A Set of Zonal Mean Equations in a Pressure–Isentrope Hybrid Vertical Coordinate

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11 February 1997 and 6 January 1998

ABSTRACT

A set of mass-weighted zonal mean equations in a pressure–isentrope hybrid vertical coordinate is derived. This formulation is able to present the nonacceleartion theorem in ageostrophic and finite-amplitude sense.

1. Introduction

Zonal mean states of the earth’s atmosphere and their maintenance mechanisms have been of central interest to many scientists for a long time because the Coriolis force and diabatic heating make the atmosphere almost axisymmetric. Once, mean meridional transports of mass, angular momentum, heat, and minor constituents were studied in the context of zonal means and eddies on pressure surfaces, the so-called conventional Eulerian mean values. The conventional zonal mean system, however, is seriously subject to the problem of Stokes drifts and prevents a physical understanding of the mean-meridional transports.

Andrews and McIntyre (1976) proposed the transformed Eulerian mean (TEM) method, which corrects Stokes drifts and represents nonacceleration theorem associated with wave momentum transports under the assumption of geostrophic balance and infinitesimally small wave amplitudes. Recently, many attempts have been made to estimate mass exchange rate between the stratosphere and the troposphere in the context of the downward control principle based on the TEM (Haynes et al. 1991). Strictly speaking, assumptions of the TEM are not valid for the real atmosphere. Diabatically derived mean meridional circulations look different from the dynamical derivations possibly due to breakdown of the above assumptions (e.g., Shine 1989; Rosenlof and Holton 1993). In order to generalize the nonacceleration theorem to ageostrophic and finite-amplitude cases, Andrews (1983) and Tung (1986) attempted to formulate zonal mean equations in isentropic coordinates. Their formulations seem to be too complicated to apply to the actual problems. Another deficiency is that the TEM cannot explicitly express the lower boundary conditions, as has already been discussed by many authors (e.g., Andrews 1980; McIntyre 1980) and commented on by Egger (1996).

A set of mass-weighted zonal mean equations is proposed in a pressure–isentrope hybrid vertical coordinate as discussed below (Iwasaki 1989, 1990, 1992, hereafter I89, I90, and I92). This note further refines the formulation and stresses that it has a very comprehensive form of generalizing the nonacceleration theorem and the downward control principle, furthermore, greatly extending its applicability to the real atmosphere.

2. Formulations

First of all, the basic formulation is briefly described. Vertical coordinates are defined as zonal mean pressures \( p_i \) on isentropes:

\[
p_i(\phi, \theta, t) = \frac{1}{2\pi} \int_0^{2\pi} p(\lambda, \phi, \theta, t) d\lambda,
\]

where \( p = p_i \) is taken in case of \( u < u_s \) or the ground surface lines when \( u > u_s \). Definitions of symbols are given in Table 1. Equation (1) is solved for the potential temperature \( \theta \) by a function of \( p_i \).

\[
\theta = \theta(\phi, p_i, t).
\]

The functional solution allows us to express any dependent variable as a function of \( \lambda, \phi, p_i, \) and \( t \). In the longitudinal direction, constant \( p_i \) lines become isentropic lines when \( \theta > \theta_i \) or the ground surface lines when \( \theta < \theta_i \). Zonal means are defined by

\[
\overline{A}^* = \frac{\partial p}{\partial p_i} A,
\]

where overbars denote zonal means along constant \( p_i \) lines and the asterisk denotes mass weighting (\( \equiv \partial p/\partial p_i \)).

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This is a mass-weighted isentropic zonal mean. When isentropes intersect the lower boundary in some longitudes, the mass weight \( \partial p/\partial p_l \) becomes zero, a kind of massless layer, in the longitudinal regions of \( \theta(\phi, p_l) < \theta_\ast \). The mass weight in the definition of zonal mean is required to express the conservation of any variable including effects of the intersection of isentropes with the lower boundary. Eddies are defined as departures from the zonal means (3) as follows:

\[
A' = A - \overline{A}. \tag{4}
\]

Based on the above definitions, we have a mean-zonal momentum equation,

\[
\begin{align*}
\left( \frac{\partial \mathbf{u}_*}{\partial t} \right)_p + \mathbf{v}_* \left[ \frac{1}{a \cos \phi} \left( \frac{\partial}{\partial \phi} \mathbf{u}_* \cos \phi \right) - f \right] + \frac{\omega_t^* \partial \mathbf{u}_*}{\partial p_l} &= D_F + \overline{X}_*, \tag{5}
\end{align*}
\]

where

\[
D_F = \nabla \cdot \mathbf{F}_p.
\]

The mass-weighted zonal mean is introduced to the zonal wind as well as the meridional wind and vertical velocity and it leads to a comprehensive form of wave momentum flux. The eddy momentum flux is derived into

\[
\mathbf{F} = (\mathbf{F}_\phi, \mathbf{F}_p),
\]

\[
= a \cos \phi \left\{ -(u'v')^* - \left( \frac{\partial p_t}{\partial \phi} \right)_p (u'v')^* \right\} + \frac{\partial \mathbf{F}_p}{\partial p_l}.
\]

(6)

The conservation of air mass is presented by

\[
\frac{d}{dt} \left( \frac{\partial}{\partial \phi} \left( a \cos \phi \mathbf{v}_* \cos \phi \right) \right)_{p_l} + \frac{\partial \mathbf{v}_*}{\partial p_l} = 0. \tag{10}
\]

Based on Eq. (10), a nondivergent mean-meridional mass circulation can be derived with an exact representation of the lower boundary condition.

