NOTES AND CORRESPONDENCE

Frontal Equilibration by Frictional Processes

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ABSTRACT

The effect of friction on frontogenesis driven by a stretching deformation field is studied analytically in both a quasigeostrophic and a semigeostrophic framework for a semi-infinite, adiabatic, Boussinesq fluid on an f plane. Friction is incorporated into the model in terms of a boundary layer pumping term.

The solutions demonstrate that the effect of friction is frontolytic. The quasigeostrophic fronts always equilibrate at a finite horizontal scale. The semigeostrophic fronts equilibrate at a finite horizontal scale if the strength of the frontogenesis is below a threshold value. Above this threshold, the front is predicted to collapse to a discontinuity in its thermal and momentum fields.

1. Introduction

Observations indicate that the horizontal scale of a surface front exhibits a wide range of values. Some fronts show scales of only a few kilometers; others are significantly wider and more diffuse. An unresolved issue in frontal dynamics regards the process or processes that control the ultimate scale of the front. The numerical experiments of Gall et al. (1987) and Snyder et al. (1993) support the idea that there is no dynamical process that prevents the collapse of a two-dimensional front and the formation of a temperature discontinuity in the absence of frictional processes.

The effects of friction have been investigated with two different types of frontogenesis. One type deals with frontogenesis driven by baroclinic instability. Two-dimensional studies (e.g., Blumen 1980; Blumen and Wu 1983; Volkert and Bishop 1990; Snyder et al. 1993; Nakamura and Held 1989; Cooper et al. 1992; Garner et al. 1992; Nakamura 1994) suggest that friction fails to prevent frontal collapse, presumably because there is an infinite energy source in the model. The latter four studies deal more with the issue of baroclinic-wave equilibration associated with the frictional generation of potential vorticity anomalies. In these wave equilibration studies, friction limits the front to scales resolvable by the numerical model and allows the subsequent evolution of the occlusion and nonlinear baroclinic instability process to be investigated. A second type deals with frontogenesis driven by a horizontal deformation field. Williams (1974) and Snyder et al. (1993) successfully simulate steady-state fronts of finite horizontal scale using numerical models that include the eddy diffusion of heat and momentum.

The purpose of this note is to present a simple analytic theory to the problem of deformation-driven frontogenesis with boundary layer pumping. The use of a deformation field as the geostrophic forcing of the front is particularly advantageous because the deformation field has no vorticity and thus has no contribution to the Ekman pumping. Thus, the geostrophic forcing is constant in our problem and frontal equilibration results solely from the interaction of surface friction on the frontal dynamics and the issue of wave equilibration does not arise.

Section 2 presents the semigeostrophic model where the effects of friction are incorporated by the inclusion of an Ekman pumping term in the lower boundary condition. Quasigeostrophic results are obtained from this model by inspection of the results in geostrophic space. Section 3 describes the properties of the solution. All the quasigeostrophic model fronts equilibrate with a finite horizontal scale. The semigeostrophic model fronts equilibrate with a finite horizontal scale provided the strength of the frontogenesis does not exceed a threshold value.

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2. The model

The model atmosphere is an adiabatic, Boussinesq, hydrostatic fluid residing on an f plane that is semi-infinite in the vertical. We use a Cartesian coordinate system (x, y, z) where gravity is $-g\hat{\mathbf{z}}$. A flat lower boundary lies at $z = 0$. Frontogenesis is forced by an imposed stretching deformation field independent of height whose horizontal stream function is $\psi = \alpha xy$. The positive constant α measures the strength of the deformation in units of inverse second. Frontal formation is anticipated along the dilatation axis $x = 0$. The potential temperature field has the form

$$\theta(x, z, t) = \theta_0(1 + N^2 z/g) + \theta_d(x, z, t), \quad (2.1)$$

where θ_d is the dynamic contribution to the potential temperature field associated with the front, θ_0 is a constant reference value, and N is the constant ambient buoyancy frequency. The dynamic component θ_d is taken to be independent of y and this assumption restricts the analysis to straight infinitely long fronts with no alongfront variation.

