On Nonlinear Symmetric Stability and the Nonlinear Saturation of Symmetric Instability

MU MU

LASG, Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing, China

THEODORE G. SHEPHERD

Department of Physics, University of Toronto, Toronto, Ontario, Canada

KYLE SWANSON

Department of Geophysical Sciences, University of Chicago, Chicago, Illinois

(Manuscript received 27 October 1995, in final form 26 March 1996)

ABSTRACT

A nonlinear symmetric stability theorem is derived in the context of the f-plane Boussinesq equations, recovering an earlier result of Xu within a more general framework. The theorem applies to symmetric disturbances to a baroclinic basic flow, the disturbances having arbitrary structure and magnitude. The criteria for nonlinear stability are virtually identical to those for linear stability. As in Xu, the nonlinear stability theorem can be used to obtain rigorous upper bounds on the saturation amplitude of symmetric instabilities. In a simple example, the bounds are found to compare favorably with heuristic parcel-based estimates in both the hydrostatic and nonhydrostatic limits.

1. Introduction

Cho et al. (1993, hereafter CSV) have recently revisited the classical problem of symmetric stability in the context of the f-plane Boussinesq equations. By using the *energy–Casimir* or *pseudoenergy* stability methodology of Fjørtoft (1950) and Arnol'd (1965, 1969) (see Holm et al. 1985; McIntyre and Shepherd 1987, section 6; Mu 1995 for a presentation of the methodology) CSV were able to recover the sufficient conditions for linear symmetric stability established by Fjørtoft (1950), Stone (1966), and Hoskins (1974). They were also able to establish a normed stability theorem for finite-amplitude disturbances, but only under somewhat more restrictive conditions on the basic flow. In an earlier study—unknown to CSV—Xu (1986) had derived a nonlinear symmetric stability theorem using direct methods whose criteria were essentially identical to the linear criteria. Xu (1986) went on to derive rigorous upper bounds on the nonlinear saturation of symmetric instability using his so-called "generalized energetics." Unaware of Xu's work, CSV also used their finite-amplitude stability theorem to obtain rigorous saturation bounds. In a simple example, these bounds were compared with heuristic parcel-based estimates obtained following the method of Emanuel (1983). In the nonhydrostatic limit, the rigorous bound was found to be close to the heuristic estimate; but in the hydrostatic limit, the rigorous bound was found to be an enormous overestimate.

The purpose of this short paper is twofold: first, to fill in the gap between the linear and nonlinear stability criteria of CSV, thereby recovering the earlier result of Xu (1986) within the more general energy–Casimir framework, and second, to provide improved saturation bounds for symmetric instability, especially in the hydrostatic limit.

2. Mathematical background

The system of equations to be considered (see CSV for more details) is the nonhydrostatic, adiabatic, Boussinesq equations on the f-plane, under symmetric conditions where all fields are independent of the horizontal coordinate y. This system may be written

$$\omega_t = -\partial(\psi, \omega) + \partial(m, fx) + \partial(\theta, gz/\theta_0), \quad (1a)$$

$$m_t = -\partial(\psi, m), \quad \theta_t = -\partial(\psi, \theta), \quad (1b,c)$$

and is considered in a simply connected domain D with the boundary condition

$$\psi = 0 \quad \text{on} \quad \partial D. \quad (1d)$$

Here ψ is the streamfunction of the motion in the x-z

Corresponding author address: Dr. Theodore G. Shepherd, Department of Physics, University of Toronto, 60 St. George Street, Toronto, ON M5S 1A7, Canada.
E-mail: tgs@atmosp.physics.utoronto.ca

plane, with associated velocity $(u, w) = (\psi_z, -\psi_x)$; $\omega = \nabla^2\psi = u_z - w_x$ is the y-component of vorticity; $m = v + fx$ is the y-component of absolute velocity; θ is the potential temperature, with constant reference value θ_0; f is the constant Coriolis parameter; g is the constant gravitational acceleration; and $\partial(a, b) \equiv a_z b_x - a_x b_z$ is the two-dimensional Jacobian operator.

