Ekman Pumping for Stratified Planetary Boundary Layers Adjacent to a Free Surface or Topography

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ABSTRACT

In this paper, the classic Ekman pumping formulas for the vertical flow out of a boundary layer are generalized for both the layer above a rigid surface of variable slope and also for the boundary layer underneath a moving free surface. The assumptions that the density is constant and the geostrophic velocity does not vary with height in the boundary layer are relaxed without compromising the simplicity of the final approximation. Similarly, it proves to be unnecessary to assume a specific eddy viscosity law.

The pumping formulas obtained here are compared with the old ones and significant differences are found.

1. Introduction

Ekman pumping is the vertical flow at the interface between the boundary layer and the interior flow that is created by cross-isobaric convergence or divergence within the boundary layer. Pedlosky (1987) studied the effect of a moving free surface or a sloping rigid surface on the pumping. (The two cases are depicted schematically in Fig. 1). In this article, we show that Pedlosky's assumptions that the density is constant and that the geostrophic velocity and eddy viscosity do not vary with height can be removed.

The new pumping formula for the boundary layer above a rigid surface, such as the bottom of the atmosphere or ocean, is

$$w_\delta = \frac{\mathbf{k}}{f\rho_\delta}\cdot\boldsymbol{\nabla}\times\boldsymbol{\tau}_B + \frac{\rho_B}{\rho_\delta}\mathbf{v}_{gB}\cdot\boldsymbol{\nabla}h_B, \qquad \text{New: rigid boundary,} \quad (1)$$

where w_δ and ρ_δ are the vertical velocity and density at the top of the boundary layer; $\mathbf{k}$ is a unit vector in the vertical; $\boldsymbol{\tau}_B$, ρ_B, and $\mathbf{v}_{gB}$ are the horizontal stress, density, and geostrophic wind at the solid surface, respectively; and $h_B(x, y)$ is the altitude of the sloping rigid boundary.

The corresponding textbook formula is Eq. (4.9.36) on p. 226 of Pedlosky (1987):

$$w_\delta = \sqrt{\frac{A_v}{2f}}\varsigma_g + \mathbf{v}_{g\delta}\cdot\boldsymbol{\nabla}h_B, \qquad \text{Old: rigid boundary,} \quad (2)$$

where A_v is the turbulent eddy viscosity, f is the Coriolis parameter, ς_g is the geostrophic vorticity (assumed independent of height), and $\mathbf{v}_{g\delta}$ is the geostrophic wind at the top of the boundary layer.

For the boundary layer under a free surface and above a deep interior flow, the new expression is

$$w_E = \frac{\mathbf{k}}{f\rho_E}\cdot\boldsymbol{\nabla}\times\boldsymbol{\tau}_S + \frac{\rho_S}{\rho_E}\frac{D_g h_S}{Dt}, \qquad \text{New: free surface,} \quad (3)$$

where w_E and ρ_E are the velocity and density at the bottom of the boundary layer, $\boldsymbol{\tau}_S$ and ρ_S are the horizontal stress and density at the free surface, and $h_S(x, y, t)$ is the height of the free surface and

$$\frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + u_{gS}\frac{\partial}{\partial x} + v_{gS}\frac{\partial}{\partial y}, \quad (4)$$

where $\mathbf{v}_{gS}$ is the geostrophic velocity at the free surface. The old formula is [Pedlosky 1987, p. 232, Eq. (4.10.21)]:

$$w_E = \frac{\mathbf{k}}{f\rho}\cdot\boldsymbol{\nabla}\times\boldsymbol{\tau}_S + \frac{Dh_S}{Dt}, \qquad \text{Old: free surface,} \quad (5)$$

where ρ is the density in the boundary layer, which is assumed constant and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_S\frac{\partial}{\partial x} + v_S\frac{\partial}{\partial y}. \quad (6)$$

Note that the advecting velocity in the surface term is the actual velocity in the old formula versus the geostrophic velocity in the new pumping expression.

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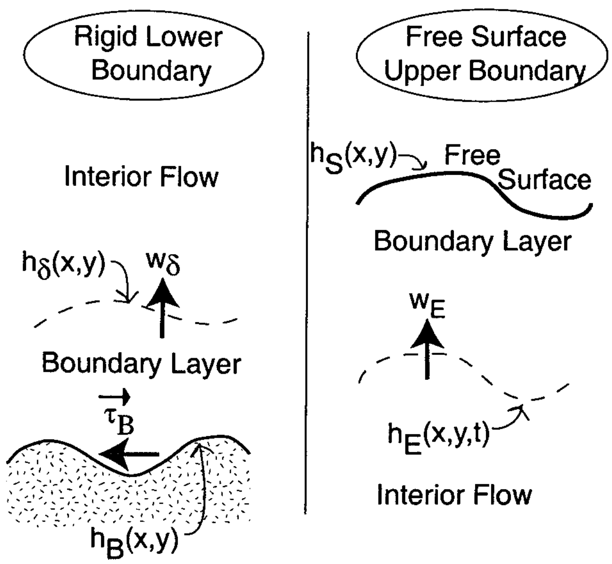

FIG. 1. Schematic of the two species of boundary layers analyzed here.

