Gamma-Weighted Discrete Ordinate Two-Stream Approximation for Computation of Domain-Averaged Solar Irradiance

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ABSTRACT

An algorithm is developed for the gamma-weighted discrete ordinate two-stream approximation that computes profiles of domain-averaged shortwave irradiances for horizontally inhomogeneous cloudy atmospheres. The algorithm assumes that frequency distributions of cloud optical depth at unresolved scales can be represented by a gamma distribution though it neglects net horizontal transport of radiation. This algorithm is an alternative to the one used in earlier studies that adopted the adding method. At present, only overcast cloudy layers are permitted.

1. Introduction

It has been demonstrated that accurate estimates of domain-averaged solar irradiances for marine boundary layer clouds can be obtained by applying a two-stream approximation to each subgrid cell and averaging the resulting irradiances (the independent pixel or column approximation; Stephens et al. 1991; Cahalan et al. 1994). Barker et al. (1998) showed that the same can be done using entire columns for large domains containing convective clouds. Since this approximation neglects net horizontal fluxes, these results suggest that if the probability density function of cloud optical thickness is known, domain-averaged irradiances can be computed as a function of a few parameters expressing the probability density function.

Satellite observations (Barker et al. 1996) and cloud-resolving models (e.g., Barker et al. 1998; Oreopoulos and Barker 1999) indicate that distributions of cloud optical thickness for domains the size of cells in typical weather and climate models can be approximated by gamma distributions. Barker (1996) used a two-stream approximation with boundary conditions of no diffuse irradiance from above and below the cloud layer to derive an analytical solution for domain-averaged reflectance and transmittance for an inhomogeneous cloud layer. Oreopoulos and Barker (1999) used the principles of invariance (Chandrasekhar 1960, 161–166) combined with Barker's (1996) analytical solution to compute irradiance at each level of a multilayered atmosphere. They used the adding method, which differs from discrete ordinate two-stream multilayered algorithms that solve a matrix equation set up by irradiance boundary conditions (e.g., Shettle and Weinman 1970; Liou 1975; Stamnes and Swanson 1981; Stamnes et al. 1988; Toon et al. 1989, Fu and Liou 1993). Since discrete ordinate two-stream algorithms are widely used to compute irradiance for multilayered atmospheres, it is worthwhile incorporating the gamma-weighted two-stream approximation into the discrete ordinate framework.

The main purpose of this paper is to derive analytical solutions for reflected and transmitted irradiance given gamma distributed cloud optical thickness that can be used with a discrete ordinate two-stream model. This provides an alternative to the adding method and entails solving a tridiagonal matrix built from boundary con-

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ditions for radiative transfer computations for multilayer atmospheres.

2. Algorithm development

a. Homogeneous cloud layer

The general two-stream solution of the radiative transfer equation for a plane-parallel, homogeneous layer (Toon et al. 1989) is

$$F^+(\xi, \tau_m) = k_1 e^{\lambda\xi\tau_m} + \eta k_2 e^{-\lambda\xi\tau_m} + C^+(\tau_m) \quad \text{and} \tag{1}$$

$$F^-(\xi, \tau_m) = \eta k_1 e^{\lambda\xi\tau_m} + k_2 e^{-\lambda\xi\tau_m} + C^-(\tau_m), \tag{2}$$

where F^+ and F^+ are upward and downward irradiance, respectively; ξ is the fraction of cloud depth measured from $\xi = 0$ at cloud top to $\xi = 1$ at cloud base, τ_m is optical thickness of the layer at $\xi = 1$,

$$\lambda = (\gamma_1^2 - \gamma_2^2)^{1/2} \quad \text{and} \tag{3}$$

$$\eta = \frac{\gamma_2}{\gamma_1 + \lambda}. \tag{4}$$

For solar radiation,

$$C^+(\tau_m) = \frac{\omega_0 \pi F_s \exp(-\xi\tau_m/\mu_0)[(\gamma_1 - 1/\mu_0)\gamma_3 + \gamma_4\gamma_2]}{\lambda^2 - 1/\mu_0^2} \quad \text{and} \tag{5}$$

