THE INSTABILITY OF WIND DISCONTINUITIES AND SHEAR ZONES IN PLANETARY ATMOSPHERES

By B. Haurwitz

College of Engineering, New York University

(Manuscript received 18 November 1948)

ABSTRACT

It is shown that perturbations forming at wind discontinuities in the atmospheres of rotating planets are unstable, even though the distribution of the angular momentum of the rotating planet exerts a stabilizing influence. Consequently disturbances can develop spontaneously. Presumably at least some of the markings observed on planets, especially on Jupiter, represent such disturbances. The velocity of these perturbations is found not to differ greatly from the mean velocity of the fluid on both sides of the boundary. Therefore, the determination of planetary velocities of rotation from visual observations of such surface markings appears justified.

At a sharp current discontinuity the amplitude of a perturbation increases faster, the shorter the wave length. At shear zones of finite width only those waves are unstable whose length is greater than five times the width of the shear zone, and those waves will develop most rapidly whose length is about eight times the width of the shear zone. Since in the large-scale atmospheric circulations different wind belts are as a rule separated by finite shear zones rather than by sharp discontinuities very short waves cannot develop because they are not unstable. An empirical check of the relation between the width of the shear zone and the length of the developing perturbations is discussed.

1. Introduction

The large-scale flow patterns of the atmospheres of the earth and the outer planets show a predominantly zonal arrangement which is caused by the rotation of the planets and by the differences in heat energy received at low and high latitudes. Outside the earth's atmosphere these zonal belts can be observed best on Jupiter because of the markings visible in its atmosphere. Details in the atmosphere of Saturn are much less distinct, although zonal belts are fairly clearly indicated. On Mars clouds have been observed too rarely to give any indication of the atmospheric circulation, and the conditions on other planets do not permit detailed studies of their atmospheric motions.

From the observations in the terrestrial atmosphere it is known that shear zones exist in which the current-velocity changes more or less rapidly in horizontal direction across the current. In the case of Jupiter these shear zones are shown by the different velocities of rotation determined for different latitudes. The average day on Jupiter is roughly 9^h55^m, but in the equatorial zone the period of rotation is about five minutes less, indicating a strong west wind in the terminology of terrestrial meteorologists. At other latitudes the periods vary normally between 9^h55^m and 9^h56^m (Russell *et al.*, 1926). These differences are more impressive if it is kept in mind that at the Jovian equator a difference of 20 sec in the rotation period corresponds to a velocity difference of 7 m sec^{-1}. The reliability of these determinations of the rotation period will be discussed below.

The amplitude of any small perturbation developing on a planetary shear zone increases with time provided that the wave length is larger than about five times the width of the shear zone. The existence of this shearing instability in the case of a shear zone was first demonstrated by Rayleigh (1894) for the case of a nonrotating fluid, but the instability conditions are not modified if the whole fluid system rotates with a constant angular velocity (Haurwitz, 1943), that is, if the Coriolis parameter is independent of the longitude.

For the study of motions in the terrestrial and other planetary atmospheres it is important to inquire to what extent these instability conditions are modified by the latitudinal variation of the Coriolis parameter. That this variation is a stabilizing factor is indicated by the fact that the distribution of the absolute angular momentum due to the planet's rotation is stable since it decreases as the square of the distance from the axis of rotation (inertia stability). It will be shown in section 4 that shearing instability still persists in spite of the inertia stability, at least at a sharp discontinuity, although the degree of instability is reduced because of the planet's rotation. Consequently perturbations can develop spontaneously in spite of the stabilizing influence of the rotation. A preliminary check indicates that the same result holds also for shear zones of finite width provided that the wave length is more than five times the width of the shear zone.

