

Subsuns and Low Reynolds Number Flow

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ABSTRACT

The phenomenon called the subsun is the specular reflection of sunlight by horizontally oriented plates of ice. Although well known in meteorological optics, the hydrodynamics of the orientation is not quantitatively understood. The theory of torques on objects at low Reynolds numbers is reviewed; coefficients C_o , C_p , and C_ψ that describe the orienting torques on discs, rods, and hexagonal prisms are defined; and the results of experiments to measure C_o and C_p are reported.

1. Introduction

From an airplane it is sometimes possible to see a nearly specular reflection of the sun from a flat layer of stratus clouds, known in meteorological optics (Minaert 1954; Tricker 1970; Greenler 1980) as a subsun. This striking phenomenon is not unusual; the author has seen it twice in a few hundred flights (on most of which the lighting or meteorological conditions were unsuitable). On the occasions on which a subsun was observed (early morning in March over New Mexico, and late afternoon in June over Ohio) it was broadened by less than a degree of arc. Where the stratus was interrupted by cumulus the subsun disappeared, and where the stratus was partly transparent and a body of water appeared below it, the nearly specular reflection from the stratus was superimposed on the sharper specular reflection from the water.

The well-known explanation of the subsun is that the stratus clouds consist of small thin plates of ice maintaining a horizontal orientation as they fall. The angular broadening reflects a small dispersion in their orientations; diffraction may also contribute but is typically smaller. If diffraction were dominant the subsun would be larger in red than in blue light, appearing blue at the center but surrounded by a red halo, while angular dispersion of the plates leads to no dependence on color. No color dependence is apparent. Cumulus clouds may consist of spherical droplets of liquid water, explaining the absence of a subsun in them.

There are related phenomena in meteorological optics. Light pillars (Sassen 1987), like the subsun, require

horizontally oriented ice plates. The better-studied halos (Fraser 1979; Pattloch and Trankle 1984) may be caused by oriented or unoriented ice crystals, either plates or columns. Elliptical rings and halos may be caused by flat ice plates rocking about their horizontal orientation (Lynch et al. 1994). The principles of the hydrodynamics of the orientation of small falling objects are understood (Cox 1965; Chester 1990), but the formidable formalism has been neither explicitly evaluated (except for spheroids of small eccentricity and long rods) nor tested experimentally.

Under what conditions will a small, thin falling plate of ice maintain an accurately horizontal orientation? At high Reynolds number $Re > 1$ a falling plate leaves a turbulent wake (Willmarth et al. 1964; Pruppacher and Klett 1978). Its center of drag lies close to its leading edge or surface; any steady orientation is unstable; it tumbles, and its path is irregular because of large horizontal forces arising during its tumbling (ice crystals, unlike airplanes, rockets, and arrows, are not equipped with stabilizing tails!). This is readily verified by dropping a penny into a jar of water, an experiment in which $Re \approx 3000$; it tumbles and usually hits the sides. For $Re \approx 100$ a falling disc may oscillate periodically about a horizontal orientation as it leaves behind a regular vortex street. This may be seen by dropping aluminum foil discs of various radii into water. At $Re < 100$, however, tumbling and oscillations are strongly damped by viscosity. Intuitive concepts from our everyday experience with high Re flows are still qualitatively applicable and show that a vertical orientation (edge on to the flow) is unstable; if the plate tilts the hydrodynamic force on its leading edge acts to amplify the tilt. However, the horizontal orientation (face on to the flow) is stable; if the plate tilts the wake of the leading edge partly shields the trailing edge from the flow, reducing the drag on it; the resulting torque restores the horizontal orientation and the disturbance is quickly damped.

