SHORTER CONTRIBUTIONS

THE EFFECT OF VISCOSITY ON CIRCULATION

By C. Truesdell

Naval Ordnance Laboratory and University of Maryland

(Manuscript received 31 July 1948)

Jeffreys (1928) has given a formula for the rate of change of the circulation Γ around an arbitrary closed material curve C in a viscous compressible fluid. A slight generalization and rearrangement of his result, written in vector notation, is

$$\frac{D\Gamma}{Dt} = \oint_C \alpha \boldsymbol{f}\cdot d\boldsymbol{r} - \oint_C \alpha\, dp + \oint_C \alpha\, d\left(\lambda \frac{D \ln \alpha}{Dt}\right) + \oint_C 2\mu\alpha\, d\left(\frac{D\ln\alpha}{Dt}\right) - \oint_C \alpha \operatorname{curl} \mu\boldsymbol{w}\cdot d\boldsymbol{r} + \oint_C 2\alpha \operatorname{grad}\mu\cdot d\boldsymbol{v}, \quad (1)$$

where D/Dt is the symbol of material differentiation, $\boldsymbol{f}$ is the extraneous force per unit volume, α is the specific volume, $\boldsymbol{r}$ is the position vector, $\boldsymbol{v}$ is the velocity, p is the pressure, λ and μ are the Cauchy–Poisson–Duhem coefficients of viscosity, and $\boldsymbol{w}$ is the vorticity curl $\boldsymbol{v}$. This result is derived from the classical linear expression $\boldsymbol{W} = \lambda \boldsymbol{I} \operatorname{div} \boldsymbol{v} + \mu \operatorname{grad} \boldsymbol{v} + \mu (\operatorname{grad} \boldsymbol{v})_c$ for the viscous stress dyadic $\boldsymbol{W}$. The last integral (and similarly the third integral in (2) below) vanishes when $\operatorname{grad}\mu = 0$, as must be assumed in order to derive the Navier–Stokes equations. It will appear in the sequel that steep gradients of viscosity, which must accompany steep gradients of temperature, can be a primary mechanism for the generation of vorticity, particularly in combination with compressibility effects. Phenomena in which such gradients occur are described by the more general dynamical equations of Duhem (1901), upon which the present analysis rests.

This note describes and explains the most general mechanism for change of circulation, subject to the usual continuity assumptions. In order to render apparent the effect of each variable we first transform (1) into a surface integral by Stokes' theorem. After some rearrangements, the terms may be grouped as follows:

$$\frac{D\Gamma}{Dt} = \int_S \operatorname{curl}(\alpha\boldsymbol{f} - \nu \operatorname{curl}\boldsymbol{w})\cdot d\boldsymbol{S} + \int_S \operatorname{grad}\left(p + \nu'\frac{D\rho}{Dt}\right)\times \operatorname{grad}\alpha\cdot d\boldsymbol{S} - \int_f \left\{\operatorname{grad}\left(2\nu\frac{D\ln\alpha}{Dt}\right)\times\operatorname{grad}\ln\mu - \operatorname{curl}(\nu[\operatorname{grad}\ln\mu]\cdot[\operatorname{grad}\boldsymbol{v} + (\operatorname{grad}\boldsymbol{v})_c])\right\}\cdot d\boldsymbol{S}, \quad (2)$$

where the "kinematic viscosities" ν and ν' are defined as $\mu\alpha$ and $(\lambda + 2\mu)\alpha$, respectively, and ρ is the density. The three integrals on the right represent three independent mechanisms for the generation of vorticity, which we shall discuss in turn.

In a homogeneous incompressible fluid of constant viscosity (2) becomes simply

$$\frac{D\Gamma}{Dt} = \int_S \operatorname{curl}(\alpha\boldsymbol{f} - \nu\operatorname{curl}\boldsymbol{w})\cdot d\boldsymbol{S}, \quad (3)$$

a result given by Jaffé (1920), the term $\nu \operatorname{curl} \boldsymbol{w}$ having been mentioned previously by V. Bjerknes (1902). In a homogeneous inviscid incompressible fluid (3) becomes

$$\frac{D\Gamma}{Dt} = \int_S \operatorname{curl}\alpha\boldsymbol{f}\cdot d\boldsymbol{S}. \quad (4)$$

Thus the only way vorticity may be created in such fluids, subject to the usual continuity assumptions, is through the action of an extraneous force which is not conservative. This mechanism has been extensively discussed by Jaffé. To understand the effect of viscosity upon the circulation in homogeneous viscous incompressible fluids, (3) shows that one need only suppose an effective extraneous force of magnitude $-\nu \operatorname{curl} \boldsymbol{w}$ per unit volume to be added to any actual extraneous force. By this mechanism viscosity cannot create vorticity, since this effective force vanishes in regions where $\boldsymbol{w} = 0$, but will modify an already existing circulation.

In a compressible fluid of constant viscosity, subject to conservative extraneous force, in a region where no vorticity exists, (2) becomes

$$\frac{D\Gamma}{Dt} = \int_S \operatorname{grad}\left(p + \nu'\frac{D\rho}{Dt}\right)\times\operatorname{grad}\alpha\cdot d\boldsymbol{S}. \quad (5)$$

In an inviscid fluid subject to conservative extraneous force we have simply

$$\frac{D\Gamma}{Dt} = \int_S \operatorname{grad} p \times \operatorname{grad}\alpha\cdot d\boldsymbol{S}. \quad (6)$$

