Collision Interactions of Envelope Rossby Solitons in a Barotropic Atmosphere

BENKUI TAN

Department of Geophysics, Peking University, Beijing, People's Republic of China

(Manuscript received 14 June 1994, in final form 26 December 1995)

ABSTRACT

A theory is developed here to describe the propagation of nonlinear Rossby wave packets in a barotropic atmospheric model and their interactions by using the multiple-scale method. It is shown that the propagation of a single Rossby wave packet can be described by the nonlinear Schrödinger equation that has envelope soliton solutions. For two interacting packets with slightly different wavenumbers they satisfy a set of two coupled nonlinear Schrödinger equations. These equations are used to study the collision interactions of two envelope Rossby solitons. It is found that despite the complexity of the interaction, the energy of each soliton is conserved, while the shapes and velocities of the two solitons may be altered significantly by the interaction. The action of one soliton on another is realized by providing a field of force or potential for it through the cross-modulation terms.

1. Introduction

The nonlinear Rossby wave packets have been the subject of considerable study in the past years. Yamagata (1980) and Boyd (1983) studied the nonlinear barotropic Rossby wave packets by use of a midlatitude beta-plane model and a equatorial beta-plane model, respectively. They found the packets can be described by the famous nonlinear Schrödinger equation that can be solved by the inverse-scattering method and possesses the envelope soliton solutions. The nonlinear baroclinic Rossby wave packets were investigated by Pedlosky (1972) and Moroz and Brindley (1981) by use of a two-layer model and a continuously stratified model, respectively. They found that the baroclinic Rossby wave packets satisfy the so-called AB equations (where A is the amplitude of the wave packet and B is a measure of the modification of the mean flow induced by the wave packet). Gibbon et al. (1979) showed that the AB equations can be transformed either into the sine–Gordon equation or the so-called self-transparency equations that can also be solved by the inverse scattering method and has a variety of soliton or envelope soliton solutions. Later, Moroz and Brindley extended their previous work by including the effects of dissipation and topographic forcing. They obtained various amplitude equations for different parameter ranges but they did not discuss the equations further.

All the work quoted above is theoretical work. Recently, the observational and numerical studies of nonlinear Rossby wave packets have attracted more and more attention (e.g., Lee and Held 1993; Swanson and Pierehumbert 1994). Lee and Held pointed out that the Rossby wave packets are observed in both hemispheres, notably in the Northern Hemisphere as the storm tracks, but more clearly in the Southern Hemisphere. They also found that the wave packets both in the real atmosphere and in the numerical models behave in the same way: the packet envelopes move with the group velocity that is amplitude independent, their shapes are retained during the propagating processes, and there is a clear negative correlation between the amplitude and the half-width of the packet. These packets resemble in nature the envelope solitons of the nonlinear Schrödinger equation.

Lee and Held (1993) further pointed out that more than one wave packet are sometimes observed simultaneously both in the real atmosphere and in the models. So there is a possibility that these wave packets may interact with each other when they move together. How do they behave when two or more packets interact with each other? Is there an exchange of energy among them? Do they survive the interaction? These problems are of fundamental importance and will be dealt with in this paper.

To make the problems be tractable by analytical approach, we will use in present study an ideal, slowly varying wave packet to model the wave packets in observations and numerical models. Here the term "slowly varying wave packet" means that the envelope of the packet varies slowly with time and space, while the wavenumber and the frequency of the carrier wave are assumed to be constant. This idealization allows us to study the wave packets systematically by means of the weakly nonlinear theory and the multiple-

Corresponding author address: Dr. Benkui Tan, Department of Geophysics, Peking University, Beijing 100871, P. R. China.

scale method and to gain a clear physical understanding of the underlying mechanism. The results obtained afterward do show that our study can reflect the main features of the real wave packets. Of course it cannot be expected that the present simple model can describe well all aspects of the dynamic features of the real packets. For example, the slowly varying wave packet is unable to reflect the asymmetric structures inside the real packets reported by Lee and Held (1993) because the wavenumber and frequency of the carrier wave for the slowly varying packet have been assumed to be constant, which represents a symmetric structure inside the packets. This needs further improvement.

For simplicity of mathematical treatment, we will work with a quasigeostrophic barotropic atmospheric model, thus neglecting baroclinic instability. Though the horizontal shear considered, the shear instability is also omitted for the same reason. These are also the main defects of the present study and deserve further improvement.

The paper is organized as following. In the next section, the wave–wave and wave-mean interaction equations for two Rossby wave packets are derived by using the multiple-scale method. In section 3 the propagation properties of a single Rossby wave packet are discussed. In section 4 the wave–wave and wave-mean interaction equations are used to study the collision interactions of two envelope Rossby solitons belonging to different modes. The final section presents the conclusions of the paper and discusses the collision behavior of the solitons.

2. Coupled evolution equations for interacting wave modes

The quasigeostrophic barotropic vorticity equation can be written in the form

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi + (\beta - U'')\frac{\partial \psi}{\partial x} = -\epsilon J(\psi, \nabla^2\psi). \tag{2.1}$$

To obtain this nondimensional equation, the total streamfunction Ψ has been expressed as

$$\Psi(x, y, t) = -\int^y U(\eta)d\eta + \epsilon\psi(x, y, t), \tag{2.2}$$

so that ϵ measures the amplitude of a disturbance superimposed on a zonal shear flow $U(y)$. In addition, we have nondimensionalized x and y with L (a characteristic length scale), velocities with a characteristic zonal flow speed U_0, and time with L/U_0. The parameter β, appearing in (2.1), is the dimensionless gradient of the Coriolis parameter.

For weakly nonlinear wave packets we introduce the slow time and space variables

$$T_1 = \epsilon t, \quad T_2 = \epsilon^2 t; \quad X_1 = \epsilon x, \quad X_2 = \epsilon^2 x. \tag{2.3}$$

Thus in Eq. (2.1)

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2},$$
$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial X_1} + \epsilon^2\frac{\partial}{\partial X_2}. \tag{2.4}$$

We seek an asymptotic solution to (2.1) by writing

$$\psi = \psi^{(1)} + \epsilon\psi^{(2)} + \epsilon^2\psi^{(3)} + \cdots, \tag{2.5}$$

where $\psi^{(1)}$, $\psi^{(2)}$, and $\psi^{(3)}$ satisfy the linear equations

$$L\psi^{(1)} = 0 \tag{2.6a}$$

$$L\psi^{(2)} = -\left(\frac{\partial}{\partial T_1} + U\frac{\partial}{\partial X_1}\right)\nabla^2\psi^{(1)} - (\beta - U'')\frac{\partial\psi^{(1)}}{\partial X_1} - 2\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial^2\psi^{(1)}}{\partial x\partial X_1} - J(\psi^{(1)}, \nabla^2\psi^{(1)}), \tag{2.6b}$$

$$\begin{aligned} L\psi^{(3)} = &-\left(\frac{\partial}{\partial T_2} + U\frac{\partial}{\partial X_2}\right)\nabla^2\psi^{(1)} \\ &- (\beta - U'')\frac{\partial\psi^{(1)}}{\partial X_2} - 2\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial^2\psi^{(1)}}{\partial x\partial X_2} \\ &- \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial^2\psi^{(1)}}{\partial X_1^2} - 2\left(\frac{\partial}{\partial T_1} + U\frac{\partial}{\partial X_1}\right) \\ &\times \frac{\partial^2\psi^{(1)}}{\partial x\partial X_1} - \left(\frac{\partial}{\partial T_1} + U\frac{\partial}{\partial X_1}\right)\nabla^2\psi^{(2)} \\ &- 2\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial^2\psi^{(2)}}{\partial x\partial X_1} - (\beta - U'') \\ &\times \frac{\partial\psi^{(2)}}{\partial X_1} - \frac{\partial\psi^{(1)}}{\partial X_1}\frac{\partial}{\partial y}\nabla^2\psi^{(1)} + \frac{\partial\psi^{(1)}}{\partial y}\frac{\partial}{\partial X_1}\nabla^2\psi^{(1)} \\ &- 2\left(\frac{\partial\psi^{(1)}}{\partial x}\frac{\partial^3\psi^{(1)}}{\partial x\partial y\partial X_1} - \frac{\partial\psi^{(1)}}{\partial y}\frac{\partial^3\psi^{(1)}}{\partial x^2\partial X_1}\right) \\ &- J(\psi^{(1)}, \nabla^2\psi^{(2)}) - J(\psi^{(2)}, \nabla^2\psi^{(1)}), \end{aligned} \tag{2.6c}$$

where the differential operator L is defined as

$$L \equiv \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 + (\beta - U'')\frac{\partial}{\partial x}. \tag{2.7}$$

We take the solution of (2.6a) to a superposition of wave packets

$$\psi^{(1)} = \sum_n A_n(T_1, T_2, X_1, X_2)\varphi_n(y) \times \exp[i(k_n x - \omega_n t)] + *, \tag{2.8}$$

where the amplitude A_n is a function of slow space and time variables and is determined by higher-order problems. The asterisk denotes the complex conjugates of the preceding terms. The modal function $\varphi_n(y)$ satisfies the eigenvalue problem

$$\frac{d^2\varphi_n}{dy^2} - \left(k_n^2 - \frac{\beta - U''}{U - c_n}\right)\varphi_n = 0, \quad \varphi_n(0) = \varphi_n(\pi) = 0. \tag{2.9}$$

The positions $y = 0, \pi$ denote the southern and northern edges of the flow. Equation (2.9) can be solved analytically only when $U(y)$ takes some specific functions. In general it is solved numerically.