From observations in the earth's atmosphere and from theoretical studies it is known that the speed of such disturbances as cyclone waves and waves in the easterlies is not identical with the mean velocity of the currents at whose boundaries they form. The speed of these disturbances is of interest not only to meteorologists, but also to astronomers in connection with the measurement of planetary rotations. As explained previously, these determinations are, especially on Jupiter, based on the observations of markings on the surface of the planet. The physical nature of these spots is not known, but it is believed that at least the more rapidly changing markings, that is, those which last only a few weeks or months, are situated in the upper atmosphere of the planet and are not caused by occurrences on its liquid or solidified surface. It is even less feasible to give an exhaustive theory of the perturbations in Jupiter's atmosphere than it is for the terrestrial atmosphere. But shearing instability and inertia stability due to the planetary rotation must play an important role in the development of these disturbances. Therefore an indication as to the accuracy of the measurements of planetary rotation periods can be obtained by determining the speed which such simplified models of atmospheric disturbances may have whose dynamics is governed by shear and planetary rotation. The velocity difference between such a disturbance and its environment is evaluated in section 4. Its maximum value is 15 per cent of the current shear. Hence, it is in general small compared to the mean absolute motion of the current. It may be concluded that visual observations of the markings on the planets permit one to determine the velocity distribution in the atmosphere with reasonable accuracy, but it must be emphasized that the result has been derived only for a rather simple model, and that there is at least the possibility of greater velocity differences because additional factors influence the life-history of these disturbances.

Moreover, some of the markings may be phenomena caused by entirely different processes, for instance by eruptions from the deeper parts of the planet into its atmosphere. In particular, the semipermanent markings are sometimes attributed to such causes, and the foregoing conclusions do not apply here, although these more stable disturbances may nevertheless also give a good indication of the rotation of the environment.

A further point to be considered in connection with the instability investigations is the degree of instability, that is, the rate at which the amplitude of a perturbation increases with time. For pure shearing waves at a sharp discontinuity this rate decreases with increasing wave length so that shorter waves grow more rapidly than longer waves. This is no longer correct when the two layers are separated by a shear zone of finite width. In this case a wave whose length is about eight times the width of the shear zone will grow most rapidly (Rayleigh, 1894). The implications of this fact for the developments of perturbations is discussed in the last section of this paper.

2. The fluid model

It will be assumed that the motion is horizontal and that the fluid is incompressible and homogeneous. Thus any possible stabilizing effects of a stable stratification are excluded, and the interaction between shearing instability and inertia stability appears in pure form.

The fluid system may be divided by a vertical surface[1] into two currents (fig. 1) whose undisturbed

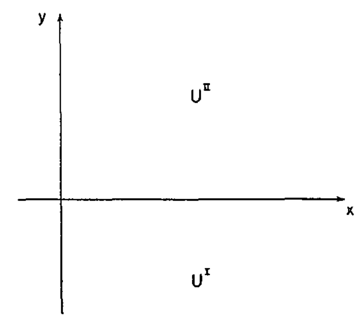

FIG. 1. Position of the two currents. The x-axis points eastward and the y-axis northward.

velocities may be constant, U^{I} and U^{II}, respectively, and parallel to the x-axis. It may further be assumed that the x-axis is directed towards east because the large-scale motions in planetary atmospheres are predominantly in the zonal direction. The transition from U^{I} to U^{II} is assumed to occur abruptly at a sharp discontinuity, thereby greatly simplifying the mathematical analysis. (The effect of a finite shear zone is discussed in the last section.) Both currents may be regarded as infinitely wide, I extending from $y = -\infty$ to 0 and II from $y = 0$ to ∞, since the existence of rigid walls would not alter the instability conditions materially.

The Coriolis parameter f will be regarded as a linear function of the latitudinal distance y,

$$f = f_0 + \beta y, \tag{1}$$

where

$$\beta = \partial f/\partial y = R^{-1} 2\omega \cos \phi. \tag{2}$$

Here R is the radius of the planet, ω its angular velocity of rotation, and ϕ is the latitude.

If P denotes the undisturbed pressure, g the acceleration of gravity, and ρ the density, the velocity U

[1] The discontinuity surface separates two currents of equal density. Therefore, it is actually not vertical, but parallel to the axis of rotation. But since the angle between the discontinuity and the horizontal is quite large, except at very low latitudes, and since the fluid motion will be assumed horizontal the kinematic effects of the inclination of the boundary are small and may be neglected.

of the undisturbed current is related to the pressure gradient by the geostrophic relation:

$$fU = -\rho^{-1}\partial P/\partial y. \tag{3}$$

Upon this general current a perturbation is superimposed which is characterized by the perturbation velocity components u, v (along the x- and y-axes) and by the perturbation pressure p. These quantities have to be added to the corresponding variables for the undisturbed motion in order to obtain the total motion. The perturbation quantities are assumed to be sufficiently small so that terms containing products of these quantities and their derivatives can be neglected in comparison with terms of the first order. This assumption is justified in stability investigations where the problem is to determine whether an originally infinitesimally small deformation of the undisturbed current will increase with time. Then the perturbation equations are

$$\begin{aligned} \frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ \partial u/\partial x + \partial v/\partial y &= 0. \end{aligned} \tag{4}$$

In the last equation the height variation of the fluid has been neglected because it has previously been shown that this is permissible (Haurwitz, 1943).