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2. Low Reynolds number flow

Very small plates will fall slowly, with $\text{Re} \ll 1$. The theory of low Re flow is presented by Happel and Brenner (1965), who include most of the results used in this section for flow in the limit $\text{Re} \rightarrow 0$. In the Navier–Stokes equation,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \frac{\eta}{\rho} \nabla^2 \mathbf{v}, \quad (1)$$

where \mathbf{v} is the fluid velocity field, ρ its density, p the pressure, and η its dynamic viscosity; the nonlinear inertial term $(\mathbf{v} \cdot \nabla) \mathbf{v} \sim v^2/l$, where l is a characteristic length, is $O(\text{Re})$ times smaller than the terms on the right-hand side and may be neglected. Similarly, the time derivative term $\partial \mathbf{v} / \partial t \sim v^2/l$ is of the same order of smallness (in the absence of time-dependent external forcing) and also may be neglected. This leaves the creeping flow equation

$$\nabla p = \eta \nabla^2 \mathbf{v}, \quad (2)$$

which is linear in velocity. Because of the linearity of (2), linear combinations of solutions that satisfy homogeneous boundary conditions (such as $\mathbf{v} = 0$ on a solid boundary) are also solutions.

An ellipsoidal body sinking under gravity in the limit $\text{Re} \rightarrow 0$ suffers no hydrodynamic torque, regardless of its orientation: Decompose the fluid velocity at infinity into components along the ellipsoid's principal axes. Each of these corresponds to a solution (Happel and Brenner 1965) of (2) that satisfies a zero-velocity boundary condition on the body's surface, has a uniform velocity at infinity, and (by symmetry) exerts no hydrodynamic torque about its geometric center. Thus the body suffers no hydrodynamic torque, whatever its orientation, and, by the symmetry of the solutions to (2), the center of drag is also its geometric center. If this is also its barycenter (as in the case of an ellipsoid of uniform density), then there is no net torque, and all orientations are neutrally stable.

The drag tensor \mathbf{D} of a spheroid or triaxial ellipsoid, defined by the relation between the applied force \mathbf{F} and its velocity \mathbf{u} ,

$$\mathbf{F} = \mathbf{D} \cdot \mathbf{u}, \quad (3)$$

is not isotropic. If an external force is not directed along a principal axis then \mathbf{u} will not be parallel to \mathbf{F} ; in a gravitational field an obliquely oriented ellipsoid will not sink straight down. An elementary calculation, using the classic results (Lamb 1945; Happel and Brenner 1965; Berg 1983) for a circular disc

$$F_{\parallel} = \frac{32}{3} \eta r u_{\parallel} \quad \text{and} \quad F_{\perp} = 16 \eta r u_{\perp},$$

where the subscripts indicate velocities and forces parallel and perpendicular to the disc's surface, and equating $F_{\parallel} = mg \sin \theta$ and $F_{\perp} = mg \cos \theta$, yields the result

for the angle ϕ between the velocity vector and the nadir:

$$\phi = \tan^{-1} \left(\frac{\sin \theta \cos \theta}{2 + \sin^2 \theta} \right), \quad (4)$$

where θ is the angle between the surface normal and the vertical. The maximum value of ϕ is $\tan^{-1}(1/24^{1/2}) \approx 11.5^\circ$, which is found for $\theta = \sin^{-1}[(2/5)^{1/2}] \approx 39.2^\circ$.

It is possible to generalize (in the limit $\text{Re} \rightarrow 0$) the result of zero torque from ellipsoids to a larger class of shapes. The general relation (Happel and Brenner 1965; Garcia de la Torre and Bloomfield 1981) between torque $\boldsymbol{\tau}$ and body velocity \mathbf{u} for a nonrotating body is

$$\boldsymbol{\tau} = \mathbf{T} \cdot \mathbf{u}, \quad (5)$$

where \mathbf{T} is a second-rank pseudotensor (it changes sign under inversion of all coordinates) because $\boldsymbol{\tau}$ is a pseudovector. It is not possible to construct a nonzero second-rank pseudotensor from a nonchiral geometric shape. Hence $\mathbf{T} = 0$ and there is no hydrodynamic torque on any nonchiral object in the limit $\text{Re} \rightarrow 0$. It is, however, possible for the centers of drag and mass to differ, so that the combination of hydrodynamic and gravitational forces may lead to a net torque and a preferred orientation; a nail is an example of such an object, which is nonetheless nonchiral.

