

Hamiltonian Description of Idealized Binary Geophysical Fluids

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ABSTRACT

A Hamiltonian formulation for the dynamics and thermodynamics of a compressible, rotating, binary fluid subject to gravity is developed. Here, binary refers to the presence of two components of the fluid, such as solids dissolved in a liquid or gaseous and liquid water existing along with dry air. These fluids are idealized in that the influences of diffusion processes are ignored and the binary flow is restricted to a single velocity.

The equations are presented in generic form applicable to an arbitrary binary geophysical flow. The relevant Poisson bracket satisfies Jacobi's identity. Three distinct Casimir invariants are described. The first reflects the conservation of entropy and concentration of the minor component. The second is a consequence of the conservation of the absolute circulation on curves formed by the intersection of surfaces of constant entropy with surfaces of constant concentration. The third is a generic potential vorticity of the form $(\boldsymbol{\omega} \cdot \nabla\lambda)/\rho$. Here, $\boldsymbol{\omega}$ is the absolute vorticity, ρ is the total density of the fluid, and λ is any thermodynamic variable. For example, λ can be the pressure, density, temperature, or mixing ratio as well as the more common choice of potential temperature.

Available energy of the system is defined locally in the finite-amplitude as well as in the small-amplitude limit. Both definitions are partitioned into available potential and available elastic energies.

A linear stability analysis indicates that the fluid is statically stable provided the square of the sound speed is positive, the total density decreases with height, and the square of a suitably defined buoyancy frequency is positive.

The formulation is applicable to a salty ocean and to a moist atmosphere. For the atmosphere, the full theory holds in the presence of either liquid water or ice in equilibrium with its vapor.

1. Introduction

Hamiltonian formulations of the equations of motion have been applied to a variety of idealized flows (see the reviews by Salmon 1988, 1998; Shepherd 1990; Morrison 1998). Particularly noteworthy is the ability of the Hamiltonian approach to systematize the derivation of the expression for the available energy (Shepherd 1993). Here available energy is the nonkinetic contribution to the pseudoenergy and has potential and elastic components.

The purpose of the present investigation is to extend the Hamiltonian description of idealized rotating compressible flows to the case where the fluid is a binary mixture of two flow components whose composition is not homogenous. The mixture is idealized to be in approximate local thermodynamic equilibrium. Diffusive processes are ignored. It is further assumed that the velocities of the two components are the same.

The next section presents the generic equations for such a binary geophysical fluid. Section 3 casts the system into a Hamiltonian formulation. This formulation extends that for the single-component system to include a prognostic equation for the relative concentration of the two components. Section 4 describes the range of Casimir invariants. These include a generalized potential vorticity that is implicit in the kinematic discussions of Truesdell (1951, 1954) and Haynes and McIntyre (1987). Section 5 describes the available energy for the system, its partitioning into available potential and available elastic energies, and its small amplitude limit. The approach follows that of Shepherd (1993) to yield a local expression like that of Andrews (1981) relative to an arbitrary reference state. Section 6 presents a linear stability analysis of the flow. Sections 7 and 8 apply these findings to a moist atmosphere and a salty ocean. The formulation for the atmosphere allows the water vapor to exist in equilibrium with either liquid or solid water and to undergo a change of phase. It generalizes the moist available potential energy of Lorenz (1979) to a local expression relative to an arbitrary reference state. It is also shown that, in principle, the nonlinear theory is applicable to the oceans.

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2. The model geophysical fluid

The equations for three-dimensional, rotating, compressible, inviscid flow of a single-temperature, single-velocity, binary system are

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + 2\mathbf{\Omega} \times \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi, \quad (2.1a)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (2.1b)$$

$$\frac{\partial \eta}{\partial t} + (\mathbf{v} \cdot \nabla) \eta = 0, \quad (2.1c)$$

$$\frac{\partial \chi}{\partial t} + (\mathbf{v} \cdot \nabla) \chi = 0, \quad (2.1d)$$

where \mathbf{v} is the three-dimensional velocity, $\mathbf{\Omega}$ is the rotation rate, p is the pressure, ρ is the total density, Φ is the geopotential, η is the total entropy per unit total mass, and χ is the generic concentration describing the relative amount of mass of the sparse component to that of the more abundant component. It is assumed that both components move with the same velocity \mathbf{v} and have the same temperature T .

We write the thermodynamic identity for the internal energy u per unit total mass as

$$du = Td\eta + \frac{p}{\rho^2} d\rho + \mu d\chi, \quad (2.2)$$

where μ is the chemical potential of the mixture. Because u is a function of ρ , η , and χ , we may write

$$du = \left(\frac{\partial u}{\partial \eta} \right)_{\rho, \chi} d\eta + \left(\frac{\partial u}{\partial \rho} \right)_{\eta, \chi} d\rho + \left(\frac{\partial u}{\partial \chi} \right)_{\rho, \eta} d\chi. \quad (2.3)$$

Comparison of these last two expressions implies

$$\left(\frac{\partial u}{\partial \eta} \right)_{\rho, \chi} = T, \quad \left(\frac{\partial u}{\partial \rho} \right)_{\eta, \chi} = \frac{p}{\rho^2}, \quad \left(\frac{\partial u}{\partial \chi} \right)_{\rho, \eta} = \mu, \quad (2.4)$$

and the commutability of the second derivatives yields Maxwell's relations in the form

$$\begin{aligned} \left(\frac{\partial T}{\partial \rho} \right)_{\eta, \chi} &= \frac{1}{\rho^2} \left(\frac{\partial p}{\partial \eta} \right)_{\rho, \chi}, & \left(\frac{\partial \mu}{\partial \rho} \right)_{\eta, \chi} &= \frac{1}{\rho^2} \left(\frac{\partial p}{\partial \chi} \right)_{\rho, \eta}, \\ \left(\frac{\partial T}{\partial \chi} \right)_{\rho, \eta} &= \left(\frac{\partial \mu}{\partial \eta} \right)_{\rho, \chi}. \end{aligned} \quad (2.5)$$

It is also convenient to introduce the enthalpy, $h = u + p/\rho$. Then the thermodynamic identity becomes

$$Td\eta = dh - \frac{1}{\rho} dp - \mu d\chi. \quad (2.6)$$

For the present we assume the definitions for u , h , η , χ , and μ are known. Their specification is postponed until sections 7 and 8 in which applications to the atmosphere and ocean are made.

