SOME EFFECTS OF WAVE-LENGTH VARIATIONS OF THE LONG WAVES IN THE UPPER WESTERLIES

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ABSTRACT

In an attempt to use the equation developed by Rossby for the motion of waves in a single-layer barotropic atmosphere as a prognostic tool, the effect of the upstream variation of wave length is studied. With the aid of the concept of group velocity an expression is obtained for trough displacement which takes into account the change of wave length with time and the acceleration of the long waves. Tests of the results indicate that the inclusion of the upstream wave-length variation in the forecast of trough displacement gives a significant improvement in the forecast verification. The results can be expressed qualitatively as follows: If the wave length of the long waves increases upstream at the initial moment, the eastward speed of the wave under consideration decreases with time. If the wave length decreases upstream at the initial moment, the eastward speed of the wave increases with time.

1. Introduction

Previous studies [5, 1] have shown that the motion of long waves in the upper westerlies can be described within a close approximation by the equation for the phase velocity of waves in a single-layer barotropic atmosphere; this equation is

$$c = U - \beta L^2/4\pi^2, \quad (1)$$

where U is the zonal wind speed (measured at 600 mb), c is the speed of propagation of the wave, β is the rate of change northward of the Coriolis parameter, and L is the wave length. If the future values of the variables U, β, and L can be specified, the future value of c can be obtained, and (1) can be used as a prognostic equation. Unfortunately, no systematic method for obtaining a quantitative forecast of U and β is known at the present time. However, in an earlier study [1] it was shown that the short-term variations of U and β are generally so small that a forecast of no change for these variables will verify well most of the time. The following study will consider the variation of L and its effect on the variation of the wave speed c. For these reasons, U and β will be considered as constants in the following treatment of the subject.[2]

2. Fundamental equations

The individual variation of the wave speed c can be obtained by differentiation of (1) with respect to time, following the wave troughs and crests, with the result that

$$\frac{dc}{dt} = -\frac{\beta L}{2\pi^2}\frac{dL}{dt} = -\frac{\beta L}{2\pi^2}\left(\frac{\partial L}{\partial t} + c\frac{\partial L}{\partial x}\right), \quad (2)$$

where x increases from west to east.

The group velocity, c_g, has been defined [3] as the velocity with which a geometrical point must travel to follow a constant value of the wave length L. It therefore follows that

$$\frac{\partial L}{\partial t} + c_g\frac{\partial L}{\partial x} = 0. \quad (3)$$

Substitution of (3) in (2) yields the result

$$\frac{dc}{dt} = \frac{\beta L}{2\pi^2}(c_g - c)\frac{\partial L}{\partial x}. \quad (4)$$

Rossby [6] has shown that the group velocity of the waves described by (1) is

$$c_g = U + \beta L^2/4\pi^2. \quad (5)$$

Substitution of (5) and (1) into (4) yields

$$\frac{dc}{dt} = \frac{\beta^2 L^3}{4\pi^4}\frac{\partial L}{\partial x}. \quad (6)$$

Equation (6) gives the acceleration of an individual wave as a function of the wave length and of the local longitudinal variation of wave length. The application of (6) to the problem of obtaining the displacement of a wave after a finite time in the future can be accomplished by an integration with respect to time. Before this can be done it is necessary to specify the variation of L and of $\partial L/\partial x$ with time. The variation of L with time can be obtained from substitution of

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[2] The use of equation (1) to describe waves having variable length (L) and period (T) is permissible providing the variations of L and T are very small in intervals of length L and time T (see [6]).

(6) in (2), which gives

$$\frac{dL}{dt} = -\frac{\beta L^2}{2\pi^2}\frac{\partial L}{\partial x}. \quad (7)$$

Integration of (6) and (7) depends on the variation of $\partial L/\partial x$ with time. Differentiation of (3) gives

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x}\right) = -c_g\frac{\partial^2 L}{\partial x^2} - \frac{\beta L}{2\pi^2}\left(\frac{\partial L}{\partial x}\right)^2, \quad (8)$$

which shows that the variation of $\partial L/\partial x$ with time depends on the longitudinal variation of $\partial L/\partial x$. However, the inclusion of these higher-order terms would contribute little to the solution of the practical problem for the time interval used; therefore, $\partial L/\partial x$ will be treated as a constant mean value in the integrations.