Hoskins and Bretherton (1972) show that, for inviscid, adiabatic, semigeostrophic flow with uniform potential vorticity, the frontogenesis problem in geostrophic space defined by

$$\mathbf{X} \equiv (X, Z, T) = (x + v_g/f, z, t), \quad (2.2)$$

is identical to the quasigeostrophic one in physical space. Here v_g is the part of the meridional wind associated with the front and not with the deformation field. Semigeostrophic theory assumes v_g is geostrophic. Exact uniformity of the potential vorticity is achieved if the dynamic component of the potential temperature field is a harmonic function of X and NZ/f (Bannon 1984). A convenient initial condition is

$$\theta_d(X, Z, T = 0) = \frac{\Delta\theta}{(\pi/2)} \tan^{-1}\left[\frac{X}{L_0 + NZ/f}\right], \quad (2.3)$$

which describes a warm airmass for $X > 0$ and a cool air mass for $X < 0$ separated by a broad transition zone of characteristic horizontal scale L_0, which is arbitrarily large but finite.

Following Hoskins and Bretherton (1972) and Blumen (1980), the presence of friction is incorporated by the application of the Ekman layer theory. Then the lower boundary condition takes the form

$$w(x, z = 0, t) = \left(\frac{\kappa}{2f}\right)^{1/2} \frac{\partial v_g}{\partial x} \quad (2.4a)$$

or, in geostrophic space,

$$w(X, Z = 0, T) = J\left(\frac{\kappa}{2f}\right)^{1/2} \frac{\partial v_g}{\partial X}, \quad (2.4b)$$

where κ is the constant eddy diffusivity, $J = (1 - \zeta_g/f)^{-1}$ is the Jacobian of the transformation, and

$$\zeta_g = \frac{\partial v_g}{\partial X}, \quad (2.5)$$

is the relative vorticity in geostrophic space. Since the deformation field is irrotational, it does not contribute to the boundary layer pumping (2.4).

We further assume that any frictional generation of potential vorticity in the interior is not advected out of the boundary layer. This approach makes the semigeostrophic problem in geostrophic space identical to the quasigeostrophic one in physical space. Wu and Blumen (1983) show that (2.4) overestimates the Ekman pumping in cyclonic regions compared to a boundary layer theory that incorporates the geostrophic momentum (GM) approximation. However, Blumen and Wu (1983) show that (2.4) yields qualitatively similar results to the GM case for a two-dimensional semigeostrophic model of baroclinic instability. We therefore adopt (2.4) here.

The method of solution follows the Fourier transform technique in Bannon (1983). The solution (Twigg 1991) is given by

$$\theta_d(X, Z, T) = \frac{\Delta\theta}{(\pi/2)} \tan^{-1}\left[\frac{X}{L + NZ/f}\right], \quad (2.6)$$

and

$$\frac{\partial v_g}{\partial X} = \frac{2b_0}{\pi N} \frac{x_g}{[(L + NZ/f)^2 + X^2]}, \quad (2.7)$$

where $b_0 = g\Delta\theta/\theta_0$.

The frontal length scale L is

$$L(t) = L_0 e^{-\alpha t} + L_0 r(1 - e^{-\alpha t}), \quad (2.8)$$

where

$$r = \frac{N}{\alpha L_0}\sqrt{\frac{\kappa}{2f}} \quad (2.9)$$

is the nondimensional parameter measuring the importance of friction on the frontogenesis. Following Williams (1974) we define the nondimensional frontal scale

$$\hat{L} \equiv \frac{\Delta\theta/L_0}{\partial\theta/\partial x|_{\max}}. \quad (2.10)$$

The definition (2.10) facilitates comparison of the semi- and quasigeostrophic results with the numerical simulations of Williams (1974).

3. Properties of the solution

We first examine the solution in geostrophic space where it corresponds to that for quasigeostrophic theory. Comparison of the solution (2.6) with the initial conditions (2.3) indicates that the two are identical apart from the length scale (2.8) of the baroclinic zone. This frontal scale is a function of the nondimensional time αt and the parameter r. For the typical values (e.g.,