3. Hamiltonian description

The prognostic system of Eqs. (2.1) is Hamiltonian in the variables $\mathbf{u} = (u_1, u_2, u_3, u_4, u_5, u_6)^T = (\mathbf{v}, \rho, \eta, \chi)^T$ with the Hamiltonian function

$$H(\mathbf{v}, \rho, \eta, \chi) = \int_V \left[\frac{1}{2} \rho |\mathbf{v}|^2 + \rho u(\rho, \eta, \chi) + \rho \Phi \right] dV \quad (3.1)$$

describing the total energy of the system. The prognostic equations may be written in Cartesian tensor notation:

$$\frac{\partial u_i}{\partial t} = J_{ij} \frac{\delta H}{\delta u_j}, \quad (3.2)$$

where $\delta H/\delta u_j$ denotes a functional derivative, and J_{ij} is the tensor:

$$\mathbf{J} = \begin{bmatrix} 0 & \frac{1}{\rho} \omega_3 & -\frac{1}{\rho} \omega_2 & -\frac{\partial}{\partial x} & \frac{1}{\rho} \frac{\partial \eta}{\partial x} & \frac{1}{\rho} \frac{\partial \chi}{\partial x} \\ -\frac{1}{\rho} \omega_3 & 0 & \frac{1}{\rho} \omega_1 & -\frac{\partial}{\partial y} & \frac{1}{\rho} \frac{\partial \eta}{\partial y} & \frac{1}{\rho} \frac{\partial \chi}{\partial y} \\ \frac{1}{\rho} \omega_2 & -\frac{1}{\rho} \omega_1 & 0 & -\frac{\partial}{\partial z} & \frac{1}{\rho} \frac{\partial \eta}{\partial z} & \frac{1}{\rho} \frac{\partial \chi}{\partial z} \\ -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} & 0 & 0 & 0 \\ -\frac{1}{\rho} \frac{\partial \eta}{\partial x} & -\frac{1}{\rho} \frac{\partial \eta}{\partial y} & -\frac{1}{\rho} \frac{\partial \eta}{\partial z} & 0 & 0 & 0 \\ -\frac{1}{\rho} \frac{\partial \chi}{\partial x} & -\frac{1}{\rho} \frac{\partial \chi}{\partial y} & -\frac{1}{\rho} \frac{\partial \chi}{\partial z} & 0 & 0 & 0 \end{bmatrix}, \quad (3.3)$$

where the velocity and absolute vorticity are

$$\begin{aligned} \mathbf{v} &\equiv (v_1, v_2, v_3), \\ \boldsymbol{\omega} &\equiv (\omega_1, \omega_2, \omega_3) = 2\mathbf{\Omega} + \nabla \times \mathbf{v}, \end{aligned} \quad (3.4)$$

respectively. We note that the functional derivatives of the Hamiltonian are

$$\frac{\delta H}{\delta \mathbf{v}} = \rho \mathbf{v}, \quad (3.5a)$$

$$\frac{\delta H}{\delta \eta} = \rho \frac{\partial u}{\partial \eta} = \rho T, \quad (3.5b)$$

$$\begin{aligned} \frac{\delta H}{\delta \rho} &= \frac{1}{2} |\mathbf{v}|^2 + \frac{\partial \rho u}{\partial \rho} + \Phi \\ &= \frac{1}{2} |\mathbf{v}|^2 + h + \Phi, \end{aligned} \quad (3.5c)$$

$$\frac{\delta H}{\delta \chi} = \rho \frac{\partial u}{\partial \chi} = \rho \mu. \quad (3.5d)$$

Then the Hamiltonian description (3.2) becomes

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{J} \frac{\delta H}{\delta \mathbf{u}} = \begin{bmatrix} \frac{1}{\rho} \left(\omega_3 \frac{\delta H}{\delta v_2} - \omega_2 \frac{\delta H}{\delta v_3} \right) - \frac{\partial}{\partial x} \left(\frac{\delta H}{\delta \rho} \right) + \frac{1}{\rho} \frac{\partial \eta}{\partial x} \frac{\delta H}{\delta \eta} + \frac{1}{\rho} \frac{\partial \chi}{\partial x} \frac{\delta H}{\delta \chi} \\ \frac{1}{\rho} \left(\omega_1 \frac{\delta H}{\delta v_3} - \omega_3 \frac{\delta H}{\delta v_1} \right) - \frac{\partial}{\partial y} \left(\frac{\delta H}{\delta \rho} \right) + \frac{1}{\rho} \frac{\partial \eta}{\partial y} \frac{\delta H}{\delta \eta} + \frac{1}{\rho} \frac{\partial \chi}{\partial y} \frac{\delta H}{\delta \chi} \\ \frac{1}{\rho} \left(\omega_2 \frac{\delta H}{\delta v_1} - \omega_1 \frac{\delta H}{\delta v_2} \right) - \frac{\partial}{\partial z} \left(\frac{\delta H}{\delta \rho} \right) + \frac{1}{\rho} \frac{\partial \eta}{\partial z} \frac{\delta H}{\delta \eta} + \frac{1}{\rho} \frac{\partial \chi}{\partial z} \frac{\delta H}{\delta \chi} \\ - \nabla \cdot \frac{\delta H}{\delta \mathbf{v}} \\ - \frac{1}{\rho} \nabla \eta \cdot \frac{\delta H}{\delta \mathbf{v}} \\ - \frac{1}{\rho} \nabla \chi \cdot \frac{\delta H}{\delta \mathbf{v}} \end{bmatrix}. \quad (3.6)$$

The first three equations represent the momentum equation in the form

$$\frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\omega} \times \mathbf{v} - \nabla \left(\frac{1}{2} |\mathbf{v}|^2 \right) - \frac{1}{\rho} \nabla p - \nabla \Phi, \quad (3.7)$$

where the pressure gradient is related to the other flow variables by the spatial derivative form of the thermodynamic identity (2.6). The fourth, fifth, and sixth equations in this set represent the equation of continuity for

total density, and the equations of conservation of entropy and concentration, respectively.

It remains to be shown that J satisfies the Jacobi condition. Its Poisson bracket is

$$[F, G] \equiv \left(\frac{\delta F}{\delta \mathbf{u}}, \mathbf{J} \frac{\delta G}{\delta \mathbf{u}} \right) \equiv \int \frac{\delta F}{\delta u_i} J_{ij} \frac{\delta G}{\delta u_j} dV, \quad (3.8a)$$

where the integration is over the total volume V of fluid. Using the divergence theorem this expression becomes

$$[F, G] = - \int \left[\frac{\delta F}{\delta \rho} \nabla \cdot \frac{\delta G}{\delta \mathbf{v}} - \frac{\delta G}{\delta \rho} \nabla \cdot \frac{\delta F}{\delta \mathbf{v}} + \frac{\nabla \times \mathbf{v}}{\rho} \cdot \left(\frac{\delta G}{\delta \mathbf{v}} \times \frac{\delta F}{\delta \mathbf{v}} \right) + \frac{\nabla \eta}{\rho} \cdot \left(\frac{\delta F}{\delta \eta} \frac{\delta G}{\delta \mathbf{v}} - \frac{\delta G}{\delta \eta} \frac{\delta F}{\delta \mathbf{v}} \right) + \frac{\nabla \chi}{\rho} \cdot \left(\frac{\delta F}{\delta \chi} \frac{\delta G}{\delta \mathbf{v}} - \frac{\delta G}{\delta \chi} \frac{\delta F}{\delta \mathbf{v}} \right) \right] dV, \quad (3.8b)$$