Since the nonlinearity is quadratic in (2.1), we limit our discussion to two interacting wave modes:

$$\psi^{(1)} = A_1\phi_1(y)\exp[i(k_1x - \omega_1t)] + A_2\phi_2(y)\exp[i(k_2x - \omega_2t)] + *. \tag{2.10}$$

Substituting (2.10) into (2.6b) yields

$$\begin{aligned} L\psi^{(2)} = &-\sum_{n=1}^{2} G_{1n}\exp[i(k_nx - \omega_nt)] \\ &- \sum_{n=1}^{2} ig_{2n}A_n^2\exp[i2(k_nx - \omega_nt)] \\ &- ig_3A_1A_2\exp[i(k_{12}x - \omega_{12}t)] \\ &- ig_4A_1A_2^*\exp[i(\alpha_{12}x - \sigma_{12}t)] + *, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} k_{12} &= k_1 + k_2, \quad \alpha_{12} = k_1 - k_2, \\ \omega_{12} &= \omega_1 + \omega_2, \quad \sigma_{12} = \omega_1 - \omega_2, \\ G_{1n} &= \left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right)\frac{\partial A_n}{\partial T_1} + \left[U\left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right) + (\beta - U'')\varphi_n - 2k_n(U - c_n)\varphi_n\right]\frac{\partial A_n}{\partial X_1}, \\ g_{2n} &= k_n\left(\varphi_n\frac{d}{dy} - \frac{d\varphi_n}{dy}\right)\frac{d^2\varphi_n}{dy^2}, \\ g_3 &= \left(k_1\varphi_1\frac{d}{dy} - k_2\frac{d\varphi_1}{dy}\right)\left(\frac{d^2\varphi_2}{dy^2} - k_2^2\varphi_2\right) + \left(k_2\varphi_2\frac{d}{dy} - k_1\frac{d\varphi_2}{dy}\right)\left(\frac{d^2\varphi_1}{dy^2} - k_1^2\varphi_1\right), \\ g_4 &= \left(k_1\varphi_1\frac{d}{dy} + k_2\frac{d\varphi_1}{dy}\right)\left(\frac{d^2\varphi_2}{dy^2} - k_2^2\varphi_2\right) - \left(k_2\varphi_2\frac{d}{dy} + k_1\frac{d\varphi_2}{dy}\right)\left(\frac{d^2\varphi_1}{dy^2} - k_1^2\varphi_1\right). \end{aligned} \tag{2.12}$$

The second, third, and fourth inhomogeneous terms in (2.11) yield the following particular solutions

$$\begin{aligned} \psi_{2n}^{(2)} &= Y_{2n}A_n^2\exp[i2(k_nx - \omega_nt)] + *, \\ \psi_3^{(2)} &= Y_3A_1A_2\exp[i(k_{12}x - \omega_{12}t)] + *, \\ \psi_4^{(2)} &= Y_4A_1A_2^*\exp[i(\alpha_{12}x - \sigma_{12}t)] + *, \end{aligned} \tag{2.13}$$

respectively, where Y_{2n}, Y_3, and Y_4 satisfy

$$\begin{aligned} &\frac{d^2Y_{2n}}{dy^2} - \left(4k_n^2 - \frac{\beta - U''}{U - c_n}\right)Y_{2n} = \frac{g_{2n}}{2(Uk_n - \omega_n)}, \\ &\qquad Y_{2n}(0) = Y_{2n}(\pi) = 0, \\ &\frac{d^2Y_3}{dy^2} - \left(k_{12}^2 - \frac{k_{12}(\beta - U'')}{Uk_{12} - \omega_{12}}\right)Y_3 = \frac{g_3}{Uk_{12} - \omega_{12}}, \\ &\qquad Y_3(0) = Y_3(\pi) = 0, \\ &\frac{d^2Y_4}{dy^2} - \left(\alpha_{12}^2 - \frac{\alpha_{12}(\beta - U'')}{U\alpha_{12} - \sigma_{12}}\right)Y_4 = \frac{g_4}{U\alpha_{12} - \sigma_{12}}, \\ &\qquad Y_4(0) = Y_4(\pi) = 0, \end{aligned} \tag{2.14}$$

respectively. The first inhomogeneous term in (2.11) yields the particular solution

$$\psi_{1n}^{(2)} = \varphi_{1n}^{(2)}\exp[i(k_nx - \omega_nt)] + *, \tag{2.15}$$

where $\varphi_{1n}^{(2)}$ satisfies

$$\begin{aligned} &\frac{d^2\varphi_{1n}^{(2)}}{dy^2} - \left(k_n^2 - \frac{\beta - U''}{U - c_n}\right)\varphi_{1n}^{(2)} \\ &= \frac{i}{k_n(U - c_n)}\left\{\left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right)\frac{\partial A_n}{\partial T_1} + \left[U\left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right) + (\beta - U'')\varphi_n - 2k_n^2(U - C_n)\right]\frac{\partial A_n}{\partial X_1}\right\}. \end{aligned} \tag{2.16}$$

Multiplying the left-hand side by φ_n, integrating over y from 0 to π and using the boundary conditions shows that the integration is equal to zero. Consequently, for Eq. (2.16) to be consistent the same operation carried out on the right-hand side must also produce zero. This leads to the solvable condition

$$\frac{\partial A_n}{\partial T_1} + c_{gn}\frac{\partial A_n}{\partial X_1} = 0, \tag{2.17}$$

where

$$c_{gn} = c_n + 2\int_0^\pi k_n^2\varphi_n^2dy \Big/ \int_0^\pi \frac{\beta - U''}{(U - c_n)^2}\varphi_n^2dy. \tag{2.18}$$

It is clear from Eq. (2.17) that at the $O(\epsilon)$ problem the amplitude A_n propagates at the speed of c_{gn}, that is,

$$A_n = A_n(X_1 - c_{gn}T_1, X_2, T_2). \tag{2.19}$$

Physically c_{gn} is the group speed. Generally $\beta - U'' > 0$ for the large-scale atmospheric motion, so $c_{gn} > c_n$.

We suppose further

$$\varphi_{1n}^{(2)} = iY_{1n}\frac{\partial A_n}{\partial X_1}. \qquad (2.20)$$

Substituting Eq. (2.20) into Eq. (2.16) and using the condition Eq. (2.17), we obtain the equation for Y_{1n}:

$$\frac{d^2Y_{1n}}{dy^2} - \left(k_n^2 - \frac{\beta - U''}{U - c_n}\right)Y_{1n} = \frac{1}{k_n(U - c_n)} \times \left[(c_n - c_{gn})\left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right) - 2k_n^2(U - c_n)\varphi_n\right],$$

$$Y_{1n}(0) = Y_{1n}(\pi) = 0. \qquad (2.21)$$

Up to now we have obtained the solutions to Eq. (2.11)

$$\psi^{(2)} = \psi_{11}^{(2)} + \psi_{12}^{(2)} + \psi_{21}^{(2)} + \psi_{22}^{(2)} + \psi_3^{(2)} + \psi_4^{(2)} + \Phi(y, T_1, X_1), \qquad (2.22)$$

where the function $\Phi(y, T_1, X_1)$ is a homogeneous solution to Eq. (2.11), which represents the zonal flow correction due to the existence of the finite-amplitude wave packets and is determined by the next-order problem.