Wave length as a function of time is obtained from (7), which yields after integration

$$\frac{1}{L} = \frac{\beta t}{2\pi^2}\frac{\partial L}{\partial x} + \frac{1}{L_0}, \quad (9)$$

since $L = L_0$ at the time t_0. Substitution of (9) into (6) gives

$$\frac{dc}{dt} = 2\pi^2\beta^2 L_0^3\frac{\partial L}{\partial x}\left(L_0\beta\frac{\partial L}{\partial x}t + 2\pi^2\right)^{-3}. \quad (10)$$

Integration of (10) gives

$$c = -\pi^2\beta L_0^2\left(2\pi^2 + L_0\beta\frac{\partial L}{\partial x}t\right)^{-2} + \text{constant}. \quad (11)$$

According to (1), at the time t_0,

$$c = U - \beta L_0^2/4\pi^2. \quad (12)$$

Substitution of (12) into (11) shows that the constant of integration is the zonal wind speed U, and (11) can be written

$$c = U - \pi^2\beta L_0^2\left(2\pi^2 + L_0\beta\frac{\partial L}{\partial x}t\right)^{-2} = \frac{dx}{dt}. \quad (13)$$

The displacement x can be obtained by integration of (13):

$$x = Ut + \frac{\pi^2 L_0}{\partial L/\partial x}\left(2\pi^2 + L_0\beta\frac{\partial L}{\partial x}t\right)^{-1} + \text{constant}. \quad (14)$$

The constant of integration can be evaluated by noting that at the time t_0, $x = 0$. The constant is therefore equal to $2L_0(\partial L/\partial x)^{-1}$; thus, the equation for the displacement can be written

$$x = \frac{\beta t}{4\pi^2}\left[L_s^2 - L_0^2\left(1 + \frac{L_0\beta}{2\pi^2}\frac{\partial L}{\partial x}t\right)^{-1}\right], \quad (15)$$

where L_s is the stationary wave length, defined [5] as $L_s \equiv 2\pi\sqrt{U/\beta}$.

Equation (1) can be rewritten in the form

$$x' = \beta t(L_s^2 - L_0^2)/4\pi^2, \quad (16)$$

x' being the displacement in the time t. Equation (16) is analogous to (15). Comparison of (16) with (15) shows that (15) may be considered as consisting of (1) with a correction term which is proportional to $\partial L/\partial x$ and to t. Examination of this correction term in (15) shows that the total displacement during the time t is increased above that given by (16) if $\partial L/\partial x$ is positive and decreased if $\partial L/\partial x$ is negative, provided that x is positive.

Equation (15) can be used more conveniently if the distances along the x-axis, namely x, L_s, and L_0 are expressed in terms of degrees longitude. If, further, the time unit is taken as one day so that $\beta \equiv 2\omega a^{-1}\cos\phi = 4\pi a^{-1}\cos\phi$, where a and ω are the earth's radius and angular velocity, then (15) becomes, ϕ being the latitude,

$$x = \frac{t\cos^2\phi}{180}\left[L_s^2 - L_0^2\left(1 + \frac{L_0 t\cos^2\phi}{90}\frac{\partial L}{\partial x}\right)^{-1}\right]. \quad (17)$$

If (17) is to be used practically, a graphical system must be employed for its solution. This can be done with the aid of the following substitutions:

$$A \equiv (t\cos^2\phi)L_s^2/180, \quad (18)$$

$$B \equiv (t\cos^2\phi)(\partial L/\partial x)/90, \quad (19)$$

$$C \equiv L_0^2/(1 + BL_0), \quad (20)$$

$$D \equiv (t\cos^2\phi)C/180. \quad (21)$$

It is evident from (17) that

$$x = A - D, \quad (22)$$

where D has been determined from the values of B and C, as indicated in (20) and (21). Figs. 1–4 provide graphical solutions of equations (18)–(21.)

3. Verification procedure

Equation (1) was verified previously [1] with the use of the wind at 600 mb, and for the testing of (17), L_s has been determined by the methods given in [1]. A test of (17) for $t = 72$ hr should give the most significant results. In this period significant changes in the speed of propogation of the long waves in the upper westerlies can be observed, and yet the distance moved by a trough is usually small compared with the wave length. The graphs for A, B, C, and D were therefore constructed for this value of t. In obtaining a graphical solution, one should determine A, B, C, and D successively and then subtract D from A. The result, x, is expressed in units of degrees longitude per three days.