We consider disturbances to a steady baroclinic basic flow (M, Θ), with the disturbed flow written as

$$\omega = \omega', \quad m = M + m', \quad \theta = \Theta + \theta'; \qquad (2)$$

the primed variables are *not* assumed to be of small amplitude. CSV showed that the system (1) conserves a disturbance pseudoenergy functional $\mathcal{A}$:

$$\frac{d\mathcal{A}}{dt} \equiv \frac{d}{dt}\iint_D \left\{\frac{1}{2}|\nabla\psi'|^2 + I(M, \Theta; m', \theta')\right\} dxdz = 0, \qquad (3)$$

where

$$I(M, \Theta; m', \theta') \equiv C(M + m', \Theta + \theta') - C(M, \Theta) - C_m(M, \Theta)m' - C_\theta(M, \Theta)\theta' \qquad (4)$$

and $C(m, \theta)$ is the function defined (up to an irrelevant constant) by $C_m(M, \Theta) = fx$, $C_\theta(M, \Theta) = gz/\theta_0$.

The small-amplitude approximation to I is given by the quadratic form

$$I(M, \Theta; m', \theta') \approx \frac{1}{2}\{C_{mm}(M, \Theta)(m')^2 + 2C_{m\theta}(M, \Theta)m'\theta' + C_{\theta\theta}(M, \Theta)(\theta')^2\}. \qquad (5)$$

Whenever (5) is positive definite, it follows that the quadratic approximation to $\mathcal{A}$ (which is conserved by the linearized dynamics) is also positive definite—which in turn implies normed stability of the linearized equations. As shown by CSV, the necessary and sufficient condition for this to be the case is that the eigenvalues of the matrix

$$\begin{pmatrix} m_0^2 C_{mm}(M, \Theta) & m_0\theta_0 C_{m\theta}(M, \Theta) \\ m_0\theta_0 C_{m\theta}(M, \Theta) & \theta_0^2 C_{\theta\theta}(M, \Theta) \end{pmatrix}, \qquad (6)$$

which we denote by $\lambda_1(M, \Theta)$ and $\lambda_2(M, \Theta)$, are both positive for all $(x, z) \in D$. (The factors m_0 and θ_0 have been introduced for dimensional consistency and do not affect the sign of the eigenvalues; m_0 is an arbitrary positive constant with the same dimension as m.) When $m_0 = g/f$, the matrix (6) is proportional to the inverse of Xu's (1986) stability matrix Π_j.

3. Nonlinear symmetric stability

In this section it is shown that linear symmetric stability implies nonlinear (normed) symmetric stability.

a. A special case

First consider the special but important case where M and Θ are linear functions of x and z (cf. Hoskins 1974), namely

$$M = ax + bz, \quad \Theta = \frac{N^2\theta_0}{g}z + \frac{f\theta_0 b}{g}x, \qquad (7)$$

where N^2, a, and b are arbitrary constants.[1] (Note that thermal-wind balance of the basic flow requires $f\theta_0 M_z = g\Theta_x$.) In this case the functional relationship between (M, Θ) and (x, z) implied by (7) may be inverted to give

$$x = \frac{N^2 M - (gb/\theta_0)\Theta}{N^2 a - fb^2}, \quad z = \frac{(ga/\theta_0)\Theta - fbM}{N^2 a - fb^2}, \qquad (8)$$

provided $N^2 a - fb^2 \neq 0$. Using (8), it follows that the function $C(m, \theta)$ defined by $C_m(M, \Theta) = fx$, $C_\theta(M, \Theta) = gz/\theta_0$ is the quadratic form

$$C(m, \theta) = \frac{1}{N^2 a - fb^2}\left\{\frac{1}{2}fN^2 m^2 - \left(\frac{fbg}{\theta_0}\right)m\theta + \frac{1}{2}\left(\frac{g}{\theta_0}\right)^2 a\theta^2\right\}, \qquad (9)$$

and the exact invariant $\mathcal{A}$ is therefore given by

$$\mathcal{A} = \iint_D \left\{\frac{1}{2}|\nabla\psi'|^2 + C(m', \theta')\right\} dxdz. \qquad (10)$$

In this special case, the small-amplitude approximation (5) is thus seen to be *exact.*