3. Not quite such low Reynolds number flow

The observations of the subsun clearly require that flat plates of ice maintain a horizontal orientation, but we have seen that in the limit $\text{Re} \rightarrow 0$ there can be no aligning torque. The explanation is that for any *finite* Re there is an aligning torque, whose magnitude may be estimated as first order in Re and in θ :

$$\tau_{\theta} = C_o F_{\perp} r \theta \text{Re}_r, \quad (6)$$

where C_o is a dimensionless coefficient of order unity applicable to discs and flattened oblate spheroids. Applying F_{\perp} for a horizontally falling disc of radius r and half-thickness h to a thin hexagonal plate, and using the resulting $u = \pi g r h \rho / (8 \eta)$ and $\text{Re}_r \equiv ur/\nu = \pi g r^2 h \rho / (8 \eta \nu)$, where ν is the kinematic viscosity, yields

$$\tau_{\theta} = \frac{\pi^2 C_o g^2 r^5 h^2 \rho^2}{4 \eta \nu} \theta. \quad (7)$$

A torque of magnitude τ acting on a disc produces a rotation rate (Perrin 1934; Happel and Brenner 1965; Berg 1983) (about any axis) in low Re flow $\dot{\theta} = 3\tau_{\theta} / (32\eta r^3)$. Using this result as an approximation for a thin hexagonal plate, and combining the rotational and translational flows (permitted in the $\text{Re} \rightarrow 0$ limit because both satisfy the homogeneous boundary condition of zero relative velocity on the solid body), yields an exponential decay time t of any deviation from the horizontal orientation:

$$t = \frac{\theta}{\dot{\theta}} = \frac{128}{3\pi^2 C_o} \frac{\eta^2 \nu}{g^2 r^2 h^2 \rho^2}. \quad (8)$$

In this time the plate falls a distance $d = ut$, conveniently expressed in terms of its radius:

$$\frac{d}{r} = \frac{2}{3C_o \text{Re}_r}. \quad (9)$$

A steady shear of the wind velocity produces a tilt of a falling plate $\theta \approx v_{ij}t$, where v_{ij} is an appropriate component of the strain rate tensor of the wind shear. In a stratus cloud the strain rate is likely to be small or negligible if the air is stably stratified.

The dominant influence disrupting the alignment of small plates in still air is orientational Brownian motion (Perrin 1934, 1936; Tricker 1970; Fraser 1979; Garcia de la Torre and Bloomfield 1981; Berg 1983). The angular diffusion coefficient $D_o = 3k_B T / (32\eta r^3)$. The root-mean-square dispersion of the angular orientation about a single horizontal axis is

$$\langle \theta^2 \rangle^{1/2} = (2D_o t)^{1/2} = \left(\frac{8}{\pi^2 C_o} \frac{k_B T \eta \nu}{g^2 r^5 h^2 \rho^2} \right)^{1/2}. \quad (10)$$

For $r = 30\mu$, $h = 1.5\mu$, and $C_o = 1$ the dispersion $\langle \theta^2 \rangle^{1/2} \approx 0.01$ radian, roughly the largest value permitted by the observations. For these dimensions $t \approx 0.2/C_o$ s⁻¹, $u \approx 0.9$ cm s⁻¹, $d \approx 0.1$ cm $\approx 30r$, and $\text{Re}_r \approx 0.02$. Significantly smaller particles would not produce a clear specular reflection because of their Brownian angular dispersion, while much larger ones would precipitate rapidly.