where we have assumed that

$$\frac{\delta G}{\delta \rho} \frac{\delta F}{\delta \mathbf{v}} \cdot \hat{\mathbf{n}} = 0 \quad (3.9)$$

on the boundaries of the domain. Here, $\hat{\mathbf{n}}$ is the unit normal on the boundaries. It is traditional (e.g., Mor-

risson 1998) to neglect these boundary contributions. The bracket (3.8b) may be put into Lie–Poisson form by a transformation from the variables \mathbf{v} , η , and χ to the variables \mathbf{M} ($=\rho\mathbf{v}$), σ ($=\rho\eta$), and X ($=\rho\chi$) in a manner similar to Morrison (1998, p. 490). This transformation amounts to writing (2.1) in flux form. The result,

$$[F, G] = - \int \left\{ \mathbf{M} \cdot \left[\left(\frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \right) \frac{\delta G}{\delta \mathbf{M}} - \left(\frac{\delta G}{\delta \mathbf{M}} \cdot \nabla \right) \frac{\delta F}{\delta \mathbf{M}} \right] + \rho \left[\frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \mathbf{M}} \cdot \nabla \frac{\delta F}{\delta \rho} \right] + \sigma \left[\frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta \mathbf{M}} \cdot \nabla \frac{\delta F}{\delta \sigma} \right] + X \left[\frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \frac{\delta G}{\delta X} - \frac{\delta G}{\delta \mathbf{M}} \cdot \nabla \frac{\delta F}{\delta X} \right] \right\} dV, \quad (3.10)$$

implies that the Jacobi identity is satisfied because the contribution of X is isomorphic with that of σ and ρ

and, apart from the contributions from F and G , (3.10) is linear in $[\mathbf{M}, \rho, \sigma, X]$ (Morrison 1982).

4. Casimir invariants

The Hamiltonian description presented in section 3 is an Eulerian one for the six variables \mathbf{u} . In principle, a Lagrangian description may be constructed for nine variables (\mathbf{u} plus the three spatial coordinates). This reduction in the number of variables from the Lagrangian to the Eulerian description implies that the Eulerian

description is noncanonical. Mathematically this reduction is reflected in the fact that the matrix \mathbf{J} is singular. As a consequence, Eulerian Hamiltonian descriptions possess global invariants. Because of the reduction in the number of invariants (Kuroda 1990), the Hamiltonian system should possess three distinct Casimir invariants. These Casimirs are functionals C that have a vanishing Poisson bracket:

$$[F, C] = - \int \left\{ \frac{\delta F}{\delta \rho} \nabla \cdot \frac{\delta C}{\delta \mathbf{v}} + \frac{\delta F}{\delta \eta} \frac{\nabla \eta}{\rho} \cdot \frac{\delta C}{\delta \mathbf{v}} + \frac{\delta F}{\delta \chi} \frac{\nabla \chi}{\rho} \cdot \frac{\delta C}{\delta \mathbf{v}} + \frac{\delta F}{\delta \mathbf{v}} \cdot \left[\nabla \left(\frac{\delta C}{\delta \rho} \right) + \frac{\nabla \times \mathbf{v}}{\rho} \times \frac{\delta C}{\delta \mathbf{v}} - \frac{\delta C}{\delta \eta} \frac{\nabla \eta}{\rho} - \frac{\delta C}{\delta \chi} \frac{\nabla \chi}{\rho} \right] \right\} dV = 0. \tag{4.1}$$

If the arbitrary functional F is a function of ρ , η , χ , or \mathbf{v} separately, then C must satisfy the four constraints

$$\nabla \cdot \frac{\delta C}{\delta \mathbf{v}} = 0, \tag{4.2a}$$

$$\frac{\nabla \eta}{\rho} \cdot \frac{\delta C}{\delta \mathbf{v}} = 0, \tag{4.2b}$$

$$\frac{\nabla \chi}{\rho} \cdot \frac{\delta C}{\delta \mathbf{v}} = 0, \tag{4.2c}$$

$$\nabla \left(\frac{\delta C}{\delta \rho} \right) + \frac{\nabla \times \mathbf{v}}{\rho} \times \frac{\delta C}{\delta \mathbf{v}} - \frac{\delta C}{\delta \eta} \frac{\nabla \eta}{\rho} - \frac{\delta C}{\delta \chi} \frac{\nabla \chi}{\rho} = 0. \tag{4.2d}$$

Three distinct solutions to (4.2) are

$$C_1 = \int \rho f(\eta, \chi) dV, \tag{4.3a}$$

$$C_2 = \int g(\eta, \chi) [\mathbf{v}_a \cdot (\nabla \eta \times \nabla \chi)] dV, \tag{4.3b}$$

$$C_3 = \int \rho q dV, \tag{4.3c}$$

where f and g are arbitrary functions of η and χ , and \mathbf{v}_a is the absolute velocity. Here, the generalized potential vorticity is

$$q \equiv \frac{\boldsymbol{\omega} \cdot \nabla \lambda}{\rho}, \tag{4.4}$$

where $\lambda = \lambda(\rho, \eta, \chi)$ is an arbitrary thermodynamic scalar function.

The first Casimir, C_1 , is a consequence of the material conservation of entropy and concentration and expresses the global invariance of these two quantities (and com-

binations thereof). This thermodynamic Casimir C_1 has the functional derivatives

$$\frac{\delta C_1}{\delta \rho} = f, \tag{4.5a}$$

$$\frac{\delta C_1}{\delta \eta} = \rho \frac{\partial f}{\partial \eta}, \tag{4.5b}$$

$$\frac{\delta C_1}{\delta \chi} = \rho \frac{\partial f}{\partial \chi}, \tag{4.5c}$$

$$\frac{\delta C_1}{\delta \mathbf{v}} = 0. \tag{4.5d}$$

The first three constraints of (4.2) are trivially satisfied while the last is satisfied by the chain rule. Thus, C_1 is a Casimir.