It may be assumed that the perturbation quantities contain a periodicity factor,

$$u, v, p \propto \exp\left[i(\mu x - \nu t)\right], \tag{5}$$

where the wave length L is given by $2\pi/\mu$ and the period T by $2\pi/\nu$. Then the wave velocity c is given by

$$c = L/T = \nu/\mu.$$

If ν and consequently c are complex, the periodicity factor contains a real exponential so that the amplitude increases or decreases with time. Since two conjugate complex values for c always appear, a complex c indicates always an increase of the amplitude with time and hence instability.

By substitution of (5) into (4) one finds from the third equation in (4) that

$$u = (i/\mu)dv/dy. \tag{6}$$

From the first equation of (4) and from (6) it follows that

$$\frac{p}{\rho} = i\frac{c - U}{\mu}\frac{dv}{dy} - \frac{i}{\mu}fv. \tag{7}$$

By substitution of (6) and (7) in the second equation of (4) the following equation is obtained for v,

$$\frac{d^2v}{dy^2} - \left(\mu^2 + \frac{\beta}{c - U}\right)v = 0. \tag{8}$$

If we put

$$\sigma = +\left[\mu^2 + \beta/(c - U)\right]^{\frac{1}{2}}, \tag{9}$$

the general solution[2] of (8) may be written

$$v = K_1 e^{\sigma y} + K_2 e^{-\sigma y}, \tag{10}$$

the periodicity factor (5) being omitted for convenience. The constants K_1 and K_2 are arbitrary. Since U is different in both layers the values of σ are also different. Further, the constants K are not the same for the two layers, although certain relations must exist in order that the boundary conditions are satisfied.

Since c may possibly be a complex quantity, $c = c_1 + ic_2$, it follows that σ may also be complex and may therefore be written in the form $\sigma = \sigma_1 + i\sigma_2$, where I and II are to be added if a distinction between both layers is necessary. If

$$\alpha_1 = \mu^2 + \frac{\beta(c_1 - U)}{(c_1 - U)^2 + c_2^2},$$

$$\alpha_2 = \frac{\beta c_2}{(c_1 - U)^2 + c_2^2},$$

it follows that

$$\sigma_1 = 2^{-\frac{1}{2}}\left[(\alpha_1^2 + \alpha_2^2)^{\frac{1}{2}} + \alpha_1\right]^{\frac{1}{2}},$$

$$\sigma_2 = 2^{-\frac{1}{2}}\left[(\alpha_1^2 + \alpha_2^2)^{\frac{1}{2}} - \alpha_1\right]^{\frac{1}{2}}.$$

The expression (10) for v may therefore be written in the form

$$v = K_1 e^{\sigma_1 y} e^{i\sigma_2 y} + K_2 e^{-\sigma_1 y} e^{-i\sigma_2 y}.$$

Since the perturbation cannot become infinitely large with increasing distance from the discontinuity (for which $y = 0$), the constant K_2 must vanish for the first layer, the constant K_1 for the second layer (σ_1 is a positive quantity). Hence

$$v^{\mathrm{I}} = K^{\mathrm{I}} e^{\sigma^{\mathrm{I}} y}, \quad v^{\mathrm{II}} = K^{\mathrm{II}} e^{-\sigma^{\mathrm{II}} y}. \tag{11}$$

The perturbation pressure is obtained from (7),

$$\begin{aligned} \frac{p^{\mathrm{I}}}{\rho} &= \frac{i}{\mu}\left[(c - U^{\mathrm{I}})\sigma^{\mathrm{I}} - f\right]K^{\mathrm{I}} e^{\sigma^{\mathrm{I}} y}, \\ \frac{p^{\mathrm{II}}}{\rho} &= -\frac{i}{\mu}\left[(c - U^{\mathrm{II}})\sigma^{\mathrm{II}} + f\right]K^{\mathrm{II}} e^{-\sigma^{\mathrm{II}} y}. \end{aligned} \tag{12}$$

In (11) and (12) the periodicity factor has again been omitted for the sake of brevity.