$$C^-(\tau_m) = \frac{\omega_0 \pi F_s \exp(-\xi\tau_m/\mu_0)[(\gamma_1 + 1/\mu_0)\gamma_4 + \gamma_2\gamma_3]}{\lambda^2 - 1/\mu_0^2}, \tag{6}$$

where $\pi F_s \mu_0$ is the solar irradiance incident at cloud top; μ_0 is the cosine of the solar zenith angle; ω_0 is the single-scattering albedo; and γ_1, γ_2, γ_3, and γ_4 are coefficients that depend on the form of the two-stream approximation (Meador and Weaver 1980).

b. Inhomogeneous cloud layer

Consider a single-layer overcast cloud for which optical thickness varies over a large horizontal domain. We assume that the optical thickness probability density function of the layer follows a gamma distribution defined as

$$P(\tau_m; \overline{\tau}_m, \nu) = \frac{1}{\Gamma(\nu)}\left(\frac{\nu}{\overline{\tau}_m}\right)^{\nu} \tau_m^{\nu-1} e^{-\nu\tau_m/\overline{\tau}_m}, \tag{7}$$

where ν and $\overline{\tau}_m$ are the shape parameter and mean optical thickness of the layer at $\xi = 1$, respectively (Wilks 1995, p. 86). The shape parameter can be estimated from moments such as $\nu = (\overline{\tau}_m/\sigma)^2$ where σ is the standard deviation of τ_m. For $\nu < 10$, however, it is recommended that the maximum likelihood method be used to estimate ν (Wilks 1995, 86–90). Domain-averaged irradiances are then computed via the independent column approximation as

$$\overline{F^{\pm}}(\xi, \overline{\tau}_m, \nu) = \int_0^{\infty} P(\tau_m; \overline{\tau}_m, \nu) F^{\pm}(\xi, \tau_m)\, d\tau_m, \tag{8}$$

where we assume that ν is independent of ξ.

The unknowns in (1) and (2), k_1 and k_2, are determined by boundary conditions,

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \frac{1}{1 - \eta^2 e^{-\lambda\tau_m}} \begin{pmatrix} \eta e^{-2\lambda\tau_m} & -e^{-\lambda\tau_m} \\ -1 & \eta e^{-2\lambda\tau_m} \end{pmatrix} \times \begin{bmatrix} C^-(0) - F^-(0) \\ C^+(\tau_m) - F^+(\tau_m) \end{bmatrix}. \tag{9}$$

Assuming no downward and upward diffuse irradiance at the top and bottom of the cloud, respectively, [i.e., $F^-(0) = F^+(\tau_m) = 0$] and expanding the denominators of k_1 and k_2 by the binomial theorem yields

$$k_1 = [\eta C^-(0)e^{-2\lambda\tau_m} - C^+(\tau_m)e^{-\lambda\tau_m}] \times (1 + \eta^2 e^{-2\lambda\tau_m} + \eta^4 e^{-4\lambda\tau_m} + \cdots) \quad \text{and} \tag{10}$$

$$k_2 = [\eta C^+(\tau_m)e^{-\lambda\tau_m} - C^-(0)] \times (1 + \eta^2 e^{-2\lambda\tau_m} + \eta^4 e^{-4\lambda\tau_m} + \cdots). \tag{11}$$

Therefore, using (10) and (11) and substituting (1) and (2) into (8), domain-averaged upward and downward irradiances at level ξ are

$$\overline{F^+}(\xi, \overline{\tau}_m, \nu) = G_1(\xi, \overline{\tau}_m, \nu) + \eta G_2(\xi, \overline{\tau}_m, \nu) + D^+(\xi, \overline{\tau}_m, \nu) \quad \text{and} \tag{12}$$

$$\overline{F^-}(\xi, \overline{\tau}_m, \nu) = \eta G_1(\xi, \overline{\tau}_m, \nu) + G_2(\xi, \overline{\tau}_m, \nu) + D^-(\xi, \overline{\tau}_m, \nu), \tag{13}$$