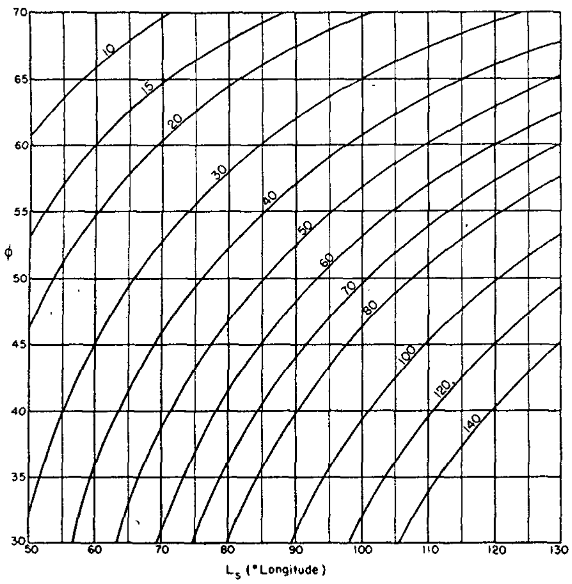

FIG. 1. Sloping curves give values of A corresponding to the latitude ϕ and the stationary wave length L_s.

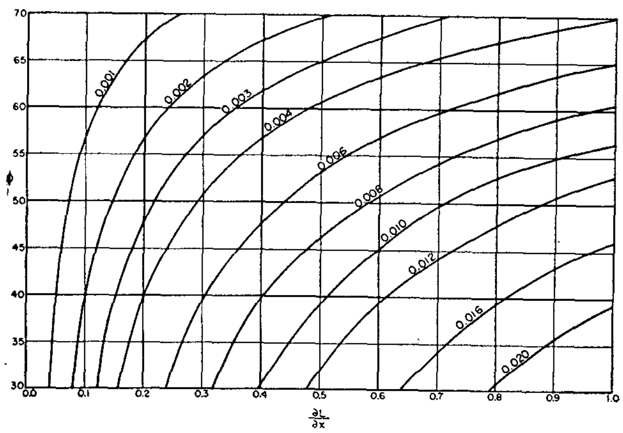

FIG. 2. Sloping curves give values of B corresponding to the latitude ϕ and to $\partial L/\partial x$. For negative values of $\partial L/\partial x$ read B as negative.

Fig. 5 illustrates the method in which the various parameters were measured in the testing of (17). The displacement of trough I is desired. The latitude $\phi = 50°$N; $L_0 = 92$ degrees longitude; and $L_1 = 60$ degrees longitude. The value of $\partial L/\partial x$ is approximated by the fraction $(L_0 - L_1) \div \frac{1}{2}(L_0 + L_1)$, which gives 0.42. Let $L_s = 92$ degrees longitude. Since $L_0 = L_s$ the wave represented by L_0, and therefore trough I, should be stationary at the time t_0. The three-day displacement of trough I is obtained from the following steps. Fig. 1 shows from the values of ϕ and L_s that $A = 58$. From the values of ϕ and $\partial L/\partial x$, fig. 2 gives $B = +0.0060$. From the values of B and L_0, fig. 3 gives $C = 5500$. Then from the values of ϕ and C, fig. 4 gives $D = 38$. From (22) it is seen that x, the three-day displacement of trough I, is equal to 20 degrees longitude. Since trough I is stationary at the initial moment, this means a large eastward acceleration of this trough is to be expected as a consequence of the decrease of wave length upstream from trough I.

