe the collision process in the way of the nonstandard initial value problem. Figure C2 shows this process. If we take $x = x_0$ as the initial position, it can be seen from panel a of Fig. C2 that at the initial position, there are two temporal waves located at different positions of the time axis. Wave 1 is located between t_{-1} and t_0, propagating right; wave 2 located between t_3 and t_4, propagating left. These two temporal waves are a reflection of the two spatial waves, which are located at different positions of space and pass through the point x_0 at different times. The later evolution of these two temporal waves with space is given in panels b, c, and d. These figures describe the collision process as clearly as in Fig. C1.

The above discussion shows that both the standard and nonstandard initial value problems can be used to describe wave motions; they describe the same wave phenomenon in different ways.

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