Ice crystals may be needlelike, and their behavior is also of interest. The drag forces on prolate ellipsoids (Perrin 1934; Happel and Brenner 1965; Berg 1983) approximate those on needlelike hexagonal cylinders. At low Reynolds numbers they are

$$F_{\parallel} = 4 \frac{\pi \eta r u_{\parallel}}{[\ln(2r/h) - 1/2]}$$

and

$$F_{\perp} = 8 \frac{\pi \eta r u_{\perp}}{[\ln(2r/h) + 1/2]},$$

where r is the longest semiaxis (half the needle's length) and h is each of its shorter semiaxes (the needle's radius), only the leading terms in h/r are taken, and the subscripts indicate velocities and forces parallel and perpendicular to the needle's length. An elementary calculation of the direction of gravitational settling yields, in analogy to Eq. (4),

$$\phi = \tan^{-1} \left(\frac{(1 - \zeta) \sin \theta \cos \theta}{\sin^2 \theta + \zeta \cos^2 \theta} \right), \quad (11)$$

where θ is the angle between the longest axis and the horizontal and $\zeta \equiv [\ln(2r/h) + \frac{1}{2}]/2[\ln(2r/h) - \frac{1}{2}]$. For $\zeta = (2/3)$ Eq. (4) is recovered; plausible values of ζ are

quite close to 2/3 and lead to a very similar dependence of ϕ on θ .

In the limit $\text{Re} \rightarrow 0$ needles will fall with indeterminate orientation, by the same argument that applies to flat plates. At small but finite Re there will similarly be an aligning torque tending to make their long axes horizontal,

$$\tau_{\theta} = C_p F_{\perp} r \theta \text{Re}_r, \quad (12)$$

where the coefficient C_p applies to needles and slender prolate spheroids. The results analogous to Eqs. (7)–(10) are

$$\tau_{\theta} = \frac{\pi C_p (\ln(2r/h) + 1/2) g^2 r^3 h^4 \rho^2 \theta}{2 \eta \nu}, \quad (13)$$

$$t = \frac{16}{3C_p (\ln(2r/h) - 1/2)(\ln(2r/h) + 1/2)} \frac{\eta^2 \nu}{g^2 h^4 \rho^2}, \quad (14)$$

$$\frac{d}{r} = \frac{\ln(2r/h) + 1/2}{3C_p (\ln(2r/h) - 1/2) \text{Re}_r}, \quad (15)$$

$$\langle \theta^2 \rangle^{1/2} = \left(\frac{4}{\pi C_p (\ln(2r/h) + 1/2) g^2 r^3 h^4 \rho^2} \frac{k_B T \eta \nu}{g^2 r^3 h^4 \rho^2} \right)^{1/2}, \quad (16)$$

and the angular diffusion coefficient of prolate spheroids $D_p = 3[\ln(2r/h) - \frac{1}{2}]k_B T / (8\pi\eta r^3)$.

A needlelike ice crystal may also rotate about its long axis. Because it is a hexagonal prism, there are preferred values of its orientation angle ψ about this axis. In analogy to the argument for thin plates, the preferred orientation is that in which two prism faces (top and bottom) are horizontal if $\theta = 0$. The previous discussion may also be applied to this rotation. Because a hexagonal prism is close to a circular cylinder, for its rotation about its long axis the relation (Lamb 1945) for a long cylinder of radius h and length $2r$ is used: $\dot{\psi} = \tau_{\psi} / (8\pi r h^2 \eta)$ (rather than that for an ellipsoid). The torque is given by

$$\tau_{\psi} = C_{\psi} F_{\perp} h \psi \text{Re}_h, \quad (17)$$

where h is used in place of r both in the lever arm and in the Reynolds number Re_h ; the coefficient C_{ψ} is expected to be small because of the resemblance of a hexagonal prism to a circular cylinder, for which $C_{\psi} = 0$. The results analogous to Eqs. (7)–(10) and (13)–(16) are

$$\tau_{\psi} = \frac{\pi C_{\psi} (\ln(2r/h) + 1/2) g^2 r h^6 \rho^2 \psi}{2 \eta \nu}, \quad (18)$$