Besides the boundary conditions at infinity the following conditions have to be satisfied at the surface of discontinuity (Haurwitz, 1941):

$$\left(\frac{\partial}{\partial t} + U^{\mathrm{I,II}}\frac{\partial}{\partial x}\right)(p_0^{\mathrm{I}} - p_0^{\mathrm{II}}) + v_0^{\mathrm{I,II}}\frac{\partial}{\partial y}(P^{\mathrm{I}} - P^{\mathrm{II}}) = 0.$$

The double superscripts indicate that this equation must hold if the appropriate values either for the first

[2] If the solution is independent of y it follows that $U - c = \beta/\mu^2$. This equation is the trough formula derived by Rossby (1939).

or for the second layer are inserted. The subscript 0 indicates that the perturbation quantities may be taken at the undisturbed position of the boundary because this substitution introduces only an error of the second order. By substitution of (10), (11), and (3) in the two preceding equations a system of two linear and homogeneous equations for the constants K^{I} and K^{II} is found. Therefore, the determinant of this system must vanish in order to have solutions different from the trivial one of $K^{I} = K^{II} = 0$. This condition leads to the frequency equation

$$(c - U^{I})^2[\mu^2 + \beta(c - U^{I})^{-1}]^{\frac{1}{2}} + (c - U^{II})^2[\mu^2 + \beta(c - U^{II})^{-1}]^{\frac{1}{2}} = 0. \quad (13)$$

If the variation of the Coriolis parameter β is neglected equation (13) has the well-known solution

$$c = \tfrac{1}{2}(U^{I} + U^{II}) \pm \tfrac{1}{2}i(U^{I} - U^{II}), \quad (14)$$

which shows the instability of pure shearing waves.

3. Short waves

If the waves are so short that

$$\mu^2 \ll \beta(c - U^{I})^{-1}, \beta(c - U^{II})^{-1},$$

an approximate solution of (13) can easily be found. In this case

$$\left(\mu^2 + \frac{\beta}{c - U^{I}}\right)^{\frac{1}{2}} \approx \mu\left[1 + \frac{\beta}{2\mu^2(c - U^{I})} - \frac{\beta^2}{8\mu^4(c - U^{I})^2}\right]$$

and a similar approximation holds for the second root. Then

$$c = \tfrac{1}{2}(U^{I} + U^{II}) - \beta/4\mu^2 \pm i[\tfrac{1}{4}(U^{I} - U^{II})^2 - 3\beta^2/16\mu^4]^{\frac{1}{2}}. \quad (15)$$

Consequently, sufficiently short waves are unstable but the degree of instability is less than for pure shearing waves because of the stabilizing effect of the rotation.

To consider a numerical example for (15) let $\beta = 1.62 \times 10^{-13}$ sec^{-1} cm^{-1} (corresponding to conditions at 45° latitude on the earth). For different wave lengths L the following values of $\beta/4\mu^2$ are obtained:

L	10 km	100 km	1000 km
$\beta/4\mu^2$	0.1025×10^{-4}	0.1025×10^{-2}	0.1025

4. Instability for arbitrary wave

In order to consider the instability conditions for waves which are longer than permitted by the inequality at the beginning of section 3, equation (13) has to be solved for c. If one of the terms in this equation is transferred to the right-hand side and squares are taken a cubic equation in c is obtained. To simplify matters it may be assumed that

$$U^{I} = U = -U^{II}.$$

Such a relation can always be obtained if the coordinate system is given a velocity $\frac{1}{2}(U^{I} + U^{II})$ in the x-direction. The instability or stability of the fluid system is not affected by this uniform translation of the coordinate system. The equation for c becomes then

$$c^3 + \frac{3}{4}\frac{\beta}{\mu^2}c^2 + U^2c + U^2\frac{\beta}{4\mu^2} = 0. \quad (16)$$

In the derivation of (16) a division by U has been performed so that the roots of (16) do not apply to the case when a wind discontinuity does not exist.