To obtain the evolution equations for Φ and A_n, we press on now to the $O(\epsilon^2)$ problem. With the insertion of $\psi^{(1)}$ and $\psi^{(2)}$ into the right-hand side of (2.6c), we can obtain all the inhomogeneous terms. Among them there are terms that are independent of x and t. Considering the form of the linear operation on the left-hand side, it is clear that these terms must vanish identically, leading to the condition relating the correction to the mean flow to the wave amplitude; that is,

$$\left(\frac{\partial}{\partial T_1} + U\frac{\partial}{\partial X_1}\right)\frac{\partial^2\Phi}{\partial y^2} + (\beta - U'')\frac{\partial\Phi}{\partial X_1} = \sum_{n=1}^{2}\left\{\frac{d^2}{dy^2}\left[k_n\left(Y_{1n}\frac{d\varphi_n}{dy} - \varphi_n\frac{dY_{1n}}{dy}\right)\right] - 4k_n^2\varphi_n\frac{d\varphi_n}{dy} + \left(\varphi_n\frac{d}{dy} - \frac{d\varphi_n}{dy}\right)\left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right)\right\}\frac{\partial}{\partial X_1}|A_n|^2. \qquad (2.23)$$

There are other kinds of inhomogeneous terms in the right-hand side of Eq. (2.6c) that are proportional to $\exp[i(k_nx - \omega_nt)]$. To obtain solutions corresponding to these terms, some solvable conditions must be placed, otherwise secular growth in $\varphi_n^{(3)}$ will appear. Multiplying (2.6c) by $\varphi_n\exp[-i(k_nx - \omega_nt)]$ and integrating over t from 0 to $2\pi/\omega_n$, over x from 0 to $2\pi/k_n$ and over y from 0 to π, the integration of the left-hand side is zero, so the integration of the right-hand side must also be zero. This leads to the following solvable conditions:

$$\left(\frac{\partial}{\partial T_2} + c_{g1}\frac{\partial}{\partial X_2}\right)A_1 - i\alpha_1\frac{\partial^2A_1}{\partial X_1^2} - i(\rho_1|A_1|^2 + \gamma_{12}|A_2|^2 + \lambda_1)A_1 = 0,$$

$$\left(\frac{\partial}{\partial T_2} + c_{g2}\frac{\partial}{\partial X_2}\right)A_2 - i\alpha_2\frac{\partial^2A_2}{\partial X_1^2} - i(\rho_2|A_2|^2 + \gamma_{21}|A_1|^2 + \lambda_2)A_2 = 0, \qquad (2.24a,b)$$

where

$$\alpha_n = \frac{\mathrm{I}_{n0}}{\mathrm{II}_n}, \quad \rho_n = \frac{\mathrm{I}_{nn}}{\mathrm{II}_n}, \quad \gamma_{12} = \frac{\mathrm{I}_{12}}{\mathrm{II}_1}, \quad \gamma_{21} = \frac{\mathrm{I}_{21}}{\mathrm{II}_2}, \quad \lambda_n = \frac{\mathrm{I}_{n3}}{\mathrm{II}_n}, \qquad (2.25)$$

while

$$\mathrm{II}_n = -\int_0^\pi \frac{\varphi_n}{U - c_n}f_ndy, \quad \mathrm{I}_{n0} = \int_0^\pi \frac{\varphi_n}{U - c_n}f_{n0}dy,$$

$$\mathrm{I}_{nn} = \int_0^\pi \frac{\varphi_n}{U - c_n}f_{nn}dy, \quad \mathrm{I}_{12} = \int_0^\pi \frac{\varphi_1}{U - c_1}f_{12}dy,$$

$$I_{21} = \int_0^\pi \frac{\varphi_2}{U - c_2}f_{21}dy, \quad I_{n3} = \int_0^\pi \frac{\varphi_n}{U - c_n}f_{n3}dy, \qquad (2.26)$$

and

$$f_n = -\left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right),$$

$$h_n = -U\left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right) - (\beta - U'')\varphi_n + 2k_n^2(U - c_n),$$

$$f_{n0} = -2k_n(U - c_{gn})\varphi_n - k_n(U - c_n)\varphi_n - (U - c_{gn})\left(\frac{dY_{1n}}{dy^2} - k_n^2Y_{1n}\right) + 2k_n^2(U - c_n)Y_{1n} - (\beta - U'')Y_{1n},$$

$$f_{nn} = -\left(2k_nY_{2n}\frac{d}{dy} + k_n\frac{dY_{2n}}{dy}\right)\left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right) + \left(k_n\varphi_n\frac{d}{dy} + 2k_n\frac{d\varphi_n}{dy}\right)\left(\frac{dY_{2n}}{dy^2} - 4k_n^2Y_{2n}\right),$$

$$f_{12} = -\left(k_{12}Y_3\frac{d}{dy} + k_2\frac{dY_3}{dy}\right)\left(\frac{d^2\varphi_2}{dy^2} - k_2^2\varphi_2\right) + \left(k_2\varphi_2\frac{d}{dy} + k_{12}\frac{d\varphi_2}{dy}\right)\left(\frac{d^2Y_3}{dy^2} - k_{12}^2Y_3\right) - \left(k_2\varphi_2\frac{d}{dy} - \alpha_{12}\frac{d\varphi_2}{dy}\right)\left(\frac{d^2Y_4}{dy^2} - \alpha_{12}^2Y_4\right) - \left(\alpha_{12}Y_4\frac{d}{dy} - k_2\frac{dY_4}{dy}\right)\left(\frac{d^2\varphi_2}{dy^2} - k_2^2\varphi_2\right),$$

$$f_{21} = -\left(k_{12}Y_3 \frac{d}{dy} + k_1 \frac{dY_3}{dy}\right)\left(\frac{d^2\varphi_1}{dy^2} - k_1^2\varphi_1\right) - \left(k_1\varphi_1 \frac{d}{dy} + \alpha_{12} \frac{d\varphi_1}{dy}\right)\left(\frac{d^2Y_4}{dy^2} - \alpha_{12}^2 Y_4\right) + \left(k_1\varphi_1 \frac{d}{dy} + k_{12} \frac{d\varphi_2}{dy}\right)\left(\frac{d^2Y_3}{dy^2} - k_{12}^2 Y_3\right) + \left(\alpha_{12}Y_4 \frac{d}{dy} + k_1 \frac{dY_4}{dy}\right)\left(\frac{d^2\varphi_1}{dy^2} - k_1^2\varphi_1\right),$$

$$f_{n3} = -k_n\left(\varphi_n \frac{\partial^3\Phi}{\partial y^3} - \left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right)\frac{\partial\Phi}{\partial y}\right). \quad (2.27)$$

Thus, we have obtained the evolution equations for A_1 and A_2. Equation (2.24), together with Eq. (2.23), is the coupled equation for Φ, A_1, and A_2. They describe the wave–wave and wave–mean flow interactions.

Equations (2.23) and (2.24) can be simplified further when we introduce the transforms by Jeffrey and Kawahara (1982):

$$T = T_2, \quad X = \frac{1}{\epsilon}(X_2 - c_{g1}T_2) = X_1 - c_{g1}T_1. \quad (2.28)$$

Under the transforms, Eq. (2.23) reduces to

$$\frac{\partial}{\partial X}\left[(U - c_{g1})\frac{\partial^2\Phi}{\partial y^2} + (\beta - U'')\Phi\right] = \sum_{n=1}^{2}\left[k_n \frac{d}{dy^2}\left(Y_{1n}\frac{d\varphi_n}{dy} - \varphi_n \frac{dY_{1n}}{dy}\right) - 4k_n\varphi_n \frac{d\varphi_n}{dy} - \left(\varphi_n \frac{d}{dy} - \frac{d\varphi_n}{dy}\right)\left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right)\right]\frac{\partial}{\partial X}|A_n|^2. \quad (2.29)$$

We seek a solution to Eq. (2.29) in the form

$$\Phi = \sum_{n=1}^{2} H_n(y)|A_n|^2, \quad (2.30)$$

where $H_n(y)$ satisfies

$$(U - c_{g1})\frac{d^2H_n}{dy^2} + (\beta - U'')H_n = k_n \frac{d^2}{dy^2}\left(Y_{1n}\frac{d\varphi_n}{dy} - \varphi_n \frac{dY_{1n}}{dy}\right) - 4k_n^2\varphi_n \frac{d\varphi_n}{dy} - \left(\varphi_n \frac{d}{dy} - \frac{d\varphi_n}{dy}\right)\left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right). \quad (2.31)$$