where

$$G_1(\xi, \overline{\tau}_m, \nu) = \sum_{i=1}^{\infty} \frac{\eta^{2i-1}C^-(0)}{\left[1 + (2i - \xi)\frac{\lambda\overline{\tau}_m}{\nu}\right]^{\nu}} - \sum_{i=1}^{\infty} \frac{\eta^{2(i-1)}C^+(0)}{\left\{1 + \left[\frac{1}{\mu_0} + (2i - 1 - \xi)\lambda\right]\frac{\overline{\tau}_m}{\nu}\right\}^{\nu}}, \tag{14}$$

$$G_2(\xi, \overline{\tau}_m, \nu) = \sum_{i=1}^{\infty} \frac{\eta^{2i-1}C^+(0)}{\left\{1 + \left[\frac{1}{\mu_0} + (2i - 1 + \xi)\lambda\right]\frac{\overline{\tau}_m}{\nu}\right\}^{\nu}} - \sum_{i=1}^{\infty} \frac{\eta^{2(i-1)}C^-(0)}{\left\{1 + [2(i - 1) + \xi]\frac{\lambda\overline{\tau}_m}{\nu}\right\}^{\nu}}, \tag{15}$$

$$D^+(\xi, \overline{\tau}_m, \nu) = \frac{\omega_0\pi F_s[(\gamma_1 - 1/\mu_0)\gamma_3 + \gamma_4\gamma_2]}{\left(\lambda^2 - \frac{1}{\mu_0^2}\right)}\left(\frac{\mu_0\nu}{\mu_0\nu + \xi\overline{\tau}_m}\right)^{\nu}, \quad \text{and} \tag{16}$$

$$D^-(\xi, \overline{\tau}_m, \nu) = \frac{\omega_0\pi F_s[(\gamma_1 + 1/\mu_0)\gamma_4 + \gamma_2\gamma_3]}{\left(\lambda^2 - \frac{1}{\mu_0^2}\right)}\left(\frac{\mu_0\nu}{\mu_0\nu + \xi\overline{\tau}_m}\right)^{\nu}. \tag{17}$$

Following the notation of Barker (1996), G_1 and G_2 at layer boundaries are

$$G_1(0, \overline{\tau}_m, \nu) = \phi_1^{\nu}\left[C^-(0)\eta\sum_{i=0}^{\infty}\frac{\eta^{2i}}{(\phi_2 + i)^{\nu}} - C^+(0)\sum_{i=0}^{\infty}\frac{\eta^{2i}}{(\phi_3 + i)^{\nu}}\right], \tag{18}$$

$$G_1(1, \overline{\tau}_m, \nu) = \phi_1^{\nu}\left[C^-(0)\eta\sum_{i=0}^{\infty}\frac{\eta^{2i}}{(\phi_6 + i)^{\nu}} - C^+(0)\sum_{i=0}^{\infty}\frac{\eta^{2i}}{(\phi_4 + i)^{\nu}}\right], \tag{19}$$

$$G_2(0, \overline{\tau}_m, \nu) = \phi_1^{\nu}\left[C^+(0)\eta\sum_{i=0}^{\infty}\frac{\eta^{2i}}{(\phi_3 + i)^{\nu}} - C^-(0)\sum_{i=0}^{\infty}\frac{\eta^{2i}}{(\phi_1 + i)^{\nu}}\right], \tag{20}$$

$$G_2(1, \overline{\tau}_m, \nu) = \phi_1^{\nu}\left[C^+(0)\eta\sum_{i=0}^{\infty}\frac{\eta^{2i}}{(\phi_5 + i)^{\nu}} - C^-(0)\sum_{i=0}^{\infty}\frac{\eta^{2i}}{(\phi_6 + i)^{\nu}}\right], \tag{21}$$

where

$$\phi_1 = \frac{\nu}{2\lambda\overline{\tau}_m}, \qquad \phi_4 = \phi_1 + \frac{1}{2\lambda\mu_0},$$
$$\phi_2 = \phi_1 + 1, \qquad \phi_5 = \phi_4 + 1,$$
$$\phi_3 = \phi_4 + \frac{1}{2}, \qquad \phi_6 = \phi_1 + \frac{1}{2}.$$