For the discussion of (16) it is convenient to use the dimensionless quantities

$$c/U = \gamma \quad \text{and} \quad \beta/4\mu^2U = \epsilon. \quad (17)$$

Then

$$\gamma^3 + 3\epsilon\gamma^2 + \gamma + \epsilon = 0. \quad (18)$$

If δ is introduced by the relation

$$\gamma = \delta - \epsilon, \quad (19)$$

equation (18) changes into

$$\delta^3 + (1 - 3\epsilon^2)\delta + 2\epsilon^3 = 0. \quad (20)$$

In the theory of cubic equations it is shown that (20) has two conjugate complex and one real root if the discriminant is positive,

$$\epsilon^6 + (\tfrac{1}{3} - \epsilon^2)^3 > 0, \quad (21)$$

while otherwise all roots are real. The discriminant can be written in the form

$$\left(\epsilon^2 - \frac{1}{6}\right)^2 + \frac{1}{108}.$$

This expression is evidently always positive since ϵ is real. Therefore two values of δ, and consequently of c, are always complex. Thus, unstable waves are possible for any wave length.

Since the discriminant of (16) is positive Cardan's formulae may be used. Then the three roots of (20) are

$$\delta_1 = M + N$$
$$\delta_2 = -\tfrac{1}{2}(M + N) + \tfrac{1}{2}i\sqrt{3}(M - N)$$
$$\delta_3 = -\tfrac{1}{2}(M + N) - \tfrac{1}{2}i\sqrt{3}(M - N),$$

where

$$M = \left\{-\epsilon^3 + \left[\left(\epsilon^2 - \frac{1}{6}\right)^2 + \frac{1}{108}\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}}$$

$$N = \left\{-\epsilon^3 - \left[\left(\epsilon^2 - \frac{1}{6}\right)^2 + \frac{1}{108}\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}}.$$

The first of these roots is real while the other two are complex. On the other hand, for short wave lengths the approximate solution (15) of the original equation (13) gives only two complex roots which correspond to the two complex roots of the cubic equation (16).

Therefore it appears that the real root of (16) is not a solution of (13), but only introduced by the squaring of the latter equation. To show rigorously that (13) has no real roots appears unnecessary here since the stable motion represented by the real root, even if it existed, could develop into a disturbance of finite amplitude only under the action of some force, a possibility which is not being considered here.

With the aid of Cardan's formulae for δ it is possible to show that the maximum value of the real wave velocity must always be small compared to the rotational velocities of the planets, a result whose astronomical implications have already been discussed in the introduction. In order to prove this statement it is preferable to consider γ rather than c and to determine the wave length L for which it has a maximum. We have for the real value of γ

$$\mathrm{Re}(\gamma) = -\tfrac{1}{2}(M + N) - \epsilon,$$

and since ϵ is proportional to L^2 a necessary condition for the maximum is that

$$\frac{d}{d\epsilon}[\mathrm{Re}(\gamma)] = 0.$$

This condition leads to an algebraic equation for ϵ which can be solved numerically, the result being $\epsilon = 0.385$. Therefore, according to (17), the wave length with the maximum real velocity is given by

$$L_{\max} \doteq 7.79(U/\beta)^{\frac{1}{2}}.$$

That this value of L is really a maximum can be shown by standard methods. If the above value of ϵ is substituted in the expression for $\mathrm{Re}(\gamma)$ it follows that

$$\mathrm{Re}(\gamma) = -0.289,$$

and the maximum real wave velocity is

$$c_{1\max} = -0.289U,$$

or slightly less than 15 per cent of the wind shear. The values of the wind shear are very much smaller than the speed of the planetary rotation so that the percentual deviation of the absolute velocity of a disturbance from the velocity of the planetary rotation is small.

5. Long waves

With the aid of Cardan's formulae it is possible to find directly the complex wave velocity for a given fluid system so that the actual speed of wave propagation and the rate of increase of the amplitude can be determined. Before giving numerical values for the general case of an arbitrary wave length the behavior of very long waves may be considered when μ is very close to zero. If terms of higher than the first order in $4\mu^2/\beta$ are neglected it follows from Cardan's formulae that

$$c = -\frac{U^2}{9}\frac{4\mu^2}{\beta} \pm i\frac{U}{\sqrt{3}}. \tag{22}$$

A comparison of this expression with (14) for the wave velocity of shearing waves unaffected by the earth's rotation shows that the instability term is smaller by a factor $3^{-\frac{1}{2}} = 0.577$ in the case of very long waves in a rotating coordinate system.