Applying Eqs. (2.28) and (2.30), Eq. (2.24) reduces to

$$i\frac{\partial A_1}{\partial T} + \alpha_1 \frac{\partial^2 A_1}{\partial X^2} + (\sigma_1|A_1|^2 + \nu_{12}|A_2|^2)A_1 = 0,$$

$$i\left(\frac{\partial}{\partial T} - \delta\frac{\partial}{\partial X}\right)A_2 + \alpha_2 \frac{\partial^2 A_2}{\partial X^2} + (\sigma_2|A_2|^2 + \nu_{21}|A_1|^2)A_2 = 0, \quad (2.32a,b)$$

where

$$\delta = \frac{1}{\epsilon}(c_{g1} - c_{g2}),$$

$$\sigma_n = \rho_n - \frac{1}{\mathrm{II}_n}\int_0^\pi \frac{k_n\varphi_n}{U - c_n} \times \left[\varphi_n \frac{d^3H_n}{dy^3} - \left(\frac{d^2\varphi_n}{dy^2} - k_n^2\varphi_n\right)\frac{dH_n}{dy}\right]dy,$$

$$\nu_{12} = \gamma_{12} - \frac{1}{\mathrm{II}_1}\int_0^\pi \frac{k_1\varphi_1}{U - c_1} \times \left[\varphi_1 \frac{d^3H_2}{dy^3} - \left(\frac{d^2\varphi_1}{dy^2} - k_1^2\varphi_1\right)\frac{dH_2}{dy}\right]dy,$$

$$\nu_{21} = \gamma_{21} - \frac{1}{\mathrm{II}_2}\int_0^\pi \frac{k_2\varphi_2}{U - c_2} \times \left[\varphi_2 \frac{d^3H_1}{dy^3} - \left(\frac{d^2\varphi_2}{dy^2} - k_2^2\varphi_2\right)\frac{dH_1}{dy}\right]dy. \quad (2.33)$$

Equations (2.32a,b) are called the coupled nonlinear Schrödinger equations that were also obtained in the wave–wave interaction problem in nonlinear optics (Manakov 1973). In these equations, α_n are the dispersion coefficients, σ_n are Landau constants, and ν_{12} and ν_{21} are the wave–wave interaction coefficients. The last terms on the left-hand side of Eq. (2.32) govern self-phase modulation and cross-phase modulation.

Here δ is assumed to be of $O(1)$, which is required physically by the fact that the two wave packets interact at the longer timescale T_2; otherwise the interactions occur at the timescale T_1. This further implies $|k_1 - k_2| \sim |m_1 - m_2| = O(\epsilon)$. It is convenient here to introduce the transform

$$A_2 = B\exp\left[i\frac{\delta}{2\alpha_2}\left(X + \frac{\delta}{2}T\right)\right]. \quad (2.34)$$

Under the transform, Eq. (2.32) reduces to

$$i\frac{\partial A_1}{\partial T} + \alpha_1 \frac{\partial^2 A_1}{\partial X^2} + (\sigma_1|A_1|^2 + \nu_{12}|B|^2)A_1 = 0,$$

$$i\frac{\partial B}{\partial T} + \alpha_2 \frac{\partial^2 B}{\partial X^2} + (\sigma_2|B|^2 + \nu_{21}|A_1|^2)B = 0. \quad (2.35)$$

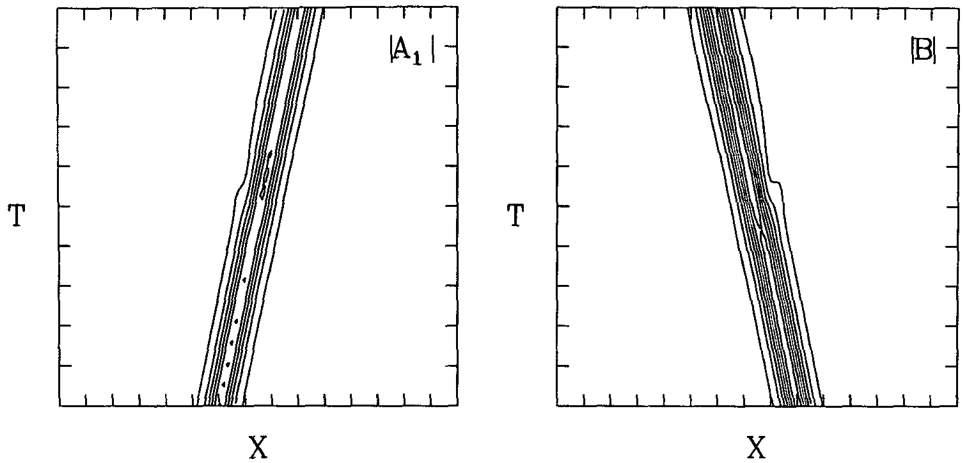

FIG. 1. Numerical simulation of the collision interaction of two envelope Rossby solitons with $\alpha_1 = 0.53$, $\sigma_1 = 1.42$, $\alpha_2 = 0.55$, $\sigma_2 = 0.93$, $\nu_{12} = 0.11$, $\nu_{21} = 0.23$, $\eta_1 = \eta_2 = 1.0$, $X_0 = 10.0$, $\Delta v \equiv 2\alpha_1\xi_1 - 2\alpha_2\xi_2 = 1.0$. The curves are the contour lines of $|A_1|$ or $|B|$.

3. Envelope solitons

It is obvious that for a single nonlinear Rossby wave packet Eq. (2.35) reduces to

$$i\frac{\partial A}{\partial T} + \alpha\frac{\partial^2 A}{\partial X^2} + \sigma|A|^2A = 0. \quad (3.1)$$

Equation (3.1) is called the nonlinear Schrödinger equation, NLS equation in short. It is one of the nonlinear evolution equations that have been studied extensively. Here we briefly present some properties of the equation.

When the product $\alpha\sigma < 0$, its asymptotic solution is simply a wave train qualitatively similar to that of the linear equation. The most striking difference is that the nonlinearity also acts to widen the wave packet, producing ''defocusing'' or ''superlinear'' dispersion in the sense that the wave train spreads out more rapidly than in the linearized theory. When the conditions

$$\alpha\sigma > 0 \quad (3.2)$$

and

$$\int_{-\infty}^{\infty} |A(X, 0)|\, dX > 0.904 \quad (3.3)$$

are satisfied, the asymptotic solution consists of a wave train plus a finite number of envelope solitons (Zakharov and Shabat 1972; Ablowitz et al. 1974). The number and nature of the solitons are determined by the initial condition and a smooth initial condition of large enough amplitude will place most of the energy in the solitons, leaving very little for the decaying wave train.

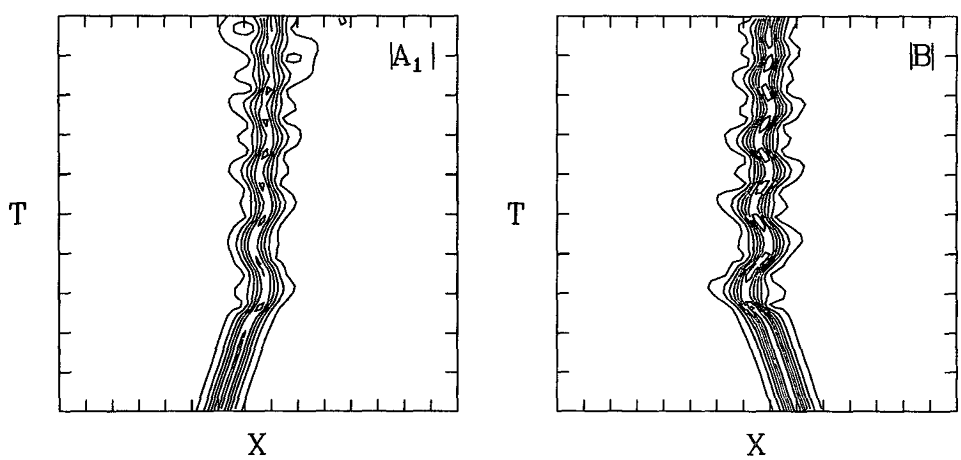

FIG. 2. As in Fig. 1 except $\Delta v = 0.2$.