$$t = \frac{16}{C_{\psi} (\ln(2r/h) + 1/2) g^2 h^4 \rho^2} \frac{\eta^2 \nu}{g^2 h^4 \rho^2}, \quad (19)$$

$$\frac{d}{h} = \frac{\ln(2r/h) + 1/2}{C_{\psi} \text{Re}_h}, \quad (20)$$

$$\langle \psi^2 \rangle^{1/2} = \left(\frac{4}{\pi C_{\psi} (\ln(2r/h) + 1/2) g^2 r h^6 \rho^2} \frac{k_B T \eta \nu}{g^2 r h^6 \rho^2} \right)^{1/2}, \quad (21)$$

and the angular diffusion coefficient $D_\psi = k_B T / (8\pi\eta r h^2)$.

For an ice needle with $r = 30\mu$, $h = 3\mu$, and $C_p = 1$, $\langle\theta^2\rangle^{1/2} \approx 0.04$ radian, while $\langle\psi^2\rangle^{1/2} \approx 0.4/C_\psi^{1/2}$ radian. Brownian angular dispersion is larger for needles than for plates of comparable dimensions and is particularly disruptive of the orientation of needles about their long axes. However, a slender needle is lighter and falls more slowly than a plate whose diameter equals the needle's length, so longer needles may be found in clouds, and these longer needles are better oriented.

4. Experiments

The coefficients C_o , C_p , and C_ψ can be measured, either in the laboratory or from numerical experiments with three-dimensional hydrodynamics calculations. Experimental measurements were performed. These turned out to be unexpectedly difficult, in part because the coefficients are small. For Re in the range 1–100 the orienting torques on falling discs and rods are large, and they become horizontal very quickly (in a few diameters). However, C_o , C_p , and C_ψ are defined as coefficients in a low- Re expansion of the torque, so that experiments must be done for $Re \ll 1$, for which the orienting torques are small. This makes the experiments sensitive to small imperfections and heterogeneities in the shape and mass distributions of the test objects. The requirement that the orienting torques be large enough to observe (and be large compared to spurious torques resulting from imperfections) conflicts with the requirement that the first term in an expansion in Re be a good approximation. In the experiments reported here $0.10 < Re_r < 0.25$, with most data obtained at the upper end of that range.

It is easy to obtain very low Re by using very viscous fluids, but the orienting torques would be too small to observe. For larger Re high viscosities imply large test objects (and increased wall effects) or higher speeds (making observations more difficult). Obtaining the required Re with test bodies of reasonable size requires a quite modest viscosity; soybean oil was used, for which $\eta = 0.6$ poise at room temperature (Weast 1967). The size of test bodies also represents a compromise; very small objects are difficult to make, but large ones would require an excessively large fluid volume to avoid wall effects on the flow, which decrease linearly in the ratio of body size to wall distance (Happel and Brenner 1965); discs with radii in the range 0.16–0.32 cm were used in a full tank of dimensions 15 cm \times 20 cm \times 30 cm. The modest viscosity requires very light test bodies to obtain Re in the desired range; the discs were punched using ordinary paper punches from hard 1-mil (0.0025 cm) thick aluminum foil (household aluminum foil is softer, and discs punched from it have ragged edges). Data were recorded using a handheld 35-mm camera at about 50-cm distance.

When working with small and light bodies, surface

forces are important. Attempts to use sugar-water solutions as the fluid failed because air bubbles invariably attached themselves to the test bodies when they were pushed through the air–liquid surface. This problem was not solved by the addition of surfactants and required the use of oil, which better wets aluminum and pencil leads. Even so, small bubbles of air often had to be dislodged by stirring the fluid near them after the test bodies were immersed.