If m_0 is a positive constant having the same dimension as m, then

$$C(m', \theta') = \frac{1}{2}c_{11}\left(\frac{m'}{m_0}\right)^2 + c_{12}\left(\frac{m'}{m_0}\right)\left(\frac{\theta'}{\theta_0}\right) + \frac{1}{2}c_{22}\left(\frac{\theta'}{\theta_0}\right)^2, \qquad (11)$$

where

$$c_{11} = \frac{fN^2 m_0^2}{N^2 a - fb^2}, \quad c_{12} = -\frac{fbgm_0}{N^2 a - fb^2}, \quad c_{22} = \frac{g^2 a}{N^2 a - fb^2}. \qquad (12)$$

The quadratic form (11) is positive definite if and only if the eigenvalues of the matrix

[1] Although Θ_z is not required to be positive on mathematical grounds, we have introduced the positive factor N^2 because the problem of symmetric stability and instability is geophysically relevant only for statically stable basic flows with $g\Theta_z > 0$.

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix} \quad (13)$$

are both positive. Since

$$c_{11} = m_0^2 C_{mm}(M, \Theta), \quad c_{12} = m_0\theta_0 C_{m\theta}(M, \Theta),$$
$$c_{22} = \theta_0^2 C_{\theta\theta}(M, \Theta), \quad (14)$$

the eigenvalues of (13) are in this case just the (constant) eigenvalues $\lambda_1(M, \Theta)$ and $\lambda_2(M, \Theta)$ of (6). As noted by CSV, these eigenvalues are both positive if and only if

$$\mathrm{Ri} \equiv \frac{N^2}{V_z^2} > \frac{f}{\zeta} > 0, \quad (15)$$

where $\zeta \equiv f + V_x$ is the z-component of the basic-flow absolute vorticity, and $V \equiv M - fx$ is the basic-flow meridional velocity. In the present special case, (15) reduces to $N^2/b^2 > f/a > 0$, so that $N^2a - fb^2 \neq 0$, as required.

We now prove *nonlinear* stability, as follows. If the eigenvalues are ordered according to $0 < \lambda_1 \leqslant \lambda_2$, then we have the *convexity condition*

$$\frac{\lambda_1}{2}\left[\left(\frac{m'}{m_0}\right)^2 + \left(\frac{\theta'}{\theta_0}\right)^2\right] \leqslant C(m', \theta')$$
$$\leqslant \frac{\lambda_2}{2}\left[\left(\frac{m'}{m_0}\right)^2 + \left(\frac{\theta'}{\theta_0}\right)^2\right], \quad (16)$$

which applies at every point $(x, z) \in D$. Now introduce the norm

$$\|\mathbf{x}'\|_\lambda^2 \equiv \iint_D \left\{\frac{1}{2}|\nabla\psi'|^2 + \frac{\lambda}{2}\left[\left(\frac{m'}{m_0}\right)^2 + \left(\frac{\theta'}{\theta_0}\right)^2\right]\right\} dxdz \quad (17)$$

with $\lambda_1 \leqslant \lambda \leqslant \lambda_2$. Using (10) and (16) with (17) then yields

$$\|\mathbf{x}'(t)\|_\lambda^2 \leqslant \frac{\lambda}{\lambda_1}\|\mathbf{x}'(t)\|_{\lambda_1}^2 \leqslant \frac{\lambda}{\lambda_1}\mathcal{A}(t) = \frac{\lambda}{\lambda_1}\mathcal{A}(0)$$
$$\leqslant \frac{\lambda}{\lambda_1}\|\mathbf{x}'(0)\|_{\lambda_2}^2 \leqslant \frac{\lambda_2}{\lambda_1}\|\mathbf{x}'(0)\|_\lambda^2. \quad (18)$$

This establishes nonlinear Liapunov (normed) stability in the norm (17) for any basic flow of the form (7) whenever (15) is satisfied.

By using a generalized potential energy [denoted by A in his (4.1)] that turns out to be identical to (9) in this case, Xu (1986, lemma 2) proved nonlinear Liapunov stability, in a norm rather similar to (17), for basic flows of the form (7) satisfying the same criterion (15). However, while (18) implies a maximum amplification factor for the norm of $\sqrt{\lambda_2/\lambda_1}$, Xu's (1986) maximum amplification factor B_0 is rather more complicated—and for $m_0 = g/f$ it can be shown to always be larger. For example, if $\lambda_1 = \lambda_2 = \lambda$ (so that our maximum amplification factor is unity), then $B_0 = \max\{1 + \sigma, 3, 1 + (2/\sigma)\}$, where $\sigma^2 = (N^2 + f\zeta)/2$. For weak stability, with $N^2a - fb^2 \to 0$ and $\lambda_1 \ll \lambda_2$, then B_0 goes like the *square* of the (large) maximum amplification factor $\sqrt{\lambda_2/\lambda_1}$ implied by (18).