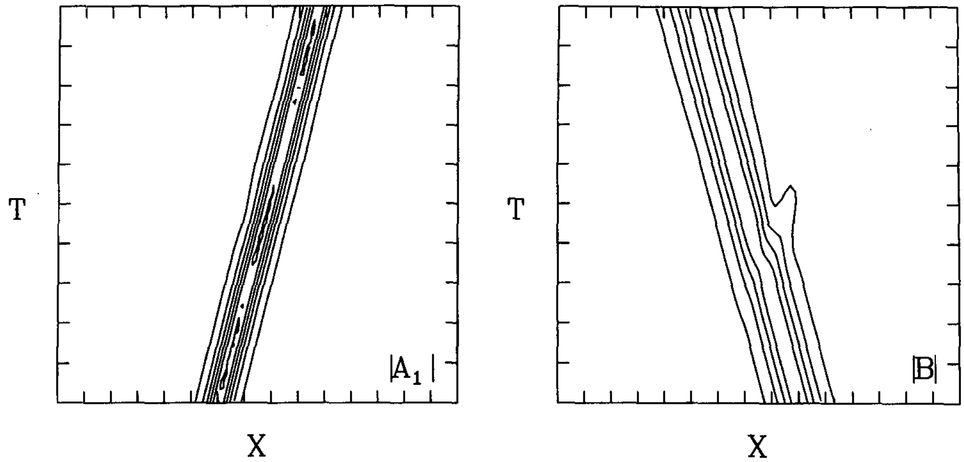

FIG. 3. As in Fig. 1 except $\eta_1 = 1.0$, $\eta_2 = 0.6$.

The condition Eq. (3.3) means that for a disturbance it can produce envelope solitons only when its initial strength is strong enough. As the values of α and σ depend on both the wavenumber k and the basic flow $U(y)$, so the condition Eq. (3.2) means that for a given wavenumber k, a disturbance can produce envelope solitons only when the basic flow is favorable. In other words, for a given basic flow, a disturbance can produce envelope solitons only when its wavenumber takes some special values. For instance, for a constant basic flow, that is, $U(y) =$ constant, a disturbance can produce envelope solitons only when $|k/m| < 0.68$ (here m is the meridional wavenumber).

An isolated single-envelope soliton solution of Eq. (3.1) is easily obtained:

$$A(T, X) = \sqrt{\frac{2\alpha}{\sigma}}\,\eta\,\mathrm{sech}[\eta(X - 2\alpha\xi T)] \times \exp[i\xi X - i\alpha(\xi^2 - \eta^2)T], \quad (3.4)$$

where η and ξ are two independent parameters related to the amplitude and speed of the envelope soliton, respectively, and they are determined by the initial condition.

Substituting Eq. (3.4) into Eq. (2.10), we obtain the streamfunction of the envelope soliton:

$$\psi^{(1)} = \sqrt{\frac{2\alpha}{\sigma}}\,\eta\,\mathrm{sech}[\epsilon\eta(x - Vt)]\exp[iKx - Wt]\varphi(y) + *, \quad (3.5)$$

where

$$V = c_g + \epsilon 2\alpha\xi,$$

$$K = k + \epsilon\xi,$$

$$W = \omega + \epsilon\xi c_g + \epsilon^2\alpha(\xi^2 - \eta^2). \quad (3.6a,b,c)$$

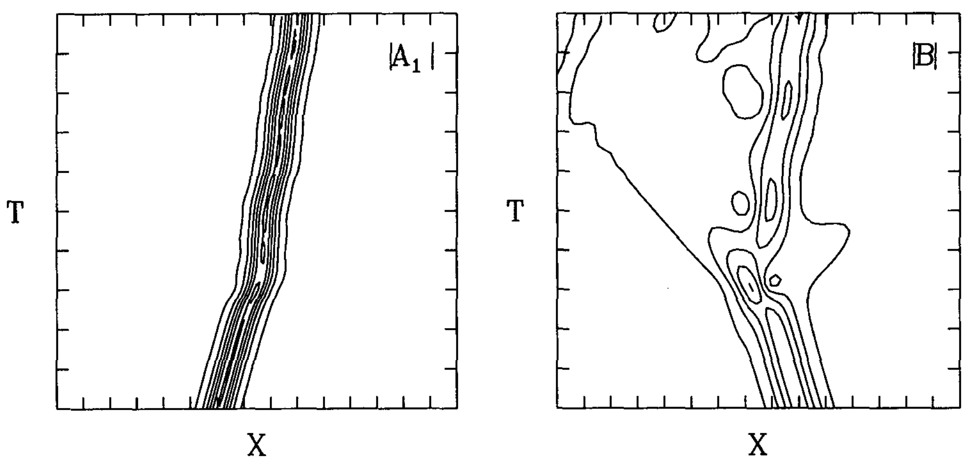

FIG. 4. As in Fig. 1 except $\eta_1 = 1.0$, $\eta_2 = 0.6$, $\Delta v = 0.2$.

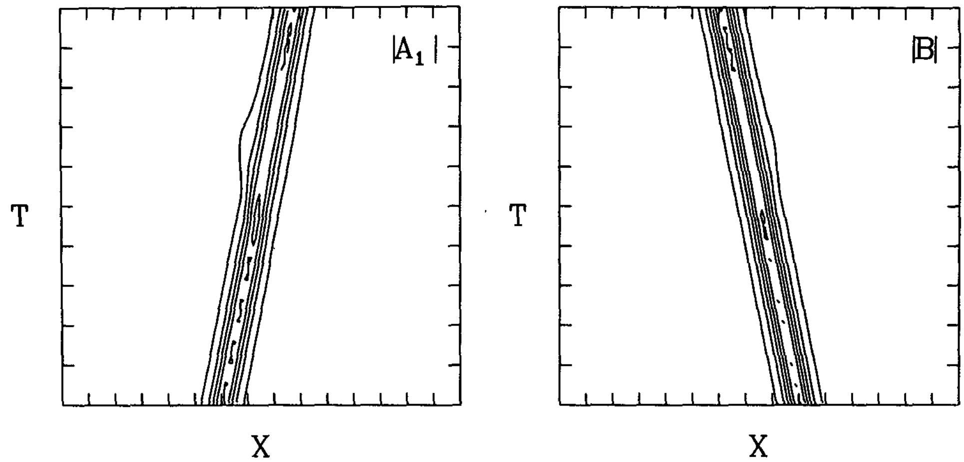

FIG. 5. Numerical simulation of the collision interaction of the two envelope Rossby solitons with $\alpha_1 = 0.53$, $\sigma_1 = 1.42$, $\alpha_2 = 0.51$, $\sigma_2 = 2.0$, $\nu_{12} = -0.13$, $\nu_{21} = -0.23$; $\eta_1 = \eta_2 = 1.0$; $X_0 = 10.0$, $\Delta v = 1.0$. The curves are the contour lines of $|A_1|$ or $|B|$.

It is clear from (3.6) that the propagation speed of the envelope is equal to the group speed of the linear Rossby wave plus an $O(\epsilon)$ correction term and it is amplitude-independent. It is also clear from Eq. (3.6) that the wavenumber of the carrier wave, K, is equal to that of the linear Rossby wave plus a $O(\epsilon)$ correction term, and the frequency of the carrier wave, W, is equal to that of the linear Rossby wave plus some small correction terms, which is amplitude dependent.

4. Collision of two-envelope solitons

Zakharov and Schulman (1982) have shown that in the case

$$\alpha_1 = \alpha_2, \quad \sigma_1 = \sigma_2 = \nu_{12} = \nu_{21} \tag{4.1}$$

or

$$\alpha_1 = -\alpha_2, \quad \sigma_1 = \sigma_2 = -\nu_{12} = -\nu_{21}, \tag{4.2}$$

the system (2.35) has an infinite set of motion invariant and may be solved by the inverse scattering method. Except for the above two cases, the system Eq. (2.35) has only four motion invariants: They are

$$E_1 = \int_{-\infty}^{\infty} |A_1|^2 dX, \quad E_2 = \int_{-\infty}^{\infty} |B|^2 dX,$$

$$E_3 = \int_{-\infty}^{\infty} \left(A_1 \frac{\partial A_1^*}{\partial X} + \frac{\nu_{12}}{\nu_{21}} B \frac{\partial B^*}{\partial X} \right) dX,$$

$$E_4 = \int_{-\infty}^{\infty} \left[\alpha_1 \left| \frac{\partial A_1}{\partial X} \right|^2 + \frac{\nu_{12}}{\nu_{21}} \alpha_2 \left| \frac{\partial B}{\partial X} \right|^2 - \frac{\sigma_1}{2} |A_1|^4 - \frac{\nu_{12}}{\nu_{21}} \frac{\sigma_2}{2} |B|^4 - \nu_{12} |A_1|^2 |B|^2 \right] dX. \tag{4.3}$$

In this case the system Eq. (2.35) cannot be solved by the inverse-scattering method.