Domain-averaged direct irradiance at ξ is given by

$$\overline{F}_d = \mu_0\pi F_s\left(\frac{\mu_0\nu}{\mu_0\nu + \xi\overline{\tau}_m}\right)^{\nu}. \tag{22}$$

Note that at the homogeneous limit when ν goes to infinity (i.e., σ goes to zero),

$$G_1(\xi, \overline{\tau}_m, \infty) = G_1(0, \overline{\tau}_m, \infty)e^{\xi\lambda\overline{\tau}_m}, \tag{23a}$$
$$G_2(\xi, \overline{\tau}_m, \infty) = G_2(0, \overline{\tau}_m, \infty)e^{-\xi\lambda\overline{\tau}_m}, \quad \text{and} \tag{23b}$$
$$\overline{F}_d = \mu_0\pi F_s e^{-\overline{\tau}_m/\mu_0}. \tag{24}$$

Since the single-scattering albedo ω_0 and asymmetry parameter g are mean values weighted by the optical thickness of constituents, they depend on cloud optical thickness. For the integration of (8), we assume that ω_0 and g, as well as λ, which is a function of ω_0 and g, are constant throughout the layer.

c. Multilayer atmosphere

For real clouds, downward irradiance at cloud top and upward irradiance at cloud base are not zero [i.e., $F^-(0)$ and $F^+(\tau_m) \neq 0$ in (11)]. Thus, assuming that when G_1 and G_2 are factored out of the unknowns, that are determined by boundary conditions, the remainder is independent of τ_m, and domain-averaged upward and downward irradiance for nonzero $F^-(0)$ and $F^+(\tau_m)$ are

$$\overline{F^+}(\xi, \overline{\tau}_m, \nu) = k_1'\frac{G_1(\xi, \overline{\tau}_m, \nu)}{G_1(1, \overline{\tau}_m, \nu)} + \eta k_2'\frac{G_2(\xi, \overline{\tau}_m, \nu)}{G_2(0, \overline{\tau}_m, \nu)} + D^+(\xi, \overline{\tau}_m, \nu) \quad \text{and} \tag{25}$$

$$\overline{F^-}(\xi, \overline{\tau}_m, \nu) = \eta k_1' \frac{G_1(\xi, \overline{\tau}_m, \nu)}{G_1(1, \overline{\tau}_m, \nu)} + k_2' \frac{G_2(\xi, \overline{\tau}_m, \nu)}{G_2(0, \overline{\tau}_m, \nu)} + D^-(\xi, \overline{\tau}_m, \nu), \quad (26)$$

where k_1' and k_2' are unknown and are determined by boundary conditions. Substituting (23) and (24) into (25) and (26) it is easy to verify that in the limit as ν goes to infinity, (25) and (26) revert to (1) and (2), except that $e^{\lambda \tau_m}$ is factored out and included in k_1.

When an inhomogeneous cloud layer is placed in a vertically inhomogeneous atmosphere consisting of n homogeneous layers, the boundary conditions are set up as a matrix equation that can be solved for $n - 1$ unknowns of k_1 and k_2 in (1) and (2) and k_1' and k_2' in (25) and (26). Toon et al. (1989) suggest that a tridiagonal matrix inversion is computationally fast and numerically stable. Therefore, a description of the elements of the matrix when an inhomogeneous cloud layer is included in the atmosphere is given in the appendix.

3. Results

Broadband, domain-averaged irradiances computed by (25) and (26) are compared to corresponding values obtained by numerically integrating (1) and (2) weighted by the appropriate optical thickness distribution [i.e., Eq. (7)]. For the test, a cloud layer of $\overline{\tau}_m = 15$ was placed between 1 and 2 km in the midlatitude summer standard atmosphere (McClatchey et al. 1972). We used a spectrally invariant surface albedo of 0.2 and a solar zenith angle of 60°. The k-distribution tables used here in estimating gaseous absorptions are those given by Kato et al. (1999). For numerical integration, lower and upper limits of optical thickness were 0.001 and 500, respectively. The population outside this range is negligible, less than 0.1% for the cases considered here.