The second Casimir, C_2 , is a consequence (Salmon 2003, personal communication) of the conservation of the absolute circulation

$$\Gamma = \int_L \mathbf{v}_a d\mathbf{x}, \tag{4.6}$$

where L is a curve formed by the intersection of a surface of constant entropy η with a surface of constant concentration χ . If the curve L is closed, then the circulation is conserved. If the curve L does not close, then the circulation is also conserved by the assumption to neglect any boundary contributions. Letting η , χ , and a be the mass-labeling coordinates such that the density is the Jacobian of the transformation between the mass-labeling and Cartesian coordinates,

$$\rho = \frac{\partial(a, \eta, \chi)}{\partial(x, y, z)}. \tag{4.7}$$

Because only a varies along the curve L , we can rewrite the circulation as

$$\Gamma = \int_L \mathbf{v}_a \cdot \frac{\partial \mathbf{x}}{\partial a} da. \tag{4.8}$$

Then

$$\begin{aligned} & \iint d\eta d\chi g(\eta, \chi) \Gamma \\ &= \iiint d\eta d\chi da g(\eta, \chi) \mathbf{v}_a \cdot \frac{\partial(\mathbf{x}, \eta, \chi)}{\partial(a, \eta, \chi)} \end{aligned} \tag{4.9}$$

is also conserved because Γ is conserved on each curve L of constant η and χ . Transforming the integral to physical space yields

$$\iiint d\mathbf{x} g(\eta, \chi) \mathbf{v}_a \cdot \frac{\partial(\mathbf{x}, \eta, \chi)}{\partial(x, y, z)}, \tag{4.10}$$

which is equivalent to (4.3b). The functional derivatives of C_2 are

$$\frac{\delta C_2}{\delta \rho} = 0, \tag{4.11a}$$

$$\frac{\delta C_2}{\delta \eta} = g \nabla \chi \cdot \boldsymbol{\omega}, \tag{4.11b}$$

$$\frac{\delta C_2}{\delta \chi} = -g \nabla \eta \cdot \boldsymbol{\omega}, \tag{4.11c}$$

$$\frac{\delta C_2}{\delta \mathbf{v}} = g(\nabla \eta \times \nabla \chi), \tag{4.11d}$$

and it is straightforward to show that these satisfy (4.2). Thus, C_2 is a Casimir.

The third Casimir, C_3 , is a consequence of the kinematic structure of potential vorticity (see Truesdell 1951, 1954; Haynes and McIntyre 1987). This kinematic Casimir arises directly from the fact that the divergence of a curl vanishes. The functional derivatives of C_3 are

$$\frac{\delta C_3}{\delta \rho} = -\frac{\partial \lambda}{\partial \rho} \nabla \cdot \boldsymbol{\omega}, \tag{4.12a}$$

$$\frac{\delta C_3}{\delta \eta} = -\frac{\partial \lambda}{\partial \eta} \nabla \cdot \boldsymbol{\omega}, \tag{4.12b}$$

$$\frac{\delta C_3}{\delta \chi} = -\frac{\partial \lambda}{\partial \chi} \nabla \cdot \boldsymbol{\omega}, \tag{4.12c}$$

$$\frac{\delta C_3}{\delta \mathbf{v}} = \nabla \times \nabla \lambda, \tag{4.12d}$$

which all vanish. This result implies that the functional derivatives of the Casimir are identically zero and C_3 satisfies the four conditions trivially. This is a very special case, but it holds for an arbitrary thermodynamic scalar function λ because of the vector identities for the vanishing of the divergence of a curl and the curl of a gradient. Thus, Eqs. (4.3) are the Casimir invariants.

5. Available energy

A pseudoenergy may be constructed following Shepherd (1993) in terms of the Hamiltonian and a Casimir C relative to a base state \mathbf{u}_0 as

$$\mathcal{A}(\mathbf{u}, \mathbf{u}_0) = H(\mathbf{u}) + C(\mathbf{u}) - [H(\mathbf{u}_0) + C(\mathbf{u}_0)]. \tag{5.1}$$

Then we define the available energy as the nonkinetic energy contribution to the pseudoenergy. To construct the pseudoenergy for a generic geophysical flow we consider a type-1 Casimir invariant (4.3a) with functional derivatives (4.5) and choose a resting base-state \mathbf{u}_0 such that

$$\mathbf{v} = 0, \quad \eta = \eta_0(z), \quad \rho = \rho_0(z), \quad \chi = \chi_0(z), \tag{5.2}$$

that is, in hydrostatic balance

$$\frac{dp_0}{dz} = -\rho_0 g. \tag{5.3}$$

Because the density is a positive quantity, this relation implies that the base-state pressure is a monotonic function of height. Thermodynamically, we have that

$$p_0 = p_0(\rho_0, \eta_0, \chi_0). \tag{5.4}$$

But with the hydrostatic relation, this implies

$$p_0 = p_0\left(\frac{dp_0}{dz}, \eta_0, \chi_0\right) \quad \text{or} \quad p_0 = p_0(\eta_0, \chi_0). \tag{5.5}$$

Similarly, $\rho_0 = \rho_0(\eta_0, \chi_0)$. Here, the Casimir C must satisfy

$$\begin{aligned} \frac{\delta H}{\delta \rho} &= -\frac{\delta C}{\delta \rho}, & \frac{\delta H}{\delta \eta} &= -\frac{\delta C}{\delta \eta}, & \frac{\delta H}{\delta \chi} &= -\frac{\delta C}{\delta \chi}, \\ \frac{\delta H}{\delta \mathbf{v}} &= -\frac{\delta C}{\delta \mathbf{v}}, \end{aligned} \tag{5.6}$$

when evaluated in the base state. The last condition is satisfied trivially. The remaining three conditions require that f satisfy

$$[f(\eta, \chi)]_0 = -gz - u_0 - \frac{p_0}{\rho_0}, \tag{5.7a}$$

$$\left[\frac{\partial f(\eta, \chi)}{\partial \eta} \right]_0 = -T_0, \tag{5.7b}$$

$$\left[\frac{\partial f(\eta, \chi)}{\partial \chi} \right]_0 = -\mu_0, \tag{5.7c}$$

where the subscript 0 indicates the quantity in parentheses has been evaluated in the base state (5.2). For example, u_0 is the base-state internal energy. In order to satisfy the first condition we choose f such that

$$f(\eta, \chi) = -gZ(\eta, \chi) - u[\rho_0(\eta, \chi), \eta, \chi] - \frac{p_0(\eta, \chi)}{\rho_0(\eta, \chi)}, \quad (5.8)$$

where

$$[Z(\eta, \chi)]_0 = z. \quad (5.9)$$

This relation is implicit in the hydrostatic relation. Clearly, (5.8) satisfies (5.7a) when evaluated in the base state. We next consider the second condition (5.7b) and evaluate

$$\frac{\partial f}{\partial \eta} = -g \frac{\partial Z}{\partial \eta} - \frac{\partial u}{\partial \eta} - \left(\frac{\partial u}{\partial \rho} \right)_0 \frac{\partial \rho_0}{\partial \eta} - \frac{1}{\rho_0} \frac{\partial p_0}{\partial \eta} + \frac{p_0}{\rho_0^2} \frac{\partial \rho_0}{\partial \eta}. \quad (5.10)$$