The most significant characteristic of the new formulas is that aside from a density factor, they are completely determined by the surface quantities: the surface stress, the surface geostrophic wind, and the surface altitude. The derivation of the new formulas is given in the next section and the difference between the new and old formulas is discussed in section 3.

2. Derivation

In common with standard Ekman theory, the boundary layer flow considered here is assumed to be in a three-force balance between the Coriolis force, the pressure gradient force, and the frictional force:

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_x}{\partial z}, \qquad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_y}{\partial z}, \quad (7)$$

where the density ρ is allowed to be a function of x, y, and z. Thus, the pressure p and therefore the geostrophic velocity $\mathbf{v}_g$ are also functions of x, y, and z even if the boundary layer flow is still in a state of hydrostatic balance. Compared to the vertical derivatives, the horizontal derivatives of the stress are negligible.

The continuity equation is approximated as

$$\frac{\partial(\rho w)}{\partial z} = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right], \quad (8)$$

where the non-Boussinesq effect, that is, the variation of density with space, is considered.

Expressing u and v in terms of the pressure gradient force and stress by using the momentum equations and substituting the result in the continuity equation yields

$$\frac{\partial(\rho w)}{\partial z} = -\left[\frac{\partial}{\partial x}\left(\frac{1}{f}\frac{\partial \tau_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{1}{f}\frac{\partial \tau_x}{\partial z}\right)\right]. \quad (9)$$

Integrating Eq. (9) with respect to z over the thickness of the boundary layer, invoking the mathematical identity

$$\frac{d}{dt}\int_{b(t)}^{a(t)} F(x, t)\,dx = F(a, t)\frac{da}{dt} - F(b, t)\frac{db}{dt} + \int_{b(t)}^{a(t)} \frac{\partial F}{\partial t}\,dx \quad (10)$$

and the boundary conditions

$$u = v = w = 0, \qquad z = h_B(x, y) \quad \text{and}$$

$$\tau_x = \tau_y = \frac{\partial \tau_x}{\partial z} = \frac{\partial \tau_y}{\partial z} = 0, \qquad z = h_\delta(x, y),$$

$$\text{rigid boundary case}, \quad (11)$$

or

$$w = w_S = \frac{Dh_S}{Dt}, \qquad z = h_S(x, y, t) \quad \text{and}$$

$$\tau_x = \tau_y = \frac{\partial \tau_x}{\partial z} = \frac{\partial \tau_y}{\partial z} = 0, \qquad z = h_E(x, y, t),$$

$$\text{free surface case}, \quad (12)$$

where D/Dt is the usual total derivative, we obtain the new Ekman pumping formulas as in Eqs. (1) and (3) above.

3. Discussion and conclusions

The new formulas are similar in form to the old ones: they contain a stress term plus a topographic term for the rigid surface case or a surface term for the free surface case. A detailed comparison shows, however, that there do exist some significant differences between them.

First, the stress term in the new formula Eq. (1) is proportional to the curl of the surface stress $\boldsymbol{\tau}_B$. It is in its general form. The same form was also obtained by Gill (1982) for a homogeneous and incompressible boundary layer with a flat boundary. So this means that both stratification and a variable boundary do not alter the form of the stress term.

The new stress term is less explicit than the old in the sense that it is not directly expressed by the wind or pressure field as is true of the stress term in the old formula (2), which is expressed by the geostrophic vorticity. However, it is not necessary to assume a constant eddy viscosity; any reasonable stress model is consistent with the new Ekman pumping formulas. For example, for the planetary atmospheric boundary layer, there is a constant-flux surface layer in its lower part in which the eddy viscosity is a function of height. For this case, our Ekman formula still works, but the old does not.

Second, the two topographic terms are different. There is a density factor, ρ_B/ρ_δ, in the new formula that is absent from the old formula. This is a result of stratification. For the atmospheric planetary boundary layer, the density factor may reach roughly 1.1∼1.2. Another difference between the two topographic terms is that the advecting wind in the new formula is the geostrophic wind at the bottom of the boundary layer versus the geostrophic wind at the top of the boundary layer in the old formula. This difference is a reflection of baroclinicity of the fluid. Generally, this difference may be not very big because the thickness of the boundary layer is usually thin; however, in a region where the baroclinicity is strong, such as a front, the difference may become significant.

Third, aside from a density factor ρ_S/ρ_E, the two surface terms are also different. The advecting wind in the new formula is the surface geostrophic wind versus the surface actual wind in the old formula. Due to the friction, the two winds are certainly different. How big this difference is on the real atmosphere–ocean interface is not known to the author. However, Pedlosky [1987, p. 229, Eq. (4.10.12)] gives theoretically the formula relating the two winds to the surface stress. It is interesting that Pedlosky (1987) uses actually the geostrophic wind instead of the actual wind in the later application of this term in his book. Here, we justify his doing so.

In conclusion, due to stratification, baroclinicity, and variable eddy viscosity, our new Ekman pumping formulas differ from the old. How big this difference is in the real atmospheric and oceanic boundary layers, and how significant the effects of the differences are for the modeling and forecasting of weather and climate deserve further investigation.

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