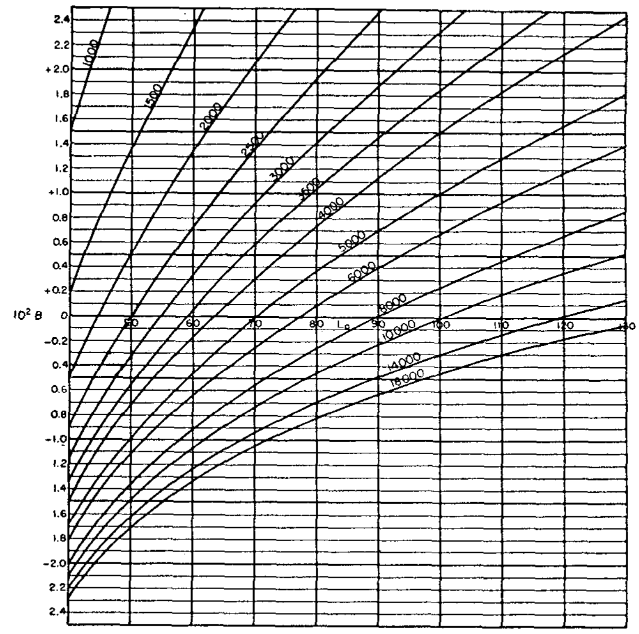

FIG. 3. Sloping curves give values of C corresponding to the values of B and the wave length L_0 (degrees longitude).

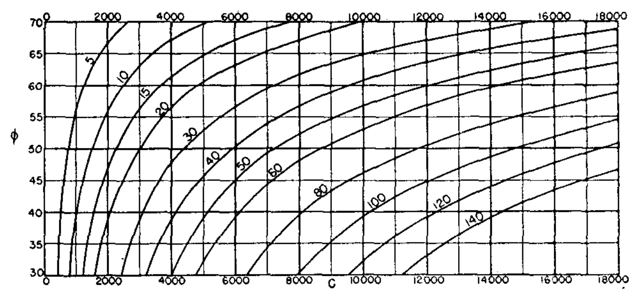

FIG. 4. Sloping curves give values of D corresponding to values of C and the latitude ϕ.

Because of the many assumptions made in the derivation of (17), the practical value of this formula can be determined only by a large number of tests on actual flow patterns. It was necessary to observe several restrictions in the testing. The first restriction was that only situations where the long-wave pattern was conserved through the 72-hr period were selected. A change in the general structure of the pattern or a change of the wave number during the period were considered sufficient to preclude a test. Situations where the measurement of wave lengths or of displacement was difficult due to the obscuring of major troughs by minor troughs of unusually large amplitude were also excluded from the testing. This restriction eliminated those patterns where the eastern trough retrograded during the test period, since the large amplitude of the minor troughs present during the retrogression of a major trough usually made the measurement of trough speed impossible.

Fig. 6 shows the results of 50 tests of equation (17). The average absolute value of the errors represented by the data in fig. 6 is 6 degrees longitude. In order to see whether this verification represented an improvement over that obtained by other methods of forecasting, tests of the same situations were made by two other methods. The first of these was made with the use of equation (1) with the data from the beginning of each period, *i.e.*, β, L_s, and L were assumed constant throughout the forecast period. The average absolute value of the errors obtained by these tests

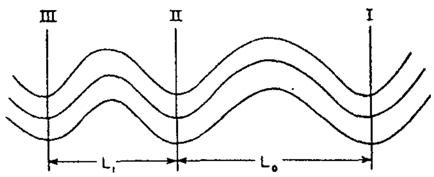

Fig. 5. Illustration of the measurement of the parameters.

was 11 degrees longitude for the three-day displacements. The final method of testing was to triple the trough displacement observed in the 24-hr period preceding the forecast period, and to take the resulting value as a forecast. This method could be used on only 42 of the patterns shown in fig. 6. The previous 24-hour trough displacement could not be evaluated for eight patterns, due to various reasons. This method of forecasting gave an average absolute error of 13 degrees longitude for the three-day displacement.

The verification of the term representing acceleration in equation (15) is presented in fig. 7, which shows that in 38 of the 50 tests the correction term in (15) contributed to the success of the displacement determination.

An example of the effect of the longitudinal variation of wave length is shown in fig. 8. The short wave

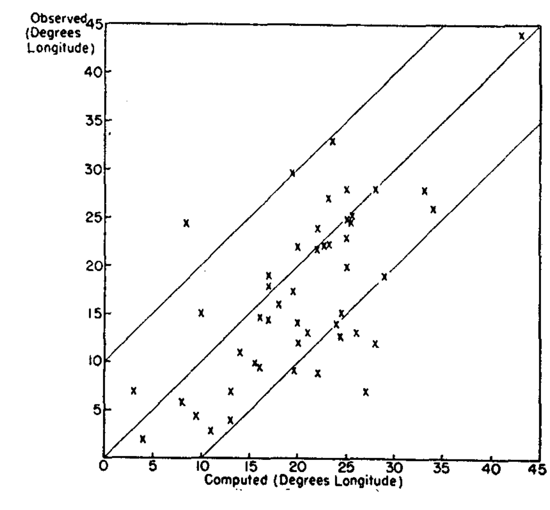

Fig. 6. Results of fifty tests of equation (17). The computed versus observed 3-day displacement is shown for each test. Outside slanting diagonals enclose points of error less than 10 degrees longitude.