Using (2.4a,b) and evaluating in the base state yields

$$\left(\frac{\partial f}{\partial \eta} \right)_0 = -T_0 - \frac{1}{\rho_0} \left[\left(\frac{\partial p_0}{\partial \eta} \right)_0 + \rho_0 g \left(\frac{\partial Z}{\partial \eta} \right)_0 \right] = -T_0. \quad (5.11)$$

The last equality holds because $p_0 = p_0(\eta, \chi)$ and $p_0 = p_0(z)$, so

$$\left(\frac{\partial p_0}{\partial \eta} \right)_0 = \left(\frac{dp_0}{dZ} \right)_0 \left(\frac{\partial Z}{\partial \eta} \right)_0 = -\rho_0 g \left(\frac{\partial Z}{\partial \eta} \right)_0, \quad (5.12)$$

and the second condition is satisfied. Similarly, using (2.4b,c),

$$\left(\frac{\partial f}{\partial \chi} \right)_0 = -\mu_0 - \frac{1}{\rho_0} \left[\left(\frac{\partial p_0}{\partial \chi} \right)_0 + \rho_0 g \left(\frac{\partial Z}{\partial \chi} \right)_0 \right] = -\mu_0, \quad (5.13)$$

and the third condition (5.7c) is satisfied.

The available energy per unit volume is the nonkinetic contribution to (5.1) or

$$A = (\rho - \rho_0)gz + \rho u(\rho, \eta, \chi) - \rho_0 u(\rho_0, \eta_0, \chi_0) + \rho f(\eta, \chi) - \rho_0 f(\eta_0, \chi_0). \quad (5.14)$$

Then using (5.8) we have

$$A = \rho g[Z(\eta_0, \chi_0) - Z(\eta, \chi)] + \rho u(\rho, \eta, \chi) - \rho_0 u[\rho_0(\eta, \chi), \eta, \chi] - \rho \frac{p_0(\eta, \chi)}{\rho_0(\eta, \chi)} + p_0(\eta_0, \chi_0), \quad (5.15)$$

which is the generalization of the result (9.12) of Shepherd (1993).

The available energy may be partitioned into available potential energy (APE) and available elastic energy (AEE), $A = \text{AEE} + \text{APE}$. Following Andrews (1981) the available elastic energy per unit volume is

$$\text{AEE} = \rho h(p, \eta, \chi) - \rho h[p_0(\eta_0, \chi_0), \eta, \chi] - p + p_0(\eta_0, \chi_0), \quad (5.16)$$

where h is the enthalpy. Then $\text{APE} = A - \text{AEE}$. If the departures of \mathbf{u} from the base state \mathbf{u}_0 are small, then (5.15) may be expanded in a Taylor series about \mathbf{u}_0 . To leading order, one finds

$$A_{\text{linear}} \equiv \frac{1}{2} A_{\rho\rho 0} (\rho - \rho_0)^2 + \frac{1}{2} A_{\eta\eta 0} (\eta - \eta_0)^2 + \frac{1}{2} A_{\chi\chi 0} (\chi - \chi_0)^2 + A_{\rho\eta 0} (\rho - \rho_0)(\eta - \eta_0) + A_{\eta\chi 0} (\eta - \eta_0)(\chi - \chi_0) + A_{\rho\chi 0} (\rho - \rho_0)(\chi - \chi_0), \quad (5.17)$$

where the subscripts ρ , η , and χ denote differentiation with respect to those variables but the subscript 0 denotes evaluation in the base state. Evaluation of the second derivatives is relatively straightforward but tedious. Some details are summarized in the appendix. In addition, use is made of the relation

$$p - p_0 = \left(\frac{\partial p}{\partial \rho} \right)_{\eta\chi 0} (\rho - \rho_0) + \left(\frac{\partial p}{\partial \eta} \right)_{\rho\chi 0} (\eta - \eta_0) + \left(\frac{\partial p}{\partial \chi} \right)_{\rho\eta 0} (\chi - \chi_0). \quad (5.18)$$

With some effort, one finds that

$$A_{\text{linear}} = \frac{1}{2} \frac{\rho_0 g^2}{N_0^2} \left[\frac{(p - p_0)}{\rho_0 c_0^2} - \frac{(\rho - \rho_0)}{\rho_0} \right]^2 + \frac{(p - p_0)^2}{2\rho_0 c_0^2}, \quad (5.19)$$

where the first term on the right-hand side is the linear APE and the second the linear AEE. Here, c_0 is the speed of sound

$$c_0^2 \equiv \left(\frac{\partial p}{\partial \rho} \right)_{\eta,\chi 0}, \quad (5.20)$$

and the square of the buoyancy frequency is

$$N_0^2 \equiv -\frac{g}{\rho_0} \frac{d\rho_0}{dz} - \frac{g^2}{c_0^2}. \quad (5.21)$$

It may also be verified that the linearized form of (5.16) is the second term in (5.19). These results are the generalization of the one-component expressions to a binary fluid.

6. Linear stability analysis

We next examine the stability of the hydrostatic base state (5.2). The pseudoenergy (5.1) vanishes when evaluated in the base state. In addition, its first variation vanishes in the base state by the construct

(5.6) of the Casimir C . Thus, the state \mathbf{u}_0 is an extremum. For linear stability, it is sufficient that the

second variation of the pseudoenergy be positive. This convexity condition is

$$\delta^2 \mathcal{A} = \int_V \left[\frac{1}{2} \rho |\delta \mathbf{v}|^2 + A_{\rho\rho} (\delta\rho)^2 + A_{\eta\eta} (\delta\eta)^2 + A_{\chi\chi} (\delta\chi)^2 + 2A_{\rho\eta} \delta\rho\delta\eta + 2A_{\rho\chi} \delta\rho\delta\chi + 2A_{\eta\chi} \delta\eta\delta\chi \right] dV > 0, \quad (6.1)$$

where the subscripts denote differentiation. For linear stability it is sufficient that the integrand be positive definite. The kinetic energy contribution is positive. The available energy contribution is a third-order quadratic form. Stability is ensured if this contribution is non-negative. Based on the theory of quadratic forms (e.g., Hildebrand 1965), this contribution is positive semi-definite if

$$A_{\rho\rho} > 0, \quad (6.2a)$$

$$A_{\rho\rho}A_{\eta\eta} - (A_{\rho\eta})^2 > 0, \quad \text{and} \quad (6.2b)$$