Data were obtained from sinking discs with $r = 0.32$ cm ($1/4''$ diameter), for which $Re_r = 0.23$, and from discs with $r = 0.24$ cm ($3/16''$ diameter), for which $Re_r = 0.13$. Measurements of eight discs with $r = 0.32$ cm and of six discs with $r = 0.24$ cm led to the results

$$C_o = \begin{cases} 0.169 \pm 0.017 & Re_r = 0.23 \\ 0.26 \pm 0.05 & Re_r = 0.13. \end{cases} \quad (22)$$

The (geometric) mean θ during the experiments was 16° for the $r = 0.32$ cm discs and 23° for the $r = 0.24$ cm discs; Eq. (22) includes a finite θ correction factor $2\theta[\cos^2\theta + (3/2)\sin^2\theta]/\sin 2\theta$ that corrects for the dependence of the vertical component of u on θ and also replaces θ in Eq. (6) by $(1/2)\sin 2\theta$, giving the correct limiting torques at $\theta = 90^\circ$ ($\tau = 0$) and at $\theta = 0$.

The experiments with $r = 0.32$ cm also yielded three discs whose relaxation to horizontal was strikingly slower (by about 50%) than that of the eight included above (whose dispersion was about 10%). These outliers could have been the consequence of unobserved small attached bubbles or manufacturing imperfections (punching leaves a region of plastic flow and tearing around a disc's edge that is not exactly symmetric, and is sometimes grossly asymmetric) that displace the center of gravity from the center of drag. If these outlying data points are included the result is $C_o = 0.154 \pm 0.031$, not significantly different from the result above. In the experiments with $r = 0.24$ cm two discs did not relax to the horizontal at all and were rejected entirely; the dispersion of relaxation rates among the remainder was greater than for the larger discs, but there were no remaining outliers.

The errors quoted are the standard deviations of the measurements of individual discs. If the scatter were Gaussian with zero mean, then the uncertainty in C_o would be about 0.4 of that quoted, but this assumption is probably unduly optimistic. The measurements at two different Reynolds numbers appear to disagree significantly; known sources of systematic error are believed to be much less than the quoted errors. This may imply that the $Re \rightarrow 0$ limit is not adequately approximated by Reynolds numbers of 0.2. It would be desirable to extend the measurements to yet lower Reynolds numbers, but smaller discs are both more sensitive to imperfections (because of their lower Reynolds numbers and smaller aligning torques) and harder to observe accurately, as is shown by the greater dispersion of the results for $r = 0.24$ cm discs.

Cylinders of mechanical pencil “lead” with $h = 0.0183$ cm (sold as 0.3-mm diameter) and lengths 0.44–0.66 cm were used to determine C_p . Despite the apparent uniformity of the cylindrical shape (their ends were flattened by mounting them in a jeweler’s lathe and grinding them against a flat grindstone) the results were rather scattered. The origin of the scatter may be heterogeneity in density or remaining imperfections of the end faces. The result was

$$C_p = 0.059 \pm 0.016, \quad (23)$$

at a mean $Re_r = 0.16$. As above, the quoted error is the dispersion of the (seven) individual measurements only. This rough result is less than the theoretical prediction of $C_p \approx 0.12$ for the rods used (Khayat and Cox 1989; Newsom and Bruce 1994) and again may reflect either manufacturing imperfections, unobserved attached bubbles, or a failure to reach the $Re \rightarrow 0$ regime at finite Re . Here C_ψ is much more difficult to measure, requiring precision hexagonal cylinders, and this was not attempted.

5. Discussion

Previous work (Lynch et al. 1994; Klett 1995) on the orientation of falling particles has used the results of Cox (1965) for the orienting torques. These results were derived for infinitesimal Reynolds numbers and spheroids of infinitesimal eccentricity and are not properly applicable to thin plates and rods, which are approximated by spheroids of unit eccentricity. Nonetheless, Cox’s result, which corresponds to $C_o = 29\pi/720 \approx 0.13$, is comparable to, though perhaps not in complete agreement with, the experimental data summarized in Eq. (22).

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