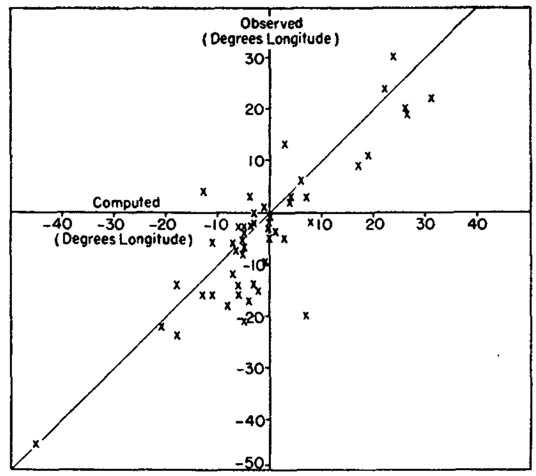

Fig. 7. Test of the acceleration term in equation (15). If the observed displacement for a 3-day period is x'', the ordinates of the points (labelled "observed") are $x'' - x'$, where x' is the 3-day displacement computed from the initial conditions from equation (16). The abscissae of the points (labelled "computed") are the values of $x - x'$, where x is the 3-day displacement computed from equation (15) or (17).

length from the trough in North America to the trough in the Atlantic on November 16 indicated a rapid downstream displacement of the latter trough. However, the quantity $\partial L/\partial x$ had an unusually large negative value on November 16, and inspection of equation (15) shows that the displacement of the Atlantic trough should be small.

The net retrogression which occurred in the United States during the three day period shown resulted in a large increase in the wave length from the central to the eastern trough, and as a result, the net displacement of the Atlantic trough was nearly zero for the period.

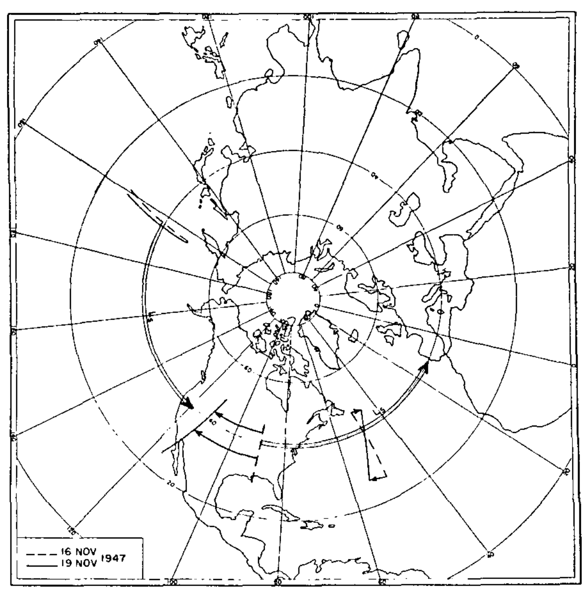

Fig. 8. Schematic chart of the net displacements of the major troughs from 16–19 November 1947, at 700 mb. The double arrows mark the distance of the stationary wave length L_s. The single arrows indicate the net 3-day displacements.

4. Conclusions

The results presented above show that the use of the concept of group velocity, applied to the long atmospheric waves, improves the accuracy with which the behavior of the long waves can be described. However, this concept is of most use only when enough data are at hand to show more than one of the long waves. This agrees with the statements made earlier by Namias [4] and others [1, 2, 7], who indicated that data from very great distances are helpful in improving the accuracy of relatively short-range forecasts.

It is important to remember that the development used in this paper can be applied only when a definite and persistent long-wave pattern exists through the period in question. The application of equation (17) in actual forecasting routine has pointed to the importance of the occurrence of changes in the wave number, *i.e.*, the formation of new or the destruction of old major troughs. If some method were known that provides a forecast of changes in wave number before such changes actually begin, much greater success could be attained in the use of equation (17).

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