$$\begin{aligned} & [A_{\rho\rho}A_{\eta\eta} - (A_{\rho\eta})^2][A_{\rho\rho}A_{\chi\chi} - (A_{\rho\chi})^2] \\ & - [A_{\rho\rho}A_{\eta\chi} - A_{\rho\eta}A_{\rho\chi}]^2 = 0, \end{aligned} \quad (6.2c)$$

when evaluated in the base state. The vanishing of the third condition (6.2c) means that the 3×3 symmetric matrix associated with the quadratic form is singular and that a 2×2 submatrix contains the necessary information. Then it is sufficient for stability that

$$A_{\rho\rho 0} > 0, \quad (6.3a)$$

$$A_{\eta\eta 0} > 0, \quad (6.3b)$$

$$A_{\rho\rho 0}A_{\eta\eta 0} - A_{\rho\eta 0}^2 > 0. \quad (6.3c)$$

Using the results in the appendix, we find

$$A_{\rho\rho 0} = \frac{1}{\rho_0} \left(\frac{\partial p}{\partial \rho} \right)_{\eta,\chi,0} = \frac{c_0^2}{\rho_0}, \quad (6.4a)$$

$$A_{\eta\eta 0} = -\frac{1}{g} \frac{d\rho_0}{dz} N_0^2 \left(\frac{\partial p}{\partial \rho} \right)_{\eta,\chi,0} \left(\frac{\partial Z}{\partial \eta} \right)_0^2, \quad (6.4b)$$

$$A_{\rho\rho 0}A_{\eta\eta 0} - A_{\rho\eta 0}^2 = N_0^2 \left(\frac{\partial p}{\partial \rho} \right)_{\eta,\chi,0} \left(\frac{\partial Z}{\partial \eta} \right)_0^2, \quad (6.4c)$$

where c_0 is the speed of sound (5.20) and the square of the buoyancy frequency is given by (5.21). Then stability is ensured if the sound speed is real, the density decreases with height, and the square of the buoyancy frequency is positive. While these stability conditions are well known, their derivation here for a compressible binary fluid using a Hamiltonian description is new. Identical stability requirements result from conditions analogous to (6.3) but for the derivatives with respect to ρ and χ . Those conditions with respect to η and χ

fail to capture all of the stability requirements. This result reflects the singularity of the 3×3 matrix.

7. Application to a moist atmosphere

In order to apply the preceding results to a moist atmosphere it is necessary to provide the equation of state and the definitions for the thermodynamic-state variables for entropy, internal energy, and enthalpy such that the fundamental thermodynamic relation (2.2) is satisfied. Assuming an ideal gas behavior, the equation of state for the total pressure p is

$$p = \rho R_a T \frac{(1 + r_v/\varepsilon)}{(1 + r)}, \quad (7.1)$$

where $\varepsilon = R_a/R_v = 0.622$ is the ratio of the molecular weight of water vapor to that of dry air, and R_a and R_v are the gas constants for dry air and water vapor. The mixing ratios of water vapor r_v and total water r are

$$r_v \equiv \frac{\rho_v}{\rho_a}, \quad (7.2a)$$

$$r \equiv \frac{\rho_w}{\rho_a}, \quad (7.2b)$$

where ρ_a , ρ_v , and ρ_w are the densities of the dry air, water vapor, and total water. The total density is related to the density of dry air by $\rho = \rho_a(1 + r)$. The vapor pressure e is

$$e = \rho_a r_v R_v T = \frac{p_a r_v}{\varepsilon} = \frac{p r_v / \varepsilon}{\left(1 + \frac{r_v}{\varepsilon}\right)}, \quad (7.3)$$

where the partial pressure due to dry air is $p_a = p - e$. Then the mixing ratio is $r_v = \varepsilon e / p_a$. In keeping with atmospheric conventions we replace the symbol η with s for atmospheric entropy and replace the symbol χ with r for the mixing ratio of total water. Because all the flow components move with the same velocity, there is no fallout and the total mixing ratio r satisfies a conservation statement of the form (2.1d). Following Lorenz (1979) and Emanuel (1994), the thermodynamic fields may be expressed in a general manner so that they are valid for unsaturated flow ($r = r_v$) without liquid water ($r_l \equiv r - r_v = 0$) and for saturated flow ($r_v = r - r_l$) with liquid water ($r_l > 0$).

The entropy s per unit mass of cloudy air is

$$(1+r)s = (s_{a0} + rs_{v0}) + (c_{pa} + rc_{pv}) \ln T - R_a \ln p_a - rR_v \ln e - r_l \frac{l_v}{T}. \quad (7.4)$$

The specific entropies for the dry air, water vapor, and liquid water are

$$s_a = s_{a0} + c_{pa} \ln T - R_a \ln p_a, \quad (7.5a)$$

$$s_v = s_{v0} + c_{pv} \ln T - R_v \ln e, \quad (7.5b)$$

$$s_l = s_{l0} + c_l \ln T, \quad (7.5c)$$

where s_{a0} , s_{v0} , and s_{l0} are reference values of the entropy of dry air, water vapor, and liquid water. Here, c_{pa} and c_{pv} are the specific heats at constant pressure for the dry air and water vapor, while c_l is the specific heat of the liquid water. We assume that these specific heats are constants independent of temperature but allow the enthalpy of vaporization l_v to satisfy Kirchhoff's relation $dl_v = -(c_l - c_{pv})dT$. In saturated conditions, the saturation vapor pressure e_{sat} is defined as a function of temperature by the Clausius–Clapeyron equation

$$\frac{d \ln e_{\text{sat}}}{dT} = \frac{l_v}{R_v T^2}, \quad (7.6)$$

such that $e_{\text{sat}} = 6.11$ mb at $T = 273.16$ K.

Similarly the internal energy is

$$(1+r)u = u_{a0} + ru_{v0} + (c_{va} + rc_{vv})T - r_l(l_v - R_v T), \quad (7.7)$$

where u_{a0} and u_{v0} are reference values of the internal energy:

$$u_{a0} \equiv u_a(T_{00}) - c_{va}T_{00}, \quad (7.8a)$$

$$u_{v0} \equiv u_v(T_{00}) - c_{vv}T_{00}, \quad (7.8b)$$

and T_{00} is a reference temperature. Then the enthalpy h per unit mass of cloudy air is

$$(1+r)h = u_{a0} + ru_{v0} + (c_{pa} + rc_{pv})T - r_l l_v. \quad (7.9)$$

Note that this relation defines the enthalpies of the dry air and water vapor as

$$h_a = u_{a0} + c_{pa}T \quad \text{and} \quad (7.10a)$$

$$h_v = u_{v0} + c_{pv}T. \quad (7.10b)$$

The thermodynamic identity (2.2) for the present system can be shown to have the form

$$du = Tds + \frac{p}{\rho^2}d\rho + \mu dr, \quad (7.11)$$

where the chemical potential for this system is $(1+r)^2\mu = \mu_v - \mu_a$ and the chemical potentials of the dry air and water vapor are

$$\mu_a = h_a - Ts_a, \quad (7.12a)$$

$$\mu_v = h_v - Ts_v. \quad (7.12b)$$

This result follows in a clear and easily interpreted fashion because the definitions for the entropy and internal energy retained the reference values of the fields (cf. Lorenz 1979). Having served their purpose, these reference values do not alter (2.1) or the Hamiltonian formulation and may be dropped in the subsequent analysis. [We note that Eq. (17) of Lorenz is valid for his problem where the differentials are material ones following a cloudy air parcel that conserves its total water, $dr = 0$.]