Clearly, E_1 and E_2 are the energies of the two waves; E_4 is the Hamiltonian of the system Eq. (2.35). This means that the energy of each wave and the Hamiltonian of the two-wave system are conserved.

For the atmosphere, the conditions (4.1) and (4.2) are not satisfied in general. So the system (2.35) is nonintegrable in general. Under the periodic boundary conditions, the system (2.35) has periodic plane wave solutions. These solutions and their stability have been studied in another paper (Tan 1993). In this paper we shall study the interactions of the two-wave packets under the infinite boundary conditions. It is clear from (2.35) that when the two-wave packets are separated well in space, each wave packet is governed by the corresponding nonlinear Schrödinger equation and it is assumed to take a form of an envelope soliton. How do they behave when these two envelope solitons collide with each other? Do they survive the collision? If they do not, in what way do they behave? These problems are of fundamental importance and will be studied numerically in the following.

To obtain the coefficients α_n, σ_n, ν_{12}, and ν_{21} in the evolution Eq. (2.35), let us take a close look at Eqs. (2.25) and (2.33). It is clear from these equations that the coefficients usually depend on the wavenumbers k_n and the basic flow $U(y)$. For an arbitrarily given basic flow, it is difficult to obtain them analytically, they are determined only by the numerical means. To demonstrate the nature of the collision solutions of the two coupled nonlinear Schrödinger equations, we consider here a simple case that the basic flow is constant. Fortunately in this case, the coefficients can be obtained analytically (see the appendix). It is found that the interaction coefficients ν_{12} and ν_{21} can be take positive

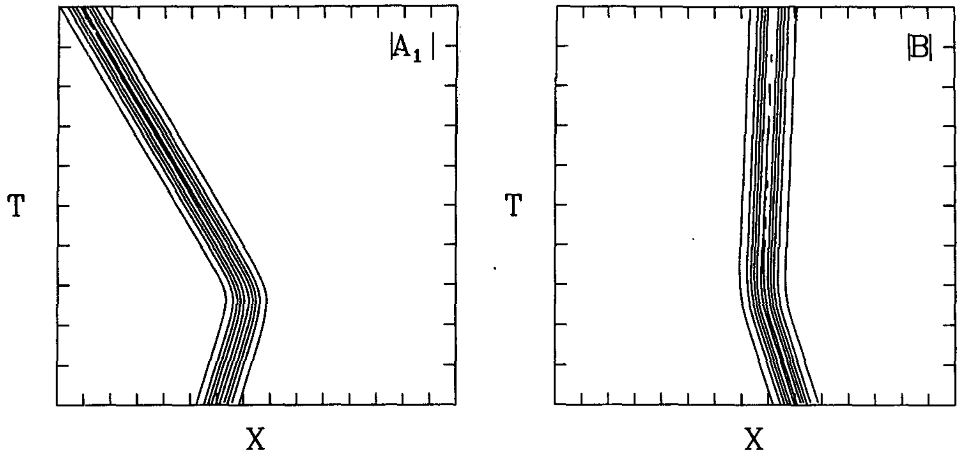

FIG. 6. As in Fig. 5 except $\Delta v = 0.2$.

or negative signs, which depends on the zonal wavenumbers k_n and the meridional wavenumbers m_n.

We take the initial conditions as

$$A_1(X, 0) = \sqrt{\frac{2\alpha_1}{\sigma_1}}\,\eta_1\,\mathrm{sech}(\eta_1 X)\exp(i\xi_1 X),$$

$$B(X, 0) = \sqrt{\frac{2\alpha_2}{\sigma_2}}\,\eta_2\,\mathrm{sech}[\eta_2(X + X_0)]\exp[i\xi_2(X + X_0)]. \quad (4.4)$$

Well before the collision, the evolution of the two-envelope solitons reads

$$A_1(X, T) = \sqrt{\frac{2\alpha_1}{\sigma_1}}\,\eta_1\,\mathrm{sech}[\eta_1(X - 2\alpha_1\xi_1 T)] \times \exp[i\xi_1 X - i\alpha_1(\xi_1^2 - \eta_1^2)T],$$

$$B(X, T) = \sqrt{\frac{2\alpha_2}{\sigma_2}}\,\eta_2\,\mathrm{sech}[\eta_2(X + X_0 - 2\alpha_2\xi_2 T)] \times \exp[i\xi_2(X + X_0) - i\alpha_2(\xi_2^2 - \eta_2^2)T]. \quad (4.5)$$

The numerical results show that the collision behavior of the solitons is complex and diverse, which is closely related to the signs of the interaction coefficients ν_{12} and ν_{21} and the initial conditions of the solitons. The numerical results are shown in Figs. 1–8.

Figures 1–4 illustrate the numerical results with $\alpha_1 = 0.53$, $\sigma_1 = 1.42$, $\alpha_2 = 0.55$, $\sigma_2 = 0.93$, $\nu_{12} = 0.11$, $\nu_{21} = 0.23$, which correspond to $U = 1.0$, $\beta = 0.8$, $k_1 = 0.55$, $k_2 = 0.52$, $m_1 = m_2 = 1$. This is the case that ν_{12} and ν_{21} are positive. Figures 1 and 2 show the collisions of the two solitons of approximately equal strength: here $\eta_1 = \eta_2 = 1.0$. Figure 1 shows the fast collision, the relative velocity between the two solitons well before the collision $\Delta v \equiv 2\alpha_1\xi_1 - 2\alpha_2\xi_2 = 1.0$. As can be seen, in this case the two solitons survive the collision, with their shapes and velocities remaining unchanged. Figure 2 shows the slow collision with $\Delta v = 0.2$. In this case, the collision makes the solitons fuse into a new bounded "breather" that oscillates between the two modes. Figures 3 and 4 show the collision of a larger soliton with a smaller soliton, here $\eta_1 = 1.0$, $\eta_2 = 0.6$. It is clear that when the collision is fast, the collision does not affect the natures of the solitons (Fig. 3). When the collision is slow, only the larger soliton survives the collision; it slightly slows down and oscillates after the collision. Whereas the smaller soliton does not survive the collision, most of its energy is captured by the larger soliton and part of its energy radiates out (Fig. 4).

Figures 5 and 6 show the numerical results with $\alpha_1 = 0.53$, $\sigma_1 = 1.42$, $\alpha_2 = 0.51$, $\sigma_2 = 2.0$, $\nu_{12} = -0.13$, $\nu_{21} = -0.23$, which correspond to $U = 1.0$, $\beta = 0.8$, $k_1 = 0.55$, $k_2 = 0.58$, $m_1 = m_2 = 1$. This is the case that ν_{12} and ν_{21} are negative. In these figures the parameters $\eta_1 = \eta_2 = 1.0$. They are the collisions of the two solitons of approximately equal strength. It is clear from the figures that when the collision is fast, the solitons maintain their shapes and velocities upon collision (Fig. 5). On the other hand, the slow collision alters the properties of the solitons significantly: though the shapes of the two solitons remain unchanged after the collision, the solitons have changed the directions of propagation after the collision (Fig. 6).

Using the same set of coefficients, we have also studied the collision of a larger soliton with a smaller soliton. It is found that the collision behavior of the solitons is similar to the situation in Figs. 5 and 6. The figures are omitted here.