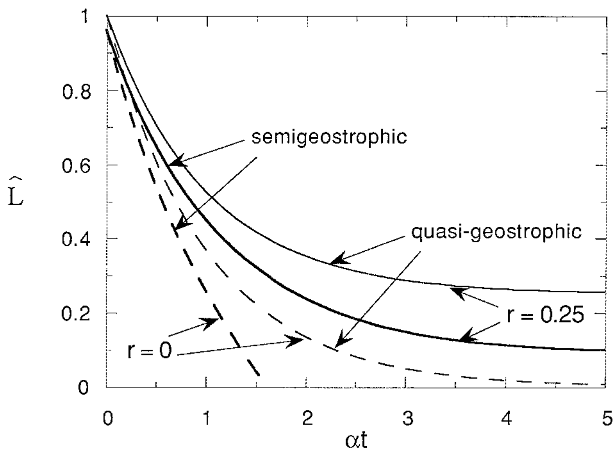

FIG. 1. Nondimensional frontal scale, $\hat{L}(t)$, as a function of time. The solid curves denote the case with friction ($r = 0.25$); the dashed curves the case without friction ($r = 0$). The heavy curves denote the semigeostrophic case; the light curves the quasigeostrophic case. The quasigeostrophic case corresponds to the semigeostrophic solution in geostrophic coordinates.

Williams 1974) $N = 10^{-2}$ s^{-1}, $f = 10^{-4}$ s^{-1}, $g = 10$ m s^{-2}, $\alpha = 10^{-5}$ s^{-1}, $\kappa = 10$ m^2 s^{-1}, $\theta_0 = 300$ K, and $L_0 = 10^6$ m, we find $r \sim 0.22$. For the definition (2.10), we find that the quasigeostrophic frontal scale is

$$\hat{L}_{QG} = \frac{L(t)}{L_0}. \quad (3.1)$$

Figure 1 is a plot of this frontal scale as a function of time with ($r = 0.25$) and without ($r = 0$) friction. The figure clearly demonstrates that the Ekman pumping slows the frontal collapse and has a frontolytic effect. At each time the frontal scale is larger with friction than without. As time tends to infinity, the frontal scale approaches the finite scale L_{QG}

$$L_{QG} = rL_0 = \frac{N}{\alpha}\sqrt{\frac{\kappa}{2f}} \approx 224 \text{ km}. \quad (3.2)$$

Thus, friction counteracts the action of the deformation field and produces an equilibrated front of finite horizontal scale. The equilibrated scale (3.2) is independent of the strength of the front, b_0, and the initial scale, L_0. An alternative expression for L_{QG} is

$$L_{QG} = \left(\frac{f}{\alpha}\right)L_E, \quad (3.3)$$

where

$$L_E = \frac{Nh_E}{f} \approx 22.4 \text{ km} \quad (3.4)$$

is the Rossby radius based on the Ekman scale

$$h_E = \sqrt{\frac{\kappa}{2f}} \approx 224 \text{ m}. \quad (3.5)$$

(Note that the depth of the Ekman layer is $D_E = 2\pi h_E = 1.41$ km.) The equilibrated frontal scale is inversely proportional to the strength of the deformation field α. In the absence of the deformation forcing, (2.8) agrees with Snyder and Keyser (1996),

$$L(t) = L_0 + h_E N t, \quad (3.6)$$

and the baroclinic zone dissolves linearly in time as the frontal scale increases at the rate $h_E N \sim 200$ km d^{-1}.

The form of the solution (2.6) reveals that the presence of the friction acts to lower the apparent ''floor'' of the atmosphere as seen by an observer aloft. For example, the solution with friction at the height Z is that of the inviscid solution at $Z + (f/\alpha)h_E$. Note that the effective vertical displacement $(f/\alpha)h_E \sim 2.24$ km is distinct from the Ekman depth $D_E = 2\pi h_E = 1.41$ km.

The semigeostrophic model predicts frontal equilibration in geostrophic space at the horizontal scale L_{QG}. Equilibration in physical space requires mathematically that the Jacobian J of the transformation does not vanish. A discontinuity develops in physical space when $\zeta_g = f$ in geostrophic space. A nondimensional measure of the strength of the frontogenesis (Bannon 1984) is

$$\sigma(t) \equiv \frac{1}{f}\frac{\partial v_g}{\partial X}\bigg|_{\max} = \frac{b_0}{\pi N f L(t)}, \quad (3.7)$$

because the maximum vorticity is at the surface at $X = L(t)$. The inviscid theory of Bannon (1984) is applicable to the present problem since the Ekman pumping only alters the lower boundary condition. A discontinuity forms and there is frontal collapse in physical space when $\sigma = 1$. The maximum value of σ, corresponding to the smallest horizontal scale, L, is