Figure 1 shows the top-of-the-atmosphere reflectance and top-of-the-atmosphere to surface transmittance for the analytic model and the numerical integration. Differences between broadband reflectances and transmittances are less than 0.015, 0.01, and 0.007 for $\nu = 1.0$, 1.5, and 2.0, respectively. As mentioned by Barker et al. (1996), since the algorithm's values result from integration over τ_m from 0 to infinity, differences between algorithmic and numerically integrated irradiances can be caused, in part, by numerical integration over a finite range. As a consequence, differences between estimated irradiances increase as ν and $\overline{\tau}_m$ decrease; the population outside the range of integration increases to more than 1% when $\nu = 0.5$ and $\overline{\tau}_m = 5$. In this case, differences in broadband reflectance and transmittance are 0.006 and 0.023, respectively.

This algorithm runs into difficulty when ω_0 approaches unity because of the binomial expansion of the denominator in (9) and because of the singularity in the two-stream approximation. Therefore, a maximum single-scattering albedo is set to avoid the problem (Wiscombe 1977). For a maximum single-scattering albedo of 0.99999 and use of 100 terms to compute G_1 and G_2 by (14) and (15), the algorithm reproduces conservative-scattering reflectances given by Barker et al. (1996). For example, for four overcast scenes in Barker et al. (1996), the sums of transmittance and reflectance are greater than 0.9995 for all solar zenith angles.

4. Discussion and summary

The result shown in the previous section demonstrates that this algorithm can compute accurate domain-averaged irradiances for multilayered atmospheres with a single-layer inhomogeneous cloud. In such cases, trivial boundary conditions are sufficient for straightforward computation of domain-averaged irradiance. If the cloud layer were to be divided into many layers, trivial boundary conditions are no longer appropriate because irradiance is horizontally variable; columns locating below columns that are optically thinner (thicker) than the mean optical thickness receive more (less) radiation than the domain-averaged irradiance at the top boundary (Stephens 1988; Oreopoulos and Barker 1999). One could, however, resort to an adjustment of optical properties similar to that proposed by Oreopoulos and Barker (1999). When overcast cloud layers are separated by a clear layer, the trivial boundary conditions can be applied once again if one is content to accept the assumption that for multilayered systems, fluctuations in cloud optical properties and irradiance are uncorrelated. Although the discrete ordinate formulation allows for overcast clouds only, the algorithm presented here could be used in a cloud overlap scheme such as that developed by Geleyn and Hollingsworth (1979).

When $\overline{\tau}_m/\nu$ is small, (14) and (15) approach (10) and (11) because the second term in the denominators of (14) and (15) are small and the denominators can be approximated by exponential functions. Therefore, the inhomogeneous solutions (25) and (26) approach the homogeneous solutions (1) and (2), respectively. Two cases that give small $\overline{\tau}_m/\nu$ are $\overline{\tau}_m \ll 1$ and $\nu \gg 1$. When the layer is optically thin, the transmittance and reflectance are linear functions of the optical thickness so that the mean optical thickness provides the domain averaged transmittance and reflectance. As the mean optical thickness increases, the difference of transmittance and reflectance increases. When ν is large, the layer is close to homogeneous; hence, the inhomogeneous solution should approach the homogeneous solution. The absorptance of the atmosphere with a homogeneous cloud layer is always greater than that with an inhomogeneous cloud layer for these cases (Fig. 2). As the mean optical thickness or ν of the cloud layer increases, the absorptance of the atmosphere with an inhomogeneous cloud layer approaches to that with a homogeneous cloud layer. However, the absorption vertical profile can be different.

