

CORRESPONDENCE

The neglect of compressibility in the flow of gas at low speeds

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It is not unusual for meteorologists to adopt a device of aerodynamicists and assume that an "adiabatic" gas flowing at speeds much less than that of sound is to a good approximation incompressible. A recent example is the stimulating article by Freeman¹ in the August 1948 issue of this JOURNAL. Because of the different orders of magnitude of the various dynamic quantities in meteorology and in aerodynamics a re-examination of this device from the meteorological point of view seems desirable.

We will follow the discussion of Milne-Thompson.² For an adiabatic fluid Bernoulli's law is:

$$\lambda p/\rho + \frac{1}{2}v^2 + gh = \text{constant}$$

along a streamline, where $\lambda \equiv \gamma/(\gamma - 1)$ and γ is one of the Poisson constants ($\gamma = 1.4$ for air). If we designate by p_0 and ρ_0 the pressure and density, respectively, at a point where the fluid has the velocity zero and height h_0 , then letting $H \equiv h - h_0$, Bernoulli's law is:

$$\lambda p/\rho + \frac{1}{2}v^2 + gH = \lambda p_0/\rho_0.$$

Solving for p/p_0 and eliminating the density by use of the adiabatic law $p/p_0 = (\rho/\rho_0)^\gamma$, we obtain

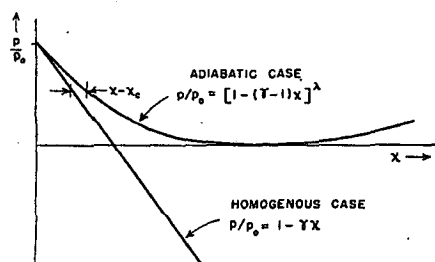
$$p/p_0 = [1 - (\gamma - 1)\chi]^\lambda, \tag{1}$$

where $\chi \equiv \frac{1}{2}c_0^{-2}(v^2 + 2gH)$ and $c_0 \equiv (\gamma p_0/\rho_0)^{1/2}$ is the speed of sound. Expanding (1),

$$p/p_0 = 1 - \gamma\chi + \text{higher-order terms.} \tag{2}$$

It is readily seen that (2) less the higher-order terms is equivalent to Bernoulli's law for a homogeneous fluid: $p/\rho_0 + \frac{1}{2}v^2 + gH = p_0/\rho_0$.

In the adjoining figure we have graphed Bernoulli's law for an adiabatic and also for an homogeneous atmosphere.



Graph of equations (1) and (2).

Suppose we proceed along a streamline from an initial point h_i to a point $h = h_i + H$ and there seek to compute the velocity with an accuracy of 20 per cent or better. Let v_0 be the velocity computed from (2), so that

$$p/p_0 = 1 - \gamma\chi_0,$$

where $\chi_0 \equiv \frac{1}{2}c_0^{-2}(v_0^2 + 2gH)$, and let v be the "true" velocity, computed from (1). Then

$$1 - \gamma\chi_0 = [1 - (\gamma - 1)\chi]^\lambda,$$

or solving for χ in powers of χ_0 ,

$$\chi = \chi_0 + \frac{1}{2}\chi_0^2 + \dots,$$

omitting powers higher than the second. Now let $\delta \equiv v - v_0$ and assume that the error in computing v is $\delta/v_0 \leq 0.20$. Then

$$\chi - \chi_0 = \frac{v_0^2}{c_0^2} \left(\frac{\delta^2}{2v_0^2} + \frac{\delta}{v_0} \right) \leq 0.22 \frac{v_0^2}{c_0^2}. \tag{3}$$

But $\chi - \chi_0 \approx \frac{1}{2}\chi_0^2$ so if (3) is to be satisfied, $\chi_0 \leq 0.7v_0/c_0$. Therefore certainly we require that

$$H \leq 0.7v_0c_0/g. \tag{4}$$

This means that if we adopt the device of considering our "adiabatic" gas as incompressible, in order to avoid errors greater than 20 per cent in the computation of the velocity by means of Bernoulli's law, we must not move along the streamline a horizontal distance from our initial point so great as to correspond to the vertical displacements given in the second column of the adjoining table (computed from (4)). For a mean vertical velocity of 1 cm sec⁻¹ these maximum distances are tabulated in the third column of the table.

v_0 (m sec ⁻¹)	max H (m)	max horizontal displacement (degrees latitude)
5	117	0.5
10	234	2.1
15	350	4.7
20	467	8.4

Now Bernoulli's law is valid only for steady-state conditions. Hence the velocities in the first column of the table should be interpreted as relative to a coordinate system fixed to a permanent-type wave. For aerodynamics, where the range of H along a streamline, past a wing say, is of the order of several meters, the device of regarding adiabatic airflow at moderate speeds as though the air were incompressible is a valid one. However for permanent-type waves in the atmosphere, where vertical velocities are frequently greater than or equal to one centimeter per second and relative winds less than 10 m sec⁻¹, this

device fails except for phenomena of horizontal dimension less than two degrees of latitude. The technique can be used for atmospheric models in which horizontal flow is postulated, but whether anything about the real atmosphere can be inferred from the properties of such a model is a moot question, to be separately decided in each case.

¹ John C. Freeman, Jr., "An analogy between the equatorial easterlies and supersonic gas flows," *J. Meteor.*, 5, 138-146, 1948.

² L. M. Milne-Thompson, "*Theoretical hydrodynamics*," London, MacMillan and Co., 552 pp., 1938.