b. The general case

We now consider the general case and return to (3) and (4). Suppose first that the initial disturbance $(m'(0), \theta'(0))$ satisfies

$$\{(M + m'(0), \Theta + \theta'(0)) | (x, z) \in D\}$$
$$\subseteq \{(M, \Theta) | (x, z) \in D\}; \quad (19)$$

in other words, the disturbance introduces no new values (m, θ). Since by (1b,c) m and θ are Lagrangian invariants, the condition (19) will be satisfied for all time if it is satisfied at $t = 0$. Then Taylor's remainder theorem implies that at each point $(x, z) \in D$ there exists some $\tilde{m} \in (M, M + m')$ and $\tilde{\theta} \in (\Theta, \Theta + \theta')$ such that

$$I(M, \Theta; m', \theta') = \frac{1}{2}\{C_{mm}(\tilde{m}, \tilde{\theta})(m')^2 + 2C_{m\theta}(\tilde{m}, \tilde{\theta})m'\theta' + C_{\theta\theta}(\tilde{m}, \tilde{\theta})(\theta')^2\}. \quad (20)$$

If we suppose further that the (M, Θ) distribution is convex, then (19) ensures that $\{(\tilde{m}, \tilde{\theta}) | (x, z) \in D\} \subseteq \{(M, \Theta) | (x, z) \in D\}$. The eigenvalues of the matrix

$$\begin{pmatrix} m_0^2 C_{mm}(\tilde{m}, \tilde{\theta}) & m_0\theta_0 C_{m\theta}(\tilde{m}, \tilde{\theta}) \\ m_0\theta_0 C_{m\theta}(\tilde{m}, \tilde{\theta}) & \theta_0^2 C_{\theta\theta}(\tilde{m}, \tilde{\theta}) \end{pmatrix} \quad (21)$$

are evidently given by $\lambda_1(\tilde{m}, \tilde{\theta})$ and $\lambda_2(\tilde{m}, \tilde{\theta})$. Define

$$\Lambda_1 \equiv \min_{(M,\Theta)}\{\lambda_1(M, \Theta), \lambda_2(M, \Theta)\},$$
$$\Lambda_2 \equiv \max_{(M,\Theta)}\{\lambda_1(M, \Theta), \lambda_2(M, \Theta)\}. \quad (22)$$

The sufficient condition for linear stability is that $\lambda_1(M, \Theta)$ and $\lambda_2(M, \Theta)$ are everywhere positive; for nonlinear stability, we require further that they are everywhere bounded away from zero and infinity, namely

$$0 < \Lambda_1 \leqslant \Lambda_2 < \infty. \quad (23)$$

[This subtle distinction between the linear and nonlinear criteria is entirely analogous to the familiar case of the barotropic vorticity equation (Arnol'd 1965, 1969; Holm et al. 1985; McIntyre and Shepherd 1987).] Under these conditions, $I(M, \Theta; m', \theta')$ is bounded at every point $(x, z) \in D$ according to the convexity condition

$$\frac{\Lambda_1}{2}\left[\left(\frac{m'}{m_0}\right)^2 + \left(\frac{\theta'}{\theta_0}\right)^2\right] \leqslant I(M, \Theta; m', \theta')$$
$$\leqslant \frac{\Lambda_2}{2}\left[\left(\frac{m'}{m_0}\right)^2 + \left(\frac{\theta'}{\theta_0}\right)^2\right]. \quad (24)$$

The proof of nonlinear stability now follows according to (18), with λ_1, λ_2 replaced by Λ_1, Λ_2 and with $\Lambda_1 \leqslant \lambda \leqslant \Lambda_2$:

$$\|\mathbf{x}'(t)\|_\lambda^2 \leqslant \frac{\Lambda_2}{\Lambda_1} \|\mathbf{x}'(0)\|_\lambda^2. \quad (25)$$

Note that the stability condition $\Lambda_1 > 0$ is equivalent to (15).