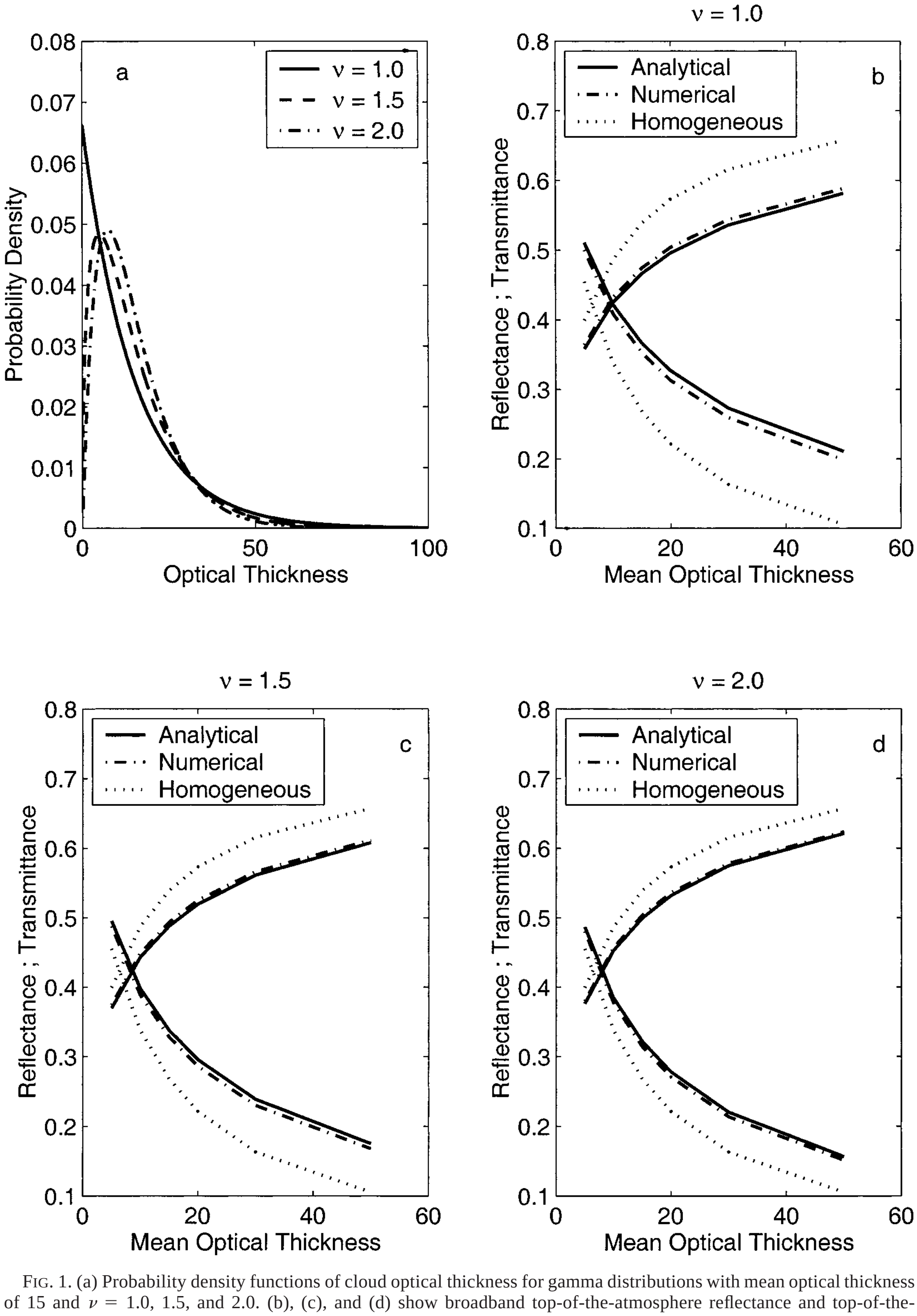

FIG. 1. (a) Probability density functions of cloud optical thickness for gamma distributions with mean optical thickness of 15 and ν = 1.0, 1.5, and 2.0. (b), (c), and (d) show broadband top-of-the-atmosphere reflectance and top-of-the-atmosphere to surface transmittance as functions of mean optical thickness for distributions shown in (a) as computed by the algorithm using analytical integration and by numerical integration of the homogeneous solution weighted by the distributions in (a). Values for homogeneous clouds computed by the standard two-stream approximation are also shown for reference.