According to (5), for a complex wave velocity $c_1 \pm ic_2$ the time factor has the form $\exp[-i\mu(c_1 \pm ic_2)t]$. Hence the rate of growth of the amplitude is represented by

$$\exp[(2\pi/L)c_2t]. \tag{23}$$

In order to see how the effect of the rotation reduces the instability consider the time during which the amplitude would increase by a factor m. If the effect of rotation is neglected it follows from (14) that

$$m = \exp[(2\pi/L)Ut_m],$$

while for long waves if the effect of the earth's rotation is taken into account,

$$m = \exp\left(\frac{2\pi}{L}\frac{U}{\sqrt{3}}t_m'\right).$$

Thus

$$t_m' = \sqrt{3}t_m = 1.73t_m,$$

and the time required for the amplitude to grow a certain fixed amount m is 73 per cent longer in the case of long waves subjected to the effects of rotation.

To show the actual rate of growth of the amplitude let $U = 15$ m sec^{-1} and $L = 10{,}000$ km, in which case the approximation formula (22) holds fairly well for conditions on the earth.

TABLE 1. Rate of growth of amplitude of long unstable waves.

Time (days)	1	2	3	4	5
No rotation	2.25	5.05	11.5	25.8	58.0
Rotation	1.60	2.55	4.10	6.52	10.4

The rate of increase of the amplitude is shown in table 1; the second line gives the values if the fluid system is not rotating, and the third line the values for a rotating system, thus showing how much smaller the rate of increase is with the stabilizing effect of rotation.

6. Determination of the wave velocity for arbitrary wave lengths

To determine the wave velocity for perturbations of arbitrary wave length divide equation (20) by ϵ^3. Then

$$\eta^3 + 2 = 3b\eta, \tag{24}$$

where

$$\eta = \delta/\epsilon = \gamma/\epsilon + 1, \text{ and}$$
$$b = 1 - 1/3\epsilon^2. \tag{25}$$

The roots η of (24) as functions of b have been tabulated by Jahnke–Emde (1943).

For the numerical discussions the following parameters may be introduced: The ratio n of the equatorial circumference of the planet to the wave length ($n = L^{-1}2\pi R$) and the velocity V_r of a point at the surface at latitude ϕ due to the planet's rotation ($V_r = \omega R \cos \phi$). With these quantities we obtain

$$\epsilon = V_r/2n^2U, \qquad (26)$$

$$\eta - 1 = 2n^2c/V_r. \qquad (27)$$

For a given value of ϵ the auxiliary quantity b can be determined from (25). Then the real and imaginary part of η, η_1 and η_2, respectively, can be found from the Jahnke–Emde table, and finally the real and imaginary parts of c, c_1 and c_2 can be obtained from (27),

$$c_1 = V_r(\eta_1 - 1)/2n^2,$$
$$c_2 = V_r\eta_2/2n^2.$$

The values of $(\eta_1 - 1)/2 = n^2c_1/V_r$ and $\eta_2/2 = n^2c^2/V_r$ for values of the parameter $\frac{1}{2}\epsilon = n^2U/V_r$ are plotted in fig. 2. Outside the range of fig. 2, (15) gives c with

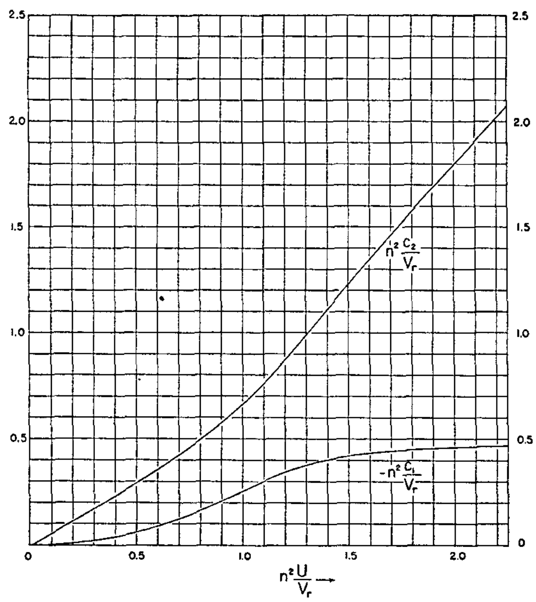

FIG. 2. Velocity as a function of latitude, wind discontinuity and wave length in nondimensional parameters.

an accuracy of about 5 per cent when $n^2U/V_r > 2.25$, corresponding to a small wave length. On the other hand, when $n^2U/V_r < 0.5$, corresponding to long waves, (22) gives c with an accuracy of 5 per cent.