Finally, we show the numerical results with $\alpha_1 = 0.038$, $\sigma_1 = 0.38$, $\alpha_2 = 0.055$, $\sigma_2 = 1.94$, $\nu_{12} = 0.11$, $\nu_{21} = -0.34$, which are obtained by choosing $U = 1.0$,

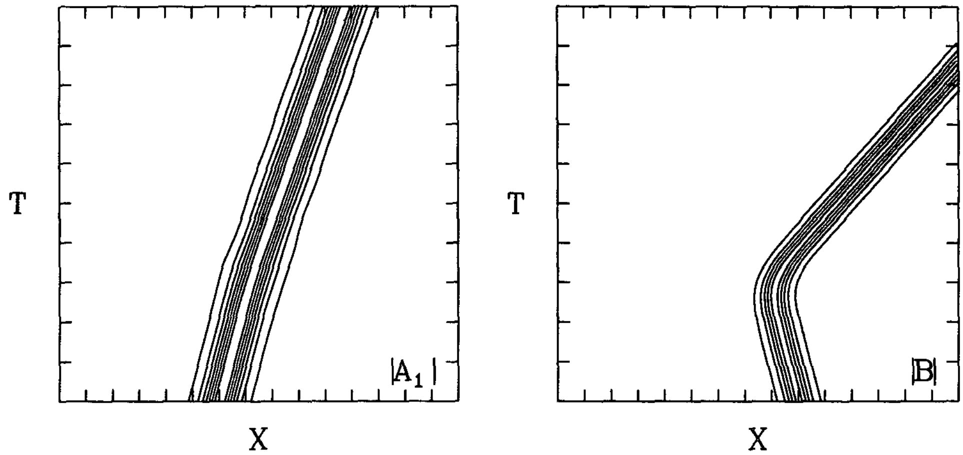

FIG. 7. Numerical simulation of the collision interaction of the two envelope Rossby solitons with $\alpha_1 = 0.038$, $\sigma_1 = 0.38$, $\alpha_2 = 0.055$, $\sigma_2 = 1.94$, $\nu_{12} = 0.11$, $\nu_{21} = -0.34$, $\eta_1 = 0.8$, $\eta_2 = 1.0$, $X_0 = 10.0$, $\Delta v = 0.2$. The curves are the contour lines of $|A_1|$ or $|B|$.

$\beta = 0.7$, $k_1 = 0.32$, $k_2 = 0.52$, $m_1 = m_2 = 2$. This is the case that the interaction coefficients ν_{12} and ν_{21} take opposite signs. It is found that like the above cases, when the collision is fast, the collision does not alter the behavior of the solitons. (The figures are omitted here.) Figure 7 shows a slow collision of two solitons of approximately equal strength. Here $\eta_1 = 0.8$, $\eta_2 = 1.0$ and $\Delta v = 0.2$. It is clear that the collision has altered the properties of the solitons. The soliton A_1 maintains its shape unchanged after the collision and propagates slightly faster than before the collision. While the soliton B is reflected back by the soliton A_1, and its velocity becomes larger in absolute value after the collision than before the collision. Figure 8 shows the collision of a larger soliton with a smaller soliton. A similar collision behavior as in Fig. 7 is observed.

5. Discussion and conclusions

The results and analysis presented here provide insight into the dynamics of the interactions between two-envelope Rossby solitons in shear flows. It is found that the energy of each wave is conserved during the interaction process; this means that there is no exchange of energy between the two waves. It is also found that the collision behaviors are affected by both the interaction coefficients ν_{12} and ν_{21} and the initial conditions (mainly the speeds of envelope solitons well before collisions). When the collision is fast, the solitons can survive the collision, maintaining their shapes and velocities unchanged while, when the collision is slow, the behavior of the solitons is complex and diverse depending on the signs of the interaction coefficients ν_{12} and ν_{21}. The solitons may be reflected back by each other or combined together to form a new bound breather oscillating between the two modes; while one soliton speeds up, another is being reflected back.

It is very interesting to note that the results presented here show that the interactions described by two coupled nonlinear Schrödinger equations are considerably more complex than envelope soliton interactions described by a single nonlinear Schrödinger equation, and they are also different from the soliton interactions described by two coupled KdV equations (Redekopp and Weidman 1978).

The following is a discussion on the mechanism of the wave–wave interaction and some physical explanation of the collision behavior of the solitons. It is clear from the two coupled nonlinear Schrödinger equations that the wave–wave interaction is realized through the cross-modulation terms. As the cross-modulation terms take the form of the product of the amplitude of one packet and the square of the absolute value of another packet, so the interactions between the packets are in action only when the two packets meet. This means that the interactions are realized through the "contacting" of the packets. In other words, there are no interactions between the packets when they separate well in space. It is also clear from the two coupled nonlinear Schrödinger equations that without the self-modulation terms, the evolution equation for each wave packet is a linear Schrödinger equation describing a particle moving in a field of external force or potential. With the self-modulation terms, the evolution equation for each wave packet is a nonlinear extension of the linear Schrödinger equation. Therefore, a soliton interacting with another soliton behaves like a particle moving in a field of external force. Here the field of force is provided by another soliton, this means that the action of one soliton on another is realized by providing a field of force for it. Since the field of force is in a form of the product of the interaction coefficient and the square of the absolute value of amplitude of another wave packet, so the nature of the field of force is de-

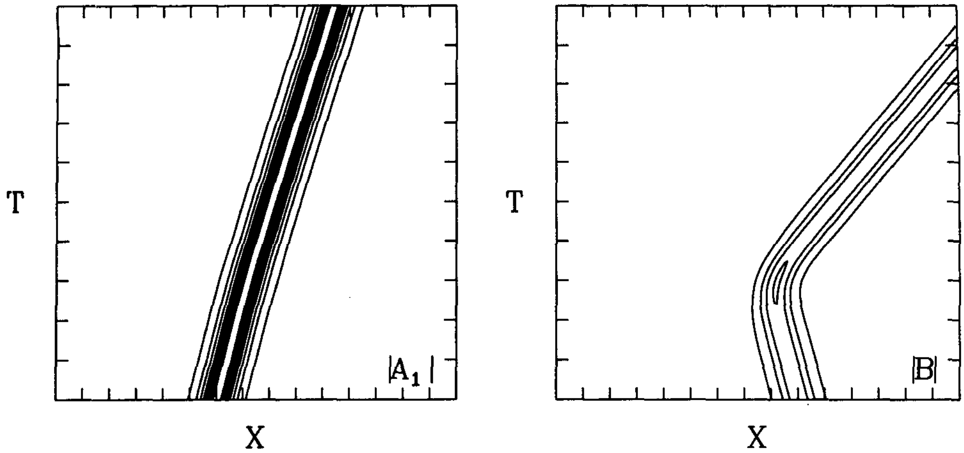

FIG. 8. As in Fig. 7 except $\eta_1 = 0.9$, $\eta_2 = 0.7$.

termined by the sign of the interaction coefficient. When the interaction coefficient is positive (negative), the field of the force is attractive (repulsive). Therefore, the collision behavior of the solitons depends on the signs of the interaction coefficients. On the other hand, the initial speed of the soliton represents its initial "momentum" or "kinetic energy." The larger the initial speed, the larger is its kinetic energy. Therefore, the behavior of the solitons depends not only on the interaction coefficients but also on the initial speeds. When the initial speeds of the solitons are high, the initial kinetic energies of the solitons are large and the effects of the external force may be minor. Therefore, the solitons can survive the fast collision whether the external forces are attractive or repulsive (see Figs. 1, 3, and 5). When the initial speeds of the solitons are slow, the initial kinetic energies are small. In this case, the effects of the external forces on the solitons may be significant depending on the signs of the interaction coefficients. In the case that both ν_{12} and ν_{21} are positive, the fields of the forces are attractive. So as the solitons collide with each other, they attract each other, forming a new bound breather (see Fig. 2) or being coupled together (see Fig. 4). In the case that both ν_{12} and ν_{21} are negative, the fields of the forces are repulsive. As the solitons collide with each other, they repel each other, being reflected back by each other after the collision (see Fig. 5). When ν_{12} and ν_{21} take opposite signs, one of the fields of the force is attractive, another is repulsive. As the solitons collide, the soliton subject to attractive force speeds up, while the soliton subject to repulsive force is being reflected back (see Figs. 7 and 8). In Figs. 7 and 8, the repulsive effect is significant, while the attractive effect is slight, this is because that ν_{12} is much smaller than ν_{21} in absolute value. We made numerical experiments with ν_{12} and ν_{21} being equal in absolute value but opposite in signs. We found that both attractive and repulsive effects are significant. One soliton speeds up; another retreats backward. Finally, they are coupled together, propagating at a speed much larger than before the collision (the figures are omitted here).

It should be pointed out that the collision behavior of the solitons reported here is obtained by taking a simple basic flow that $U(y)$ is constant. It would be helpful to do more numerical experiments with some basic flows that are functions of y. For this, the numerical method is needed, and the critical-layer problem will be faced for obtaining the coefficients in the coupled nonlinear Schrödinger equations. We will report the results on this aspect in another paper.