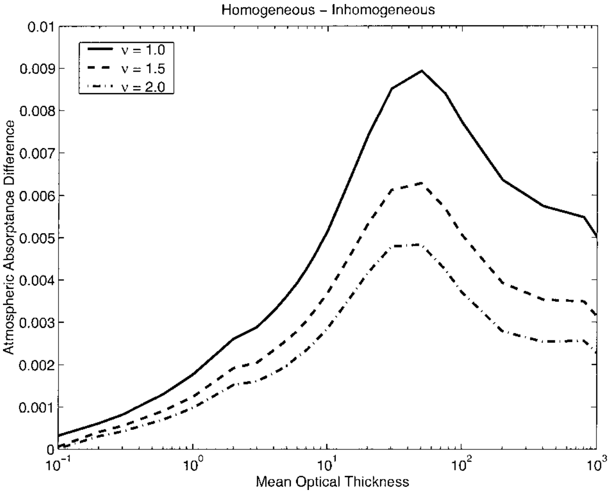

FIG. 2. Difference of the atmospheric absorptance for homogeneous cloud and inhomogeneous cloud shown as a function of mean optical thickness for distributions shown in Fig. 1a.

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APPENDIX

Matrix Elements of the Domain-Averaged Irradiance for a Multilayer Atmosphere

Using the notation of Toon et al. (1989), when an inhomogeneous cloud is placed in the jth layer, upward and downward irradiance for the layer are

$$\overline{F_j^+}(\xi, \overline{\tau}_m, \nu) = Y_{1j}\left[\frac{G_1(\xi, \overline{\tau}_m, \nu)}{G_1(1, \overline{\tau}_m, \nu)} + \eta_j \frac{G_2(\xi, \overline{\tau}_m, \nu)}{G_2(0, \overline{\tau}_m, \nu)}\right] + Y_{2j}\left[\frac{G_1(\xi, \overline{\tau}_m, \nu)}{G_1(1, \overline{\tau}_m, \nu)} - \eta_j \frac{G_2(\xi, \overline{\tau}_m, \nu)}{G_2(0, \overline{\tau}_m, \nu)}\right] + D_j^+(\xi, \overline{\tau}_m, \nu), \quad \text{and} \tag{A1}$$

$$\overline{F_j^-}(\xi, \overline{\tau}_m, \nu) = Y_{1j}\left[\eta_j \frac{G_1(\xi, \overline{\tau}_m, \nu)}{G_1(1, \overline{\tau}_m, \nu)} + \frac{G_2(\xi, \overline{\tau}_m, \nu)}{G_2(0, \overline{\tau}_m, \nu)}\right] + Y_{2j}\left[\eta_j \frac{G_1(\xi, \overline{\tau}_m, \nu)}{G_1(1, \overline{\tau}_m, \nu)} - \frac{G_2(\xi, \overline{\tau}_m, \nu)}{G_2(0, \overline{\tau}_m, \nu)}\right] + D_j^+(\xi, \overline{\tau}_m, \nu), \tag{A2}$$

where

$$Y_{1j} = (k_1' + k_2')/2 \quad \text{and} \tag{A3}$$

$$Y_{2j} = (k_1' + k_2')/2. \tag{A4}$$

Matching the irradiances at the top and bottom with those of neighboring layers provides $2n$ equations in $2n$ unknowns. Each of these equations has four unknowns. Eliminating either Y_1 and Y_2 for one of the layers results in three unknowns in each equations and, thus, leads to a tridiagonal matrix similar to that given by Toon et al. (1989). Following their notation, the tridiagonal matrix is in the form

$$A_l Y_{l-1} + B_l Y_l + H_l Y_{l+1} = E_l,$$
$$Y_l = Y_{1n}, \qquad l = 2n + 1$$
$$Y_l = Y_{2n}, \qquad l = 2n. \tag{A5}$$