This is the circulation theorem of V. Bjerknes, who has given an interpretation of the surface integral as the number of solenoids bounded by the unit surfaces p = constant, α = constant, embraced by the circuit in question, and has applied it to explain many meteorological phenomena of the greatest interest (Bjerknes, 1898; 1900). From (5) one may see that in a viscous fluid an effective pressure of magnitude $\nu' D\rho/Dt$ must be added to the ordinary pressure in order to describe the mechanism for the change of circulation. To understand the way in which this new pressure can generate vorticity, one must first observe from the dynamical equations that a viscous fluid resists dilatation as well as shearing, the resistance being proportional to $\nu' D\rho/Dt$. If grad $(\nu' D\rho/Dt) \neq 0$, this resistance is not the same in all directions. Now it is a difference of acceleration between two neighboring particles which produces vorticity. The acceleration is proportional to α. Thus to produce vorticity, grad $(\nu' D\rho/Dt)$ must point in a direction different from grad α. In an inhomogeneous but incompressible fluid $D\rho/Dt = 0$, and viscosity modifies the circulation only through the mechanism present in homogeneous incompressible fluids.

In summary, to understand the mechanism by which circulation is changed in a viscous compressible fluid of constant viscosity, one need only to understand the familiar mechanisms (4) and (6) for inviscid fluids and modify them by the addition of an effective extraneous force $-\nu$ curl $\boldsymbol{w}$ and an effective pressure $\nu' D\rho/Dt$, respectively.

In fluids whose viscosity is not constant, a third and rather complicated mechanism for the change of circulation is available. To see how nonuniform viscosity may create circulation, consider an initially irrotational plane radial flow, as from a source, in which the speed v is initially the same at all points on each of a family of concentric circles, but varies from circle to circle. Now since there is a radial velocity gradient, even in a homogeneous incompressible fluid there will be a stress of magnitude $2\mu\, dv/dr$ opposing the flow. If μ varies not only from one radial line to another but also along these lines (as for example in a suitable heated fluid) the accelerating forces at corresponding points on the two lines will be different, so that vorticity will be established. The magnitude of this effect may be computed from (1), which yields

$$\frac{D\Gamma}{Dt} = 2\alpha \oint \text{grad}\, \mu \cdot d\boldsymbol{v}, \tag{7}$$

the extraneous force being supposed conservative. Formula (7) shows that not only the viscosity but also the velocity must vary from point to point if this mechanism is to operate. Thus in a uniform irrotational flow of a homogeneous incompressible fluid, vorticity is not created even by a variation of viscosity.

Finally, it is sometimes desirable to analyze the change of circulation in a rotating coordinate system. A purely kinematic calculation using the familiar treatment of relative motion (Joos, 1934) yields the result

$$\frac{D'\Gamma}{Dt} = \frac{D\Gamma}{Dt} - \oint \left(\frac{d^2\boldsymbol{c}}{dt^2} + \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}' + 2\boldsymbol{\omega} \times \boldsymbol{v}' \right) \cdot d\boldsymbol{r}', \tag{8}$$

where $D'\Gamma/Dt$ is the rate of change of circulation as apparent to an observer in the rotating coordinate system, $D\Gamma/Dt$ is that apparent to an observer in a fixed system and hence given by (1) for viscous compressible fluids, $\boldsymbol{c}$ is the position of the moving origin relative to the fixed coordinate system, $\boldsymbol{\omega}$ is the angular velocity of the moving coordinate system, and primes denote quantities observed in the moving system. In the case when the origin $\boldsymbol{c}$ of the moving system is fixed and the angular velocity $\boldsymbol{\omega}$ is constant, V. Bjerknes (1902) has shown that (8) reduces to

$$\frac{D'\Gamma}{Dt} = \frac{D\Gamma}{Dt} - 2\omega \frac{D\Sigma}{Dt}, \tag{9}$$

where Σ is the area of the projection of the circuit upon a plane perpendicular to $\boldsymbol{\omega}$. Thus the rate of change of circulation apparent to an observer on the earth is equal to that apparent to a fixed observer minus twice the earth's angular speed times the rate of change of the projection of the circuit upon the earth's equatorial plane. With the aid of this result V. Bjerknes has shown how an initial inhomogeneity in an atmosphere at rest will shortly establish a circulation around every circuit.

Acknowledgments.—This work was carried out under Office of Naval Research contract no. 53–47. The writer is indebted to Dr. P. F. Neményi for several helpful discussions of the subject treated here.

REFERENCES

Bjerknes, V., 1898: Über die Bildung von Cirkulationsbewegungen und Wirbeln in reibungslosen Flüssigkeiten. *Videnskabsselskabets Skrifter I, Mat.-naturvid. Klasse*, no. 5.

——, 1898: Über einen hydrodynamischen Fundamentalsatz und seine Anwendung besonders auf die Dynamik der Atmosphäre und des Weltmeeres. *Kongliga Svenska Vetenskaps-Akad. Handlingar*, n.s. **31**, no. 4.

——, 1900: Das dynamische Princip der Cirkulationsbewegungen in der Atmosphäre. *Meteor. Z.*, **17**, 97–106, 145–156.

——, 1902: Cirkulation relativ zu der Erde. *Meteor. Z.*, **19**, 97–108.

Duhem, P., 1901: Recherches sur l'hydrodynamique, première partie. Sur les principes fondamentaux de l'hydrodynamique. *Ann. Fac. Sci. Toulouse* (2), **3**, 315–377.

Jaffé, G., 1920: Bemerkung über die Entstehung von Wirbeln in Flüssigkeiten. *Phys. Z.*, **21**, 541–543.

Jeffreys, H., 1928: The equations of viscous motion and the circulation theorem. *Proc. Cambridge phil. Soc.*, **24**, 477–479.

Joos, G., 1934: *Theoretical physics* (translated from the first German edition by Ira M. Freeman). New York, Stechert & Co., 748 pp. (see ch. 10, sec. 3).