In order to utilize the expression for the available energy (5.15), it is necessary to express the internal energy in terms of the state variables ρ , s , and r . Formally this requirement is met by the preceding set of relations. The ideal gas laws may be used to eliminate the partial pressures from the definition of the entropy (7.4) to yield an equation for the temperature field as a function of ρ , s , and r . Then the internal energy (7.7) and total pressure (7.1) are also functions of ρ , s , and r . This relation for the pressure holds in the unsaturated case because $r_v = r$ and in the saturated case because $r_v = r_{\text{sat}}$, where

$$r_{\text{sat}} = \frac{\varepsilon e_{\text{sat}}(T)(1+r)}{\rho R_a T}, \quad (7.13)$$

and the saturation vapor pressure is only a function of temperature by the Clausius–Clapeyron Eq. (7.6). In order to partition the available energy into available elastic and potential energies, the enthalpy (7.9) must be expressed in terms of the pressure, entropy, and total mixing ratio. Again this is formally possible. Because (5.15) and (5.16) involve differences in the internal energy and enthalpy for the same values of $\chi = r$, the reference values u_{a0} and u_{v0} need not be specified and may be dropped from the subsequent analysis.

The small-amplitude expressions for the available energies (5.19) are more readily accessible as they only require expressions for the speed of sound and buoyancy frequency. It is a tedious exercise to demonstrate from the above relations that these are given by

$$c_0^2 = \gamma_m R_m T_0, \quad (7.14a)$$

$$N_0^2 = \frac{g}{\gamma_m} \frac{d \ln p_0}{dz} - g \frac{d \ln \rho_0}{dz}, \quad (7.14b)$$

for the unsaturated case. Here,

$$\gamma_m \equiv \frac{c_{pa} + r_{v0}c_{pv}}{c_{va} + r_{v0}c_{vv}}, \quad (7.15a)$$

$$R_m \equiv R_a + r_{v0}R_v. \quad (7.15b)$$

In the saturated case we find results that agree with Lalas and Einaudi (1974) and Durran and Klemp (1982).

We note that the formulation presented here for a moist atmosphere with water vapor and liquid water is readily modified for one with water vapor and solid water.

8. Application to the ocean

To apply the Hamiltonian description to the ocean we choose the fundamental thermodynamic state variables to be the density ρ , the entropy η , and the concentration is taken to be the salinity $\chi \equiv S$. The salinity S is the mass of sea salt per unit mass of seawater. In contrast to the atmospheric thermodynamics that may be expressed completely and analytically, the oceanic case is complicated by the reliance on tabular expressions for the state variables (e.g., Fofonoff 1962, 1985). The approach of Feistel (1993; see also Feistel and Hagen 1994, 1995) is adopted here because it presents a synthesis of earlier results that is internally mathematically consistent. In this approach, the equation of state for seawater and the Gibbs free energy G are given as functions of the measurable state variables of salinity, temperature, and pressure:

$$\rho = \rho(S, T, p) \quad \text{and} \quad (8.1a)$$

$$G = G(S, T, p). \quad (8.1b)$$

In addition the temperature is expressed as a function of the salinity, entropy, and pressure:

$$T = T(S, \eta, p). \quad (8.2)$$

These three relations are sufficient in principle to express the internal energy and enthalpy in forms suitable for use in the preceding results. Use of the temperature relation (8.2) in (8.1a) implies that the density is a function of salinity, entropy, and pressure. Formally, this may be inverted to yield an expression for the pressure as a function of density, entropy, and salinity. Similarly, (8.2) may be used in the Gibbs function (8.1b) to eliminate temperature. Thus, we find

$$p = p(\rho, \eta, S) \quad \text{and} \quad (8.3a)$$

$$G = G(p, \eta, S). \quad (8.3b)$$

Using (8.3) we find that the Gibbs free energy is also a function of density, entropy, and salinity:

$$G = G(\rho, \eta, S). \quad (8.4)$$

Again, using (8.3) in (8.2) we have

$$T = T(\rho, \eta, S). \quad (8.5)$$

Because the internal energy $u = G + T\eta - p/\rho$, the results (8.3)–(8.5) imply

$$u = u(\rho, \eta, S). \quad (8.6)$$

and the thermodynamic identity (2.2) becomes

$$du = Td\eta + \frac{p}{\rho^2}d\rho + \mu_{\text{sw}}dS, \quad (8.7)$$

where μ_{sw} is the chemical potential of seawater (i.e., the difference between the specific chemical potential of salt is seawater and that of water in seawater; Fofonoff 1962; Feistel 1993). Formally then, this idealized ocean system is Hamiltonian.

Lastly because the enthalpy $h = G + T\eta$, it may be expressed using (8.2) and (8.3) as $h = h(p, \eta, S)$. Thus, the available energy relations of section 5 are applicable to the ocean in both their finite- and small-amplitude forms.

9. Conclusions

The Hamiltonian description of a compressible, rotating flow has been extended to the case of a binary fluid. This generic description in section 3 is applicable to any fluid system satisfying the thermodynamic identity (2.2). This constraint is satisfied by an ocean idealized to contain a single solute, the salinity, and by a moist atmosphere idealized to contain water in gaseous and either liquid or solid form. [A moist atmosphere containing water vapor and liquid and solid water is not Hamiltonian in the present description because the thermodynamic identity is more general than that of (2.2).] The Casimir invariants of the generic binary fluid are given by (4.3). They reflect the conservation of entropy and concentration, the conservation of circulation on lines formed on the intersection of surfaces of constant entropy and surfaces of constant concentration, and the conservation of a generalized potential vorticity. In (4.3c) the thermodynamic scalar function λ may be the pressure, density, temperature, salinity, or water mixing ratio.