As an example for the use of fig. 2 consider the case of a wave 5000 km long on the earth at 46° latitude at a wind discontinuity of 10 m sec^{-1}. Then $U = 5$ m sec^{-1}, $n = 8$, and $V_r = 320$ m sec^{-1}. It follows that $n^2U/V_r = 1$; therefore, according to fig. 2,

$$\tfrac{1}{2}(\eta_1 - 1) = -0.25, \quad c_1 = -1.25 \text{ m sec}^{-1},$$
$$\tfrac{1}{2}\eta_2 = 0.66, \quad c_2 = 3.32 \text{ m sec}^{-1}.$$

It should be noted that the real part of the wave velocity, c_1, has to be added to the mean velocity $\frac{1}{2}(U^{\text{I}} + U^{\text{II}})$ in order to obtain the speed relative to the earth. Since the rate of growth of the amplitude with time is $\exp(2\pi c_2 t/L)$ it follows that the amplitude has increased to e times its original value in the time $t_e = L(2\pi c_2)^{-1} = 67$ hr. For other waves whose length differs from 5000 km while the other parameters are the same as before, the values of c_1 and of t_e can be read from fig. 3. The difference c_1 between the

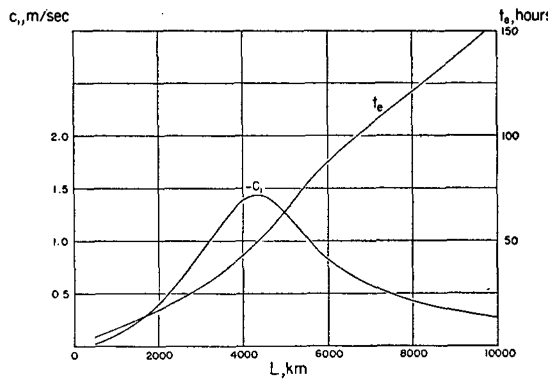

FIG. 3. Real wave velocity (c_1) and rate of increase of the amplitude t_e (the time during which the amplitude increases by a factor e) as a function of wave length L. Wind shear 10 m sec^{-1}, angular velocity of rotation 7.29×10^{-5} sec^{-1}, latitude 46°.

wave velocity and the mean of the two current velocities is small for all wave lengths. In the example considered here it reaches a maximum for a wave length between 4000 km and 4500 km. The time t_e which measures the rate of growth of the amplitude increases with increasing wave length so that longer waves would develop much more slowly than shorter ones. If this were actually correct in the atmosphere it would imply that long waves could hardly be observed since their growth would always be impeded by shorter ones whose amplitudes increase faster. However, as mentioned in the introduction, conditions are different when the sharp wind discontinuity is replaced by a shear zone, as is the rule in planetary circulations.

7. Effect of a shear zone on the degree of instability

If the current velocity changes linearly in a shear zone of the width l from the value U^{I} in the first layer to U^{II} in the second layer, the wave velocity is

$$c = \tfrac{1}{2}(U^{\text{I}} + U^{\text{II}}) \pm \tfrac{1}{2}i(U^{\text{I}} - U^{\text{II}})\,\Theta, \qquad (28)$$

where

$$\Theta^2 = L/l\pi - 1 - L^2\{2\pi^2l^2[1 + \coth(2\pi l/L)]\}^{-1}. \qquad (29)$$

This formula has been derived by Rayleigh for a nonrotating fluid, but the same result is obtained when a constant rotation is assumed (Haurwitz, 1943). The effect of the variation of the Coriolis parameter with the latitude will not be taken into account here, but the modification in the instability conditions which are about to be discussed correspond to those in the case of a constant Coriolis parameter.

The exponential time factor of the amplitude is, according to (28), $\exp\left[\frac{1}{2}\mu(U^{\text{II}} - U^{\text{I}})\Theta t\right]$. If $l = 0$, that is for a sharp discontinuity in U, then $\Theta = 1$ and the exponential decreases with decreasing μ or increasing L. For a finite shear zone of width l the increase of the amplitude with time for a given wind discontinuity will be largest for the wave length for which $\mu\Theta$ or $\mu^2\Theta^2$ is a maximum. To determine this wave length of maximum instability let $\xi = (2\pi/L)l$. Then

$$\mu^2\Theta^2 = \frac{F(\xi)}{l^2} = \xi^2\left(\frac{2}{\xi} - 1 - \frac{2}{\xi^2}\frac{1}{1 + \coth \xi}\right).$$

This maximum of $F(\xi)$ occurs when $\xi = 0.797$ or when $L = 7.88l$, as has already been shown by Rayleigh (1894). Thus the wave will increase most rapidly with time when the wave length is about eight times the width of the shear zone. On the other hand, when $L < 4.9l$, Θ becomes imaginary, and the waves are no longer unstable, as shown by Rayleigh (1894).