Xu (1986, Lemma 4) also proved nonlinear stability for general basic flows satisfying (23), though without providing explicit bounds on the disturbance amplification. The marginal case $\Lambda_1 = 0$ was also treated by Xu (1986, Theorem 6).

CSV established a nonlinear stability result analogous to (25) but that applied to a more restricted class of basic flows and that had a different (in general larger) amplification factor. It is interesting to see the connection between the two results. The eigenvalues of (21) are given by

$$\lambda_{1,2} = \frac{1}{2} \{ m_0^2 \tilde{C}_{mm} + \theta_0^2 \tilde{C}_{\theta\theta} \pm [(m_0^2 \tilde{C}_{mm} - \theta_0^2 \tilde{C}_{\theta\theta})^2 + 4(m_0\theta_0 \tilde{C}_{m\theta})^2]^{1/2} \}, \quad (26)$$

where the tilde indicates that the function is evaluated at $(\tilde{m}, \tilde{\theta})$. Now the square root in the above expression can be bounded according to

$$[(m_0^2 \tilde{C}_{mm} - \theta_0^2 \tilde{C}_{\theta\theta})^2 + 4(m_0\theta_0 \tilde{C}_{m\theta})^2]^{1/2} \leqslant |m_0^2 \tilde{C}_{mm} - \theta_0^2 \tilde{C}_{\theta\theta}| + 2|m_0\theta_0 \tilde{C}_{m\theta}|, \quad (27)$$

which leads to the following bounds on the eigenvalues:

$$\lambda_{1,2} \geqslant \frac{1}{2} \{ m_0^2 \tilde{C}_{mm} + \theta_0^2 \tilde{C}_{\theta\theta} - |m_0^2 \tilde{C}_{mm} - \theta_0^2 \tilde{C}_{\theta\theta}| - 2|m_0\theta_0 \tilde{C}_{m\theta}| \}, \quad (28a)$$

$$\lambda_{1,2} \leqslant \frac{1}{2} \{ m_0^2 \tilde{C}_{mm} + \theta_0^2 \tilde{C}_{\theta\theta} + |m_0^2 \tilde{C}_{mm} - \theta_0^2 \tilde{C}_{\theta\theta}| + 2|m_0\theta_0 \tilde{C}_{m\theta}| \}. \quad (28b)$$

Under the stability assumptions $\tilde{C}_{mm} > 0$, $\tilde{C}_{\theta\theta} > 0$, (28) is equivalent to

$$\min(m_0^2 \tilde{C}_{mm}, \theta_0^2 \tilde{C}_{\theta\theta}) - |m_0\theta_0 \tilde{C}_{m\theta}| \leqslant \lambda_{1,2} \leqslant \max(m_0^2 \tilde{C}_{mm}, \theta_0^2 \tilde{C}_{\theta\theta}) + |m_0\theta_0 \tilde{C}_{m\theta}|. \quad (29)$$

It is easy to verify that using these bounds on Λ_1 and Λ_2 in (25), with $m_0 = g/N$, CSV's corresponding normed stability result (3.12) is obtained. Thus, while the present results involve the actual eigenvalues of the Hessian of C, CSV's results—while not presented that way—effectively used *bounds* on those eigenvalues. This explains why CSV's results are weaker than the present ones.

What can be done if the condition (19) is not satisfied? In this case, the disturbance introduces new (m, θ) values beyond those present in the basic flow. The expression (20) is still well defined, and the conservation law (3) is still valid provided one suitably extends the domain of definition of the function $C(m, \theta)$. Such extension is clearly always possible; the question is whether it can be done in such a way that (22) remains true. In the case of one-dimensional functions, such extension can always be done (cf. Arnol'd 1969). But for two-dimensional functions, it remains to our knowledge an open question. (It is, however, not a problem in the example considered in section 4 below, which follows section 3a.) This means that in any particular application, one would have to determine the eigenvalues of (21) for the range of (m, θ) values present in the initial flow [based on a suitable extension of $C(m, \theta)$] and use those values in (25).