The result (5.15) for the available energy extends the result of Andrews (1981) to a binary system. Like that of Andrews, the available energy is a local, positive-definite definition of the nonkinetic contribution to the pseudoenergy. Following Andrews (1981, section 5) the present moist result is similarly related to the moist available potential energy of Lorenz (1979) as his dry result is to that of Lorenz (1955). The utility of the present result in an Eulerian framework is severely restricted because it is required that every value of entropy and concentration be present in the reference state. In a Lagrangian framework, one could use (5.15) to track the evolution and partitioning of the available and kinetic energies for individual fluid parcels. Such an application to moist parcel dynamics will be presented in another forum.

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APPENDIX

Derivatives of the Available Energy

This appendix presents some details of the evaluation of the second derivatives of the available energy per

unit volume A . It employs the thermodynamic relations (2.4), as well as Maxwell's relations (2.5), and the fact that

$$\frac{\partial p_0(\eta, \chi)}{\partial \eta} = \frac{dp_0}{dz} \frac{\partial Z}{\partial \eta}, \tag{A.1}$$

implies

$$\frac{\partial^2 p_0(\eta, \chi)}{\partial \eta^2} = \frac{\partial p_0(\eta, \chi)}{\partial \eta} \frac{\partial \rho_0}{\rho_0 \partial \eta} + \frac{dp_0}{dz} \frac{\partial^2 Z}{\partial \eta^2}. \tag{A.2}$$

Similar relations hold for differentiation with respect to χ . The second derivative with respect to η is one of the most involved calculations. An intermediate result is

$$\begin{aligned} \left(\frac{\partial^2 A}{\partial \eta^2}\right)_0 &= \frac{1}{\rho_0} \left[\frac{\partial \rho_0(\eta, \chi)}{\partial \eta} \right]_0 \\ &\times \left\{ \left[\frac{\partial p_0(\eta, \chi)}{\partial \eta} \right]_0 - 2 \left(\frac{\partial p}{\partial \eta}\right)_{\rho, \chi, 0} \right. \\ &\quad \left. - \left(\frac{\partial p}{\partial \rho}\right)_{\eta, \chi, 0} \left[\frac{\partial \rho_0(\eta, \chi)}{\partial \eta} \right]_0 \right\}. \tag{A.3} \end{aligned}$$

We note that, in general,

$$dp = \left(\frac{\partial p}{\partial \eta}\right)_{\rho, \chi} d\eta + \left(\frac{\partial p}{\partial \rho}\right)_{\eta, \chi} d\rho + \left(\frac{\partial p}{\partial \chi}\right)_{\rho, \eta} d\chi, \tag{A.4}$$

but for the base state $p_0 = p_0(\eta, \chi)$, so the composite derivatives (Callen 1985) are

$$\left(\frac{\partial p_0}{\partial \eta}\right)_0 = \left(\frac{\partial p}{\partial \eta}\right)_{\rho, \chi, 0} + \left(\frac{\partial p}{\partial \rho}\right)_{\eta, \chi, 0} \left(\frac{\partial \rho_0}{\partial \eta}\right)_0 \quad \text{and} \tag{A.5a}$$

$$\left(\frac{\partial p_0}{\partial \chi}\right)_0 = \left(\frac{\partial p}{\partial \chi}\right)_{\rho, \eta, 0} + \left(\frac{\partial p}{\partial \rho}\right)_{\eta, \chi, 0} \left(\frac{\partial \rho_0}{\partial \chi}\right)_0. \tag{A.5b}$$

Thus, (A.3) may be written as

$$\left(\frac{\partial^2 A}{\partial \eta^2}\right)_0 = -\frac{1}{\rho_0} \left(\frac{\partial \rho_0(\eta, \chi)}{\partial \eta}\right)_0 \left(\frac{\partial p}{\partial \eta}\right)_{\rho, \chi, 0}, \quad \text{or} \tag{A.6a}$$

$$\begin{aligned} \left(\frac{\partial^2 A}{\partial \eta^2}\right)_0 &= -\frac{1}{\rho_0} \left[\frac{\partial \rho_0(\eta, \chi)}{\partial \eta} \right]_0 \\ &\times \left\{ \left[\frac{\partial p_0(\eta, \chi)}{\partial \eta} \right]_0 - \left(\frac{\partial p}{\partial \rho}\right)_{\eta, \chi, 0} \left[\frac{\partial \rho_0(\eta, \chi)}{\partial \eta} \right]_0 \right\} \tag{A.6b} \end{aligned}$$

In summary, we find

$$A_{\rho\rho 0} = \frac{1}{\rho_0} \left(\frac{\partial p}{\partial \rho}\right)_{\eta, \chi, 0}, \tag{A.7a}$$

$$A_{\eta\eta 0} = -\frac{1}{\rho_0} \left(\frac{\partial \rho_0}{\partial \eta}\right)_0 \left(\frac{\partial p}{\partial \eta}\right)_{\rho, \chi, 0}, \tag{A.7b}$$

$$A_{\chi\chi 0} = -\frac{1}{\rho_0} \left(\frac{\partial \rho_0}{\partial \chi}\right)_0 \left(\frac{\partial p}{\partial \chi}\right)_{\rho, \eta, 0}, \quad \text{and} \tag{A.7c}$$

$$A_{\rho\eta 0} = \frac{1}{\rho_0} \left(\frac{\partial p}{\partial \eta}\right)_{\rho, \chi, 0}, \tag{A.8a}$$

$$A_{\rho\chi 0} = \frac{1}{\rho_0} \left(\frac{\partial p}{\partial \chi}\right)_{\rho, \eta, 0}, \tag{A.8b}$$

$$A_{\chi\eta 0} = -\frac{1}{\rho_0} \left(\frac{\partial p}{\partial \eta}\right)_{\rho, \chi, 0} \left(\frac{\partial \rho_0}{\partial \chi}\right)_0. \tag{A.8c}$$

Then we have

$$A_{\rho\rho 0} A_{\eta\eta 0} - A_{\rho\eta 0}^2 = \frac{g}{\rho_0} \left(\frac{\partial p}{\partial \eta}\right)_{\rho, \chi, 0} \frac{\partial Z}{\partial \eta}, \tag{A.9a}$$

$$A_{\rho\rho 0} A_{\chi\chi 0} - A_{\rho\chi 0}^2 = \frac{g}{\rho_0} \left(\frac{\partial p}{\partial \chi}\right)_{\rho, \eta, 0} \frac{\partial Z}{\partial \chi}, \tag{A.9b}$$

$$A_{\rho\rho 0} A_{\eta\chi 0} - A_{\rho\eta 0} A_{\rho\chi 0} = \frac{1}{\rho_0^2} \left(\frac{\partial p}{\partial \eta}\right)_{\rho, \chi, 0} \left(\frac{\partial \rho_0}{\partial \chi}\right)_0. \tag{A.9c}$$

Lastly, one can show that (6.2c) holds using these results and (A.5).

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