3. Characteristics of the formulation

\subsection{a. Finite-amplitude expression of the nonacceleration theorem}

The above formulation expresses comprehensively the nonacceleration theorem in ageostrophic and finite-amplitude conditions. In the eddy momentum flux, vertical wave momentum transports \( p(\partial \Phi/\partial \lambda)_{p_l} \) separate from diffusive momentum transports due to diabatic mixings of material surfaces \( [u' \partial \overline{\theta}/\partial \lambda]^* \). Remember that \( p(\partial \Phi/\partial \lambda)_{p_l} \) is an exact expression of the form drag over the mountain and one can see that the \( p(\partial \Phi/\partial \lambda)_{p_l} \) is the form drag, that is, finite-amplitude wave momentum flux, over the isentropic surface when \( \theta > \theta_\ast \). It should be noted that this form can be derived from a mass-weighted zonal mean of the zonal momentum equation (5).

\subsection{b. Lower boundary conditions and their effects on baroclinic waves}

Following Eq. (1), the lower boundary value of \( p_1 \) at each latitude becomes

\[
p_{1\text{max}}(\phi, t) = p_1(\phi, \theta_{\text{max}}, t) = \frac{1}{2\pi} \int_{\theta_{\text{max}}}^{\theta_{\text{min}}} p_1 d\lambda. \tag{11}
\]

Zonal means and eddies of any variables can easily be
computed by using the definitions (3) and (4) even for isentropes intersecting the ground surfaces. The mass stream function of mean-meridional flows can express the mass conservation exactly. The wave momentum \( p \left( \partial \mathbf{V} / \partial \lambda \right)_p \) becomes form drag over mountains for \( \theta < \theta_0 \) or over the atmospheric isentropes otherwise. The \( p \) analysis is of great benefit to the analysis of the tropospheric circulations, particular of baroclinic waves, in which the lower boundary condition plays an essential role.

Edmon et al. (1980) applied the TEM method to Eady’s solution of baroclinic instability waves and showed that mass streamlines intersect the upper and lower boundaries at right angles and no mean-meridional flows appear in the entire domain. The intersection of streamlines with the lower boundaries tends to be misunderstood as source and sink of the air mass. The \( p \) analysis of Eady’s ideal solution showed that mean-meridional flows bounded by the upper and lower boundaries (190) appear as a result of the intersection of isentropes with the boundaries and they make mass streamlines parallel to the lower boundary and closed within the domain considered. The Coriolis acceleration of the mean-meridional flows balances with the eddy-induced forces. For Eady’s ideal solution, we confirm the following relationship between the Coriolis acceleration and mean-meridional flows even near the lower boundary

\[
\frac{\partial \mathbf{v}}{\partial t} = -\frac{\partial F}{\partial p},
\]

The \( p \) analysis of the GCM (189; 192) showed a similar type of mean-meridional flows bounded by the lower boundary. It suggests that actual baroclinic waves cause wave–mean flow interactions and mean meridional flows bounded by the lower boundary though intersections of isentropes.

c. Residual circulations versus diabatic circulations

As was discussed by Matsuno (1980), extratropical air parcels show perpetual elliptic motions in a meridional (latitude–pressure) plane, which mainly cause conventional Eulerian mean meridional circulations with indirect cells due to baroclinic waves and Rossby waves. Adopting \( p \) reduces the elliptic motions to oscillatory motions along isentropic lines and introducing the mass-weighted zonal mean eliminates effects of the oscillatory motions completely from mean-meridional circulations. It means that the \( p \) analysis completely corrects the effects of Stokes drifts on mean-meridional circulations.

In this formulation, the thermodynamic equation (8) does not have eddy term so that mean-meridional circulations only cause meridional heat transports. The eddy-free thermodynamic equation, in principle, makes the dynamically derived mean meridional circulations equivalent to the diabatic circulations. Note that such equivalence is retained by taking the mass-weighted zonal mean of diabatic heating as well.

In practice, the \( p \) analysis of the GCM showed mean-meridional circulations very similar to the so-called Brewer–Dobson circulations. In addition, dynamically derived circulations agree well with diabatically diagnosed circulations (189). The global mass exchange rate between the troposphere and stratosphere was shown to have a typical annual variation and its mechanism was interpreted under the downward control principle (192). The annual variation of the exchange rate also explains the variation of the lower stratospheric temperature around the equator. Thus, the above-mentioned formulation is feasible for making consistent analyses of mean meridional transports of mass, angular momentum, heat, and minor constituents in a realistic atmosphere.

Acknowledgments. The author is indebted to valuable discussions with Dr. A. Kasahara of National Center for Atmospheric Research and staff members of Numerical Prediction Division at Japan Meteorological Agency. Comments from anonymous reviewers were very helpful for improving the manuscript.

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