c. Extension to moist adiabatic flow

A brief remark may be made concerning the extension of this analysis to moist adiabatic flow. Section 2b of CSV showed that the linear symmetric stability criteria extended to moist adiabatic flow (cf. Bennetts and Hoskins 1979), with θ replaced by the equivalent potential temperature $\theta_e = \theta_e(\theta, z)$. In the finite-amplitude context, (3) remains valid but the function I now depends explicitly on z as well as on m and θ_e. This means that the eigenvalues of the moist adiabatic version of (21) will not be simply related to the eigenvalues of the moist adiabatic version of (6), or even to the eigenvalues of (21) at $t = 0$. However, one could still determine the minimum and maximum eigenvalues of the moist version of (21) over all possible m, θ_e, and z (the range of those values being determined by the initial conditions) and use those values as Λ_1 and Λ_2. When $\Lambda_1 > 0$, nonlinear stability would hold. But this would certainly be a much more restrictive condition than that for linear stability.

4. Nonlinear saturation of symmetric instability

CSV used the method of Shepherd (1988) to obtain rigorous upper bounds on the saturation amplitudes of symmetric instabilities. The method is very simple and is based on the fact that, from (3), the kinetic energy in the x-z plane is bounded in terms of the initial conditions according to

$$\iint_D \frac{1}{2} |\nabla\psi|^2(t)\,dxdz = \iint_D \frac{1}{2} |\nabla\psi'|^2(t)\,dxdz \leqslant \mathcal{A}(t) = \mathcal{A}(0) \quad (30)$$

for any stable basic state. Given an initial condition consisting of an arbitrary perturbation $(m^{(1)}, \theta^{(1)}, \omega^{(1)})$ to an *unstable* steady flow $(m^{(0)}, \theta^{(0)})$, we may regard this initial condition as a disturbance (m', θ', ω') to a stable basic flow (M, Θ). The initial disturbance is then given by $m'(0) = m^{(0)} - M + m^{(1)}$, $\theta'(0) = \theta^{(0)} - \Theta + \theta^{(1)}$, and $\omega'(0) = \omega^{(1)}$, and (30) becomes

$$\iint_D \frac{1}{2} |\nabla\psi|^2(t)dxdz \leqslant \iint_D \left\{\frac{1}{2} |\nabla\psi'(0)|^2 + I(M, \Theta; m'(0), \theta'(0))\right\} dxdz. \quad (31)$$

For any given initial condition, the right-hand side of (31) is a functional of the basic flow, and one may seek to minimize it over all possible stable basic flows.

To demonstrate the method, CSV analyzed the example of an infinitesimal initial perturbation to a pure baroclinic flow

$$m^{(0)} = \tilde{N}(1 + \epsilon)z + fx, \quad \omega^{(0)} = 0, \quad (32a,b)$$

$$\theta^{(0)} = \frac{\theta_0\tilde{N}^2}{g} z + \frac{\theta_0\tilde{N}f}{g}(1 + \epsilon)x, \quad (32c)$$

considered in the rectangular domain

$$-\frac{L}{2} \leqslant x \leqslant \frac{L}{2}, \quad -\frac{H}{2} \leqslant z \leqslant \frac{H}{2}. \quad (33)$$

The flow (32) is unstable for $\epsilon > 0$, so we may regard ϵ as a "supercriticality" parameter.

Now introduce the basic flow

$$M = \tilde{N}(1 - \delta)z + (1 + \alpha)fx, \quad (34a)$$

$$\Theta = \frac{\theta_0\tilde{N}^2}{g}(1 + \gamma)z + \frac{\theta_0\tilde{N}f}{g}(1 - \delta)x, \quad (34b)$$

with potential vorticity

$$Q = \partial(\Theta, M) = \frac{f\theta_0\tilde{N}^2}{g}[\gamma + \alpha + \gamma\alpha + \delta(2 - \delta)]. \quad (35)$$

(This is a more general form than that used by CSV.) This basic flow is stable to symmetric disturbances provided that

$$\gamma > -1 \quad \text{and} \quad (1 - \delta)^2 < (1 + \gamma)(1 + \alpha). \quad (36)$$

The initial disturbance relative to this basic flow is then (apart from the infinitesimal part) given by

$$m'(0) = m^{(0)} - M = \tilde{N}(\epsilon + \delta)z - \alpha fx, \quad (37a)$$

$$\theta'(0) = \theta^{(0)} - \Theta = \frac{\theta_0\tilde{N}f}{g}(\epsilon + \delta)x - \frac{\theta_0\tilde{N}^2}{g}\gamma z. \quad (37b)$$