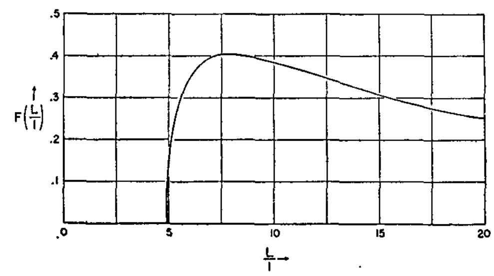

FIG. 4. The instability function $F(L/l)$ as function of the ratio of wave length L to the width l of the shear zone.

The function F is plotted in fig. 4 with L/l as the abscissa. Since $F(L/l)$ is proportional to the time factor in the exponential, the graph shows that waves which are shorter than about five times the width of the shear zone cannot form spontaneously. As waves longer than this critical value are considered, the time factor increases rapidly to a maximum when the wave length is slightly less than eight times the width of the shear zone. Beyond this maximum value F decreases less rapidly with increasing wave length.

Since changes in the current velocities of planetary circulations occur always in shear zones of finite width rather than at sharp discontinuities, only waves of a certain minimum length will form spontaneously. Locally, of course, narrow shear zones may exist so that short waves can develop as small-scale phenomena. The much wider shear zones in the large-scale atmospheric circulations permit only the development of longer waves, and the smaller waves which might otherwise interfere with their development are excluded. Since waves whose length is about eight times the width of the shear zone have the largest time factor these waves may be expected to develop most frequently. But as fig. 4 shows, the time factor does not decrease very rapidly for longer waves so that those waves should also be found. On the other hand, when the wave length is less than eight times the width of the shear zone the time factor decreases rapidly and conditions are therefore not favorable for the origin of such waves even though they are theoretically possible.

An empirical check of these conclusions is difficult. In the first place it is very difficult to measure the width of a shear zone. In the second place in order to observe the wave length of a perturbation sufficient development must at least have taken place so that the disturbance can be observed. G. Cressman and E. Schacht of the Department of Meteorology at the University of Chicago have investigated some cases of development of perturbations at shear zones while at the Institute of Tropical Meteorology of the University of Puerto Rico. Their results are summarized in table 2.

TABLE 2. Relation between development and L/l.

L/l	Development	No development
<5	2	2
>5	12	3

Thus, there are 5 cases contradicting the statement that development should take place when $L/l > 5$ and not take place when $L/l < 5$, while 14 cases bear out this statement. However, in view of the uncertainty in determining the various parameters too much weight should not be attributed to this empirical verification.

Acknowledgment.—Thanks are due to Dr. H. Panofsky for numerous stimulating discussions especially concerning the astronomical implications of the hydrodynamical problem.

REFERENCES

Haurwitz, B., 1941: *Dynamic Meteorology.* New York, McGraw-Hill Book Co., 365 pp. (see pp. 274–276).

——, 1943: The effect of a gradual wind change on the stability of waves. *Ann. N. Y. Acad. Sci.*, **44**, 69–80.

Jahnke, E., and F. Emde, 1943: *Tables of Functions.* New York, Dover Publications, 303 pp. plus Addenda (see Addenda, p. 26).

Lamb, H., 1945: *Hydrodynamics*, 6 ed. New York, Dover Publications, 738 pp. (see p. 374).

Rayleigh, Lord, 1894: *Theory of sound*, 2 ed., vol. 2. New York, Dover Publications (1945), 504 pp. (see p. 394).

Rossby, C.-G., and collaborators, 1939: Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centers of action. *J. marine Res.*, **2**, 38–55.

Russell, H. N., R. S. Dugan, and J. A. Stewart, 1926: *Astronomy*, vol. 1. Boston, Ginn and Co., 470 + 21 pp. (see p. 364).