Finally, we point out some limitations of the asymptotic approach adopted here: we needed to assume two wave packets with slightly different wavenumbers to achieve a slow interaction to derive the nonlinear Schrödinger equation. The neglected subharmonic modes with the wavenumbers $2k_1$, $2k_2$, $k_1 \pm k_2$ at $O(\epsilon)$ will affect the main component in a longer timescale $O(\epsilon^{-3})$ [compare with the scale $O(\epsilon^{-2})$ for the nonlinear interactions of two packets]. If we take the principal period of the wave as ~5 days and the period for the packet ~ 10 days, for example, the two packets interact at the scale of 20 days, whereas the subharmonics affect the wave packets at the scale of 40 days. This formal estimate appears to severely restrict the applicability of this asymptotic approximation. However, we emphasise that the asymptotic approaches can describe the full system modestly well in the various problems (e.g., Karoly 1983; Yano 1992). Also this kind of approaches can give good physical insights to the problem. Yet the present results need to be further verified by direct numerical integration of the equation. This deserves further investigation.

Acknowledgments. The author is very grateful to the reviewers for their helpful comments.

APPENDIX

Constants in the Evolution Equations

In the case that the basic flow $U(y)$ is constant then, the coefficients in Eq. (2.35) can be obtained analytically. In the following we give a brief outline for obtaining the coefficients.

The solutions to Eq. (2.9) are

$$\varphi_n = \sin m_n y, \tag{A.1}$$

with the dispersion relation

$$\omega_n = Uk_n - \frac{\beta k_n}{K_n^2}, \tag{A.2}$$

or

$$c_n = U - \frac{\beta}{K_n^2}, \tag{A.3}$$

where $K_n^2 = k_n^2 + m_n^2$, and m_n is the meridional wavenumber of the waves. Substituting Eq. (A.1) into Eq. (2.14) and Eq. (2.21), we obtain their solutions

$$Y_{1n} = 0,$$
$$Y_{2n} = 0,$$
$$Y_3 = D_1 \sin(m_1 + m_2)y + D_2 \sin(m_1 - m_2)y,$$
$$Y_4 = D_3 \sin(m_1 + m_2)y + D_4 \sin(m_1 - m_2)y, \tag{A.4}$$

where

$$D_1 = \frac{(k_1m_2 - k_2m_1)(K_1^2 - K_2^2)}{2\{\beta k_{12} - (Uk_{12} - \omega_{12})[k_{12}^2 + (m_1 + m_2)^2]\}},$$
$$D_2 = \frac{(k_1m_2 + k_2m_1)(K_1^2 - K_2^2)}{2\{\beta k_{12} - (Uk_{12} - \omega_{12})[k_{12}^2 + (m_1 - m_2)^2]\}},$$
$$D_3 = \frac{(k_1m_2 + k_2m_1)(K_1^2 - K_2^2)}{2\{\beta \alpha_{12} - (U\alpha_{12} - \sigma_{12})[\alpha_{12}^2 + (m_1 + m_2)^2]\}},$$
$$D_4 = \frac{(k_1m_2 - k_2m_1)(K_1^2 - K_2^2)}{2\{\beta \alpha_{12} - (U\alpha_{12} - \sigma_{12})[\alpha_{12}^2 + (m_1 - m_2)^2]\}}. \tag{A.5}$$

The solutions to Eq. (2.31) are

$$H_n = \frac{2k_n^2 m_n}{(U - c_{g1})(m_1 + m_2)^2 - \beta} \sin 2m_n y. \tag{A.6}$$

Substituting Eqs. (A.1), (A.2), and (A.6) into Eq. (2.25) and Eq. (2.23), we obtain the coefficients in the evolution equations (2.35):

$$\alpha_n = \frac{\beta k_n(3m_n^2 - k_n^2)}{(k_n^2 + m_n^2)^3}, \quad \sigma_n = \frac{2k_n^3 m_n^2(3m_n^2 - k_n^2)K_n^2}{\beta(3m_n^4 - 6k_n^2m_n^2 - k_n^4)},$$

$$\begin{aligned} \nu_{12} = \frac{1}{2K_1^2}[&D_1(k_1m_2 - k_2m_1)(K_1^2 + 2k_1k_2 + 2m_1m_2) \\ &+ D_2(k_1m_2 + k_2m_1)(K_1^2 + 2k_1k_2 - 2m_1m_2) \\ &+ D_3(k_1m_2 + k_2m_1)(K_1^2 - 2k_1k_2 + 2m_1m_2) \\ &+ D_4(k_1m_2 - k_2m_1)(K_1^2 - 2k_1k_2 - 2m_1m_2)], \\ &\qquad m_1 \neq m_2, \end{aligned}$$

$$\begin{aligned} \nu_{12} = \frac{1}{2K_1^2}[&D_1m(k_1 - k_2)(K_1^2 + 2k_1k_2 + 2m^2) \\ &+ D_3m(k_1 + k_2)(K_1^2 - 2k_1k_2 + 2m^2)] \\ &- \frac{2k_1k_2^2m^2(4m^2 - K_1^2)}{K_1^2(4m^2(c_{g1} - U) + \beta)}, \quad m_1 = m_2 = m, \end{aligned}$$

$$\begin{aligned} \nu_{21} = \frac{1}{2K_2^2}[&D_1(k_2m_1 - k_1m_2)(K_2^2 + 2k_1k_2 + 2m_1m_2) \\ &+ D_2(k_2m_1 + k_1m_2)(K_2^2 + 2k_1k_2 - 2m_1m_2) \\ &+ D_3(k_2m_1 + k_1m_2)(K_2^2 - 2k_1k_2 - 2m_1m_2) \\ &+ D_4(k_2m_1 - k_1m_2)(K_2^2 - 2k_1k_2 - 2m_1m_2)], \\ &\qquad m_1 \neq m_2, \end{aligned}$$

$$\begin{aligned} \nu_{21} = \frac{1}{2K_2^2}[&D_1m(k_2 - k_1)(K_2^2 + 2k_1k_2 + 2m_1m_2) \\ &+ D_3m(k_1 + k_2)(K_2^2 - 2k_1k_2 + 2m^2)] \\ &- \frac{2k_1^2k_2m^2(4m^2 - K_2^2)}{K_2^2[4m^2(c_{g1} - U) + \beta]}, \quad m_1 = m_2 = m. \end{aligned} \tag{A.7}$$

REFERENCES

Ablowitz, M. J., D. J. Kaup, A. C. Newell, and H. Segur, 1974: The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud. Appl. Math.*, **53,** 259–315.

Boyd, J. P., 1983: Equatorial solitary waves. Part II: Envelope solitons. *J. Phys. Oceanogr.*, **13,** 428–449.

Gibbon, J. D., I. N. James, and I. M. Moroz, 1979: An example of soliton behavior in a rotating baroclinic fluid. *Proc. Roy. Soc. London,* **A367,** 219–237.

Jeffrey, A., and T. Kawahara, 1982: *Asymptotic Methods in Nonlinear Wave Theory.* Pitman, 256 pp.

Karoly, D. J., 1983: Rossby wave propagation in a barotropic atmosphere. *Dyn. Atmos. Oceans,* **7,** 111–125.

Lee, S., and I. Held, 1993: Baroclinic wave packets in models and observations. *J. Atmos. Sci.*, **50,** 1413–1428.

Manakov, S. V., 1973: On the theory of two-dimensional stationary self-focusing of electromagnetic waves. *Sov. Phys. JETP,* **38,** 248–253.

Moroz, I., and J. Brindley, 1981: Evolution of baroclinic wave packets in a flow with continuous shear and stratification. *Proc. Roy. Soc. London,* **A667,** 374–404.

——, and ——, 1984: Nonlinear amplitude evolution of baroclinic wave trains and wave packets. *Stud. Appl. Math.*, **70,** 21–62.

Pedlosky, J., 1972: Finite amplitude baroclinic wave packets. *J. Atmos. Sci.,* **29,** 680–686.

Redekopp, L. G., and P. D. Weidman, 1978: Solitary Rossby waves in zonal shear flows and their interactions. *J. Atmos. Sci.,* **35,** 790–804.

Swanson, K., and R. T. Pierrehumbert, 1994: Nonlinear wave packet evolution on a baroclinically unstable jet. *J. Atmos. Sci.,* **51,** 384–396.

Tan, B., 1993: Nonlinear Rossby waves and their interactions Part II. Periodic wave solution and its stability (in Chinese). *J. Peking Univ.,* **29,** 378–384.

Yamagata, T., 1980: Stability, modulation and wave resonance of a planetary wave in a rotating, two-layer fluid on a channel beta plane. *J. Meteor. Soc. Japan,* **58,** 160–171.

Yano, J.-I., 1992: Asymptotic theory of thermal convection in the rapidly rotating systems. *J. Fluid Mech.,* **243,** 103–131.

Zakharov, V. E., and A. B. Shabat, 1972: Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP,* **34,** 62–69.