

NOTES AND CORRESPONDENCE

On Potential Vorticity Flux Vectors

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ABSTRACT

Dynamical, rather than kinematical, considerations indicate that a generalized potential vorticity in terms of the gradient of an arbitrary scalar function requires that the potential vorticity flux vector contain a contribution due to gravity and the pressure gradient force. It is shown that such a potential vorticity flux vector has a simpler definition in terms of the gradient of the kinetic energy rather than that of a Bernoulli function. This result is valid for multicomponent fluids. Flux vectors for a salty ocean and a moist atmosphere with hydrometeors are presented.

1. Introduction

Truesdell (1951, 1954) and Haynes and McIntyre (1987) present kinematic arguments to show that a generalized potential vorticity can be defined for the gradient of an arbitrary scalar function. This potential vorticity Q_λ is defined as the vector dot product of the absolute vorticity ω and the gradient of a scalar function, say, λ , divided by the total density ρ of the flow:

$$Q_\lambda \equiv \frac{\omega \cdot \nabla \lambda}{\rho}, \quad (1)$$

and its time evolution is governed by an equation of the form

$$\frac{\partial \rho Q_\lambda}{\partial t} + \nabla \cdot \mathbf{J}_\lambda = 0, \quad (2)$$

where \mathbf{J}_λ is the flux of potential vorticity. In this context, the choice of \mathbf{J}_λ is arbitrary to within the inclusion of the curl of an arbitrary vector function. In particular, the addition of the vector product of two arbitrary scalar functions (e.g., $\nabla f \times \nabla g$) will not change (2). However, Haynes and McIntyre (1987) show that a particular choice of \mathbf{J}_λ enables the existence of an impermeability condition

$$\frac{\partial \lambda}{\partial t} + \frac{\mathbf{J}_\lambda \cdot \nabla \lambda}{\rho Q_\lambda} = 0, \quad (3)$$

such that the flux of ρQ_λ across any constant λ surface vanishes. If \mathbf{J}_λ is required to satisfy (3) in addition to (2), then the arbitrariness of \mathbf{J}_λ is further restricted. For example, an addition of the form $\nabla \lambda \times \nabla g$ to \mathbf{J}_λ will not change (2) or (3). This arbitrariness allows some flexibility in the choice of \mathbf{J}_λ that can be invoked in specific situations to provide convenient formulations. Other desirable properties of the flux vector are that it contains a convective (sometimes called advective) component, that it satisfies a generalization of the uniqueness theorem of Bretherton and Schär (1993), and that it vanishes for a fluid at rest.

Schär (1993) has shown that one choice of potential vorticity flux may be expressed in terms of a Bernoulli function for a single-component flow when λ is taken to be the potential temperature θ . This result has been successfully utilized in a dry atmosphere (e.g., Schär and Durran 1997) and in a Boussinesq ocean (e.g., Marshall et al. 2001). An extension to handle multicomponent fluids (e.g., a salty ocean or a moist atmosphere with hydrometeors) is desirable.

It is the purpose of this note to show that dynamical considerations indicate that the specification of the generalized potential vorticity flux should include a contribution due to gravity and the pressure gradient force. Specifically the generalized flux vector

$$\mathbf{J}_\lambda = \rho Q_\lambda \mathbf{u} - \dot{\lambda} \boldsymbol{\omega} + \nabla \lambda \times \mathbf{F}_\lambda \quad (4)$$

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satisfies (2) and (3). Here, \mathbf{u} is the three-dimensional velocity, \mathbf{F}_λ is an arbitrary force field per unit mass, and the evolution equation for the scalar λ is

$$\frac{D\lambda}{Dt} = \dot{\lambda}, \quad (5)$$

where $\dot{\lambda}$ is the sum of the sources of λ , and D/Dt is the material derivative. We note that (4) has the same form as that of Haynes and McIntyre (1987) if λ is the potential temperature and \mathbf{F}_λ is the frictional force. The next section introduces an alternative definition of (4) with \mathbf{F}_λ representing the net force field per unit mass:

$$\mathbf{F}_\lambda = -\frac{1}{\rho}\nabla p - \nabla\Phi + \mathbf{F} \equiv \mathbf{F}_{\text{net}}. \quad (6)$$

Section 3 shows that, with this definition, \mathbf{J}_λ is related to the kinetic energy of the flow by

$$\mathbf{J}_\lambda = \nabla\lambda \times \left[\frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) \right] - \frac{\partial\lambda}{\partial t}\boldsymbol{\omega}, \quad (7)$$

where the quantity in parentheses is the specific kinetic energy. This result is closely related to the generalized Bernoulli theorem (Schär 1993) but has a broader range of validity. Section 4 compares the present analysis to that of Schär (1993). We show that the alternative flux vector

$$\mathbf{J}_\lambda^* = \nabla\lambda \times \left(\frac{\partial\mathbf{u}}{\partial t} + \nabla B \right) - \frac{\partial\lambda}{\partial t}\boldsymbol{\omega} \quad (8)$$

is equivalent to \mathbf{J}_λ in terms of its divergence. Here, B is the Bernoulli function. Sections 5 and 6 present the flux vectors for a salty ocean and a cloudy atmosphere that are consistent with (4) and (