$$\sigma_{\max} = \frac{\alpha b_0}{\pi N^2 f h_E}. \quad (3.8)$$

Thus, an equilibrated front with finite horizontal scale will form in physical space provided $\sigma_{\max}$ is less than unity. The expression (3.8) indicates that the equilibration threshold depends on the strength of the synoptic forcing, α, on the magnitude of the frontal baroclinic zone, b_0, and inversely on the magnitude of the friction, h_E. The criterion for frontal collapse is

$$\sigma_{\max} \geq 1 \quad \text{or} \quad \sigma_0 \geq r, \quad (3.9)$$

where $\sigma(t = 0) = \sigma_0$ is the initial strength of the front. This criterion may also be written as

$$\Delta\theta \geq \Delta\theta_c \equiv \frac{\pi N^2 f h_E \theta_0}{\alpha g}. \quad (3.10)$$

Recall that $2\Delta\theta$ is the total change in potential temperature across the front. For the parameter values given above we find that $\Delta\theta_c = 21.1$ K. While large, this critical value is halved to a more representative frontal value of 10.6 K if the strength α of the deformation field is doubled to 2×10^{-5} s^{-1}. Alternatively the criterion for frontal collapse may be written as

$$h_E = \sqrt{\frac{\kappa}{2f}} \leq \frac{\alpha b_0}{\pi N^2 f}. \qquad (3.11)$$

Then for a given temperature difference and deformation rate, the front collapses to a discontinuity if there is insufficient friction. We note that the criteria (3.9), (3.10), or (3.11) are independent of the initial scale of the front L_0.

Following Bannon (1984) we can find an expression for the nondimensional frontal scale $\hat{L}$ in physical space where the maximum thermal gradient is $\Delta\theta/[(1 - \sigma^2)L(t)]$. We find

$$\hat{L}_{SG}(t) = (1 - \sigma^2)\frac{L(t)}{L_0}. \qquad (3.12)$$

Figure 1 compares the quasi- (3.1) and semigeostrophic (3.12) frontal scales. Initially we take $\sigma(t = 0) = \sigma_0 = 0.20$. Without friction ($r = 0$) the frontogenesis is stronger for semigeostrophic theory and frontal collapse occurs in a finite time. With friction ($r = 0.25$), both fronts equilibrate, but the semigeostrophic front equilibrates to a smaller scale: $\hat{L} = 0.09$ compared to 0.25 for quasigeostrophic theory. It is important to note that (3.12) is a nonlinear result. If we choose $\sigma(t = 0) \geq r = 0.25$, the semigeostrophic front will collapse despite the presence of a given magnitude of Ekman friction.

4. Conclusions

The analytic solutions presented here indicate that the effects of surface friction are frontolytic. The quasigeostrophic results indicate that the vertical motion associated with boundary layer pumping is frontolytic. The semigeostrophic results extend this finding to include the influence of the divergent ageostrophic motion above the boundary layer associated with the pumping. The results presented here for the front given by (2.3) also hold for the class of uniform potential vorticity solutions developed by Davies and Muller (1988). Our results for a prognostic semigeostrophic model indicate that the front will either collapse or equilibrate at a finite scale. In contrast, the semigeostrophic diagnosis in physical space by Thorpe and Nash (1984) suggests that Ekman pumping would dominate the synoptic forcing and prevent frontal formation. (Their global frontogenesis timescale becomes negative, implying frontal decay, for a deformation driven front when the eddy diffusivity is only 0.5 $m^2\ s^{-1}$.)

The present analysis supports the numerical simulations of Williams (1974) that frictional processes can offset the inviscid frontogenetic dynamics and produce equilibrated fronts with a finite horizontal scale. In particular, for his case in Fig. 5 and Table 1 with an eddy diffusivity of 10 $m^2\ s^{-1}$ and a small thermal diffusivity but no horizontal diffusion, we find $r = 0.26$ and $\sigma_{max} = 0.45$. Then the result (3.12) predicts a nondimensional frontal scale of 0.20, which agrees well with the scale of 172 km for $L_0 = 1000$ km reported by Williams.

Future research should assess the role of the ageostrophic convergent flow in the boundary layer as well as the role of boundary-layer-generated potential vorticity anomalies on the frontogenesis.

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