Upon using CSV's relations (3.6) together with (34), the second derivatives of C are found to be

$$C_{mm}(M, \Theta) = \frac{f\theta_0\tilde{N}^2}{gQ}(1 + \gamma), \quad (38a)$$

$$C_{\theta\theta}(M, \Theta) = \frac{fg}{\theta_0 Q}(1 + \alpha), \quad (38b)$$

$$C_{m\theta}(M, \Theta) = -\frac{f\tilde{N}}{Q}(1 - \delta). \quad (38c)$$

These second derivatives are constants, which means that $I(M, \Theta; m', \theta')$ is a quadratic form in m' and θ'; in other words, the small-amplitude expression (5) is exact in this case, and the nonlinear analysis of section 3a applies. One may therefore substitute (5), with (37) and (38) inserted, into (31) to obtain

$$\iint_D \frac{1}{2} |\nabla\psi|^2(t)dxdz \leqslant \frac{HL}{24[\gamma + \alpha + \gamma\alpha + \delta(2 - \delta)]} \times \{\tilde{N}^2H^2[(1 + \gamma)(\epsilon + \delta)^2 + 2(1 - \delta)(\epsilon + \delta)\gamma + (1 + \alpha)\gamma^2] + f^2L^2[(1 + \alpha)(\epsilon + \delta)^2 + 2(1 - \delta)(\epsilon + \delta)\alpha + (1 + \gamma)\alpha^2]\}. \quad (39)$$

The rigorous upper bound on the kinetic energy represented by (39) is a function of the parameters of the initial unstable flow, as well as of the free parameters α, γ, and δ, and one may seek to minimize the bound over all values of those free parameters that are consistent with the constraints (36).

The general analysis of this minimization problem appears to be complex and is of little intrinsic interest, but two particularly simple bounds may be obtained as follows. First, take $\delta = -\epsilon$ and $\alpha = 0$, leaving only γ free; this corresponds to modifying only $\theta'(0)$ with $m'(0) = 0$. In this case the right-hand side of (39) takes the simple form

$$\frac{1}{24} H^3 L\tilde{N}^2\left(\frac{\gamma^2}{\gamma - \epsilon(2 + \epsilon)}\right), \quad (40)$$

which is minimized for the choice $\gamma = 2\epsilon(2 + \epsilon)$ and yields the rigorous upper bound

$$\iint_D \frac{1}{2} |\nabla\psi|^2(t)dxdz \leqslant \frac{1}{6} H^3 L\tilde{N}^2\epsilon(2 + \epsilon). \quad (41)$$

This bound is similar to the analogous bound (4.20) of CSV but is smaller by a factor of $(2 + \epsilon)$. It may be compared with the heuristic parcel-based saturation estimate of Emanuel (1983), which is based on the assumption of mixing of θ along m-surfaces (thereby releasing gravitational potential energy) and might be thought to be relevant in the nonhydrostatic limit $f^2L^2 \gg \tilde{N}^2H^2$. This estimate is given by CSV's (4.31), namely

$$\frac{1}{12} H^3 L\tilde{N}^2\epsilon(2 + \epsilon). \quad (42)$$

Evidently (41) is just twice (42) for all ϵ.

Xu (1986) derived an upper bound on the transverse kinetic energy in the case of a horizontally unbounded domain, which falls within this context of $f^2L^2 \gg \tilde{N}^2H^2$. It may be shown that Xu's (1986) bound (4.9)—taking $K_2^0 = 0$—is exactly twice (41).

A second simple bound obtains on taking $\delta = -\epsilon$ and $\gamma = 0$, leaving only α free; this corresponds to modifying only $m'(0)$ with $\theta'(0) = 0$. In this case the right-hand side of (39) takes the simple form

$$\frac{1}{24} HL^3 f^2 \left(\frac{\alpha^2}{\alpha - \epsilon(2 + \epsilon)} \right), \quad (43)$$

which is minimized for the choice $\alpha = 2\epsilon(2 + \epsilon)$ and yields the rigorous upper bound

$$\iint_D \frac{1}{2} |\nabla\psi|^2(t) dxdz \leq \frac{1}{6} HL^3 f^2 \epsilon(2 + \epsilon). \quad (44)$$

This bound has no counterpart in CSV since the parameter α was not used there. It may be compared with the heuristic parcel-based saturation estimate obtained on the assumption of mixing of m along θ-surfaces (thereby releasing centrifugal potential energy), which might be thought to be relevant in the hydrostatic limit $f^2L^2 \ll \tilde{N}^2H^2$. This estimate is given by CSV's (4.33), namely

$$\frac{1}{12} HL^3 f^2 \epsilon(2 + \epsilon). \quad (45)$$

Evidently (44) is just twice (45) for all ϵ. This is an enormous improvement on CSV's best bound (4.15) in the hydrostatic limit.