The coefficients for $l = 1$ are

$$A_1 = 0, \quad B_1 = e_{51}, \quad H_1 = -e_{61},$$
$$E_1 = F_0^-(0) - C_1^-(0).$$

The coefficients for odd values of l from 3 to $2N - 1$ are

$$A_l = e_{2n}e_{7n} - e_{8n}e_{1n}, \quad B_l = e_{1n}e_{5n+1} - e_{7n}e_{3n+1},$$
$$H_l = e_{7n}e_{4n+1} - e_{1n}e_{6n+1},$$
$$E_l = e_{7n}[C_{n+1}^+(0) + C_n^+(\tau_n)] + e_{1n}[C_n^-(\tau_n) - C_{n+1}^-(0)], \tag{A6}$$

while those for even values of l from 2 to $2N - 2$ are

$$A_l = e_{6n+1}e_{1n} - e_{7n}e_{4n+1}, \quad B_l = e_{2n}e_{6n+1} - e_{8n}e_{4n+1},$$
$$H_l = e_{4n+1}e_{5n+1} - e_{3n+1}e_{6n+1},$$
$$E_l = e_{6n+1}[C_{n+1}^+(0) + C_n^+(\tau_n)] - e_{4n+1}[C_{n+1}^-(0) - C_n^-(\tau_n)]. \tag{A7}$$

For $l = 2N$ the coefficients are

$$A_{2N} = e_{1N} - R_{\rm sfc}e_{7N}, \quad B_{2N} = e_{2N} - R_{\rm sfc}e_{8N},$$
$$H_{2N} = 0, \quad E_{2N} = R_{\rm sfc}C_N^-(\tau_n) - C_N^+(\tau_n), \tag{A8}$$

where $R_{\rm sfc}$ is surface albedo.

Elements for the cloud-free layers are

$$e_{1n} = e_{5n} = 1 + \eta_n \exp(-\lambda_n\tau_n), \tag{A9}$$
$$e_{2n} = e_{6n} = 1 - \eta_n \exp(-\lambda_n\tau_n), \tag{A10}$$
$$e_{3n} = e_{7n} = \eta_n + \exp(-\lambda_n\tau_n), \tag{A11}$$
$$e_{4n} = e_{8N} = \eta_n - \exp(-\lambda_n\tau_n), \tag{A12}$$

while for inhomogeneous cloud (jth layer), they are

$$e_{1j} = 1 + \eta_j \frac{G_2(1, \overline{\tau}_m, \nu)}{G_2(0, \overline{\tau}_m, \nu)}, \tag{A13}$$

$$e_{2j} = 1 - \eta_j \frac{G_2(1, \overline{\tau}_m, \nu)}{G_2(0, \overline{\tau}_m, \nu)}, \tag{A14}$$

$$e_{3j} = \eta_j + \frac{G_1(0, \overline{\tau}_m, \nu)}{G_1(1, \overline{\tau}_m, \nu)}, \quad \text{(A15)}$$

$$e_{4j} = \eta_j - \frac{G_1(0, \overline{\tau}_m, \nu)}{G_1(1, \overline{\tau}_m, \nu)}, \quad \text{(A16)}$$

$$e_{5j} = 1 + \eta_j \frac{G_1(0, \overline{\tau}_m, \nu)}{G_1(1, \overline{\tau}_m, \nu)}, \quad \text{(A17)}$$

$$e_{6j} = 1 - \eta_j \frac{G_1(0, \overline{\tau}_m, \nu)}{G_1(1, \overline{\tau}_m, \nu)}, \quad \text{(A18)}$$

$$e_{7j} = \eta_j + \frac{G_2(1, \overline{\tau}_m, \nu)}{G_2(0, \overline{\tau}_m, \nu)}, \quad \text{(A19)}$$

$$e_{8j} = \eta_j - \frac{G_2(1, \overline{\tau}_m, \nu)}{G_2(0, \overline{\tau}_m, \nu)}, \quad \text{(A20)}$$

and $C_j^{\pm}(\tau_j)$ are replaced by (16) and (17).

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