Xu's (1986) bound (4.7) on the transverse kinetic energy, when applied to this case with $\mathbf{h}^0 = 0$ and $\mathbf{v}^0 = 0$, is rather complicated.[2] But for $\epsilon \ll 1$, it is approximately given by

$$(L^2 + H^2)LH \frac{\tilde{N}^2 f^2 \epsilon}{\tilde{N}^2 + f^2}. \quad (46)$$

If we make the further (not unreasonable) assumptions $\tilde{N}^2 \gg f^2$ and $L^2 \gg H^2$, then (46) is roughly three times the right-hand side of (44).

In the intermediate regime $f^2L^2/\tilde{N}^2H^2 = O(1)$, parcel-based estimates are difficult to obtain due to a lack of knowledge about the slope of the parcel trajectories. In contrast, the upper bound (39) is rigorously valid throughout this regime and may be minimized using numerical methods if necessary.

Acknowledgments. MM is supported by the National Natural Science Foundation of China. TGS is supported by the Natural Sciences and Engineering Research Council and the Atmospheric Environment Service of Canada.

[2] N.B. There is a typographical error in Xu's (4.7): $\omega_{\min}$ should be $\omega^2_{\min}$.

REFERENCES

Arnol'd, V. I., 1965: Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid. *Sov. Math.*, **6,** 773–777. First published in *Dokl. Akad. Nauk. SSSR,* **162,** 975–978, 1965.

——, 1969: On an a priori estimate in the theory of hydrodynamical stability. *Amer. Math. Soc. Transl. Ser. 2,* **79,** 267–269. First published in *Izv. Vyssh. Uchebn. Zaved. Matematika,* **54**(5), 3–5, 1966.

Bennetts, D. A., and B. J. Hoskins, 1979: Conditional symmetric instability—A possible explanation for frontal rainbands. *Quart. J. Roy. Meteor. Soc.*, **105,** 945–962.

Cho, H.-R., T. G. Shepherd, and V. A. Vladimirov, 1993: Application of the direct Liapunov method to the problem of symmetric stability in the atmosphere. *J. Atmos. Sci.*, **50,** 822–836.

Emanuel, K. A., 1983: The Lagrangian parcel dynamics of moist symmetric instability. *J. Atmos. Sci.*, **40,** 2368–2376.

Fjørtoft, R., 1950: Application of integral theorems in deriving criteria of stability for laminar flows and for the baroclinic circular vortex. *Geofys. Publ.*, **17**(6), 1–52.

Holm, D. D., J. E. Marsden, T. Ratiu, and A. Weinstein, 1985: Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.*, **123,** 1–116.

Hoskins, B. J., 1974: The role of potential vorticity in symmetric stability and instability. *Quart. J. Roy. Meteor. Soc.*, **100,** 480–482.

McIntyre, M. E., and T. G. Shepherd, 1987: An exact local conservation theorem for finite-amplitude disturbances to non-parallel shear flows, with remarks on Hamiltonian structure and on Arnol'd's stability theorems. *J. Fluid Mech.*, **181,** 527–565.

Mu, M., 1995: Some advances in the study of nonlinear instability of the atmospheric motions. *Chin. J. Atmos. Sci.*, **19,** 318–334.

Shepherd, T. G., 1988: Rigorous bounds on the nonlinear saturation of instabilities to parallel shear flows. *J. Fluid Mech.*, **196,** 291–322.

Stone, P. H., 1966: On non-geostrophic baroclinic instability. *J. Atmos. Sci.*, **23,** 390–400.

Xu, Q., 1986: Generalized energetics for linear and nonlinear symmetric instabilities. *J. Atmos. Sci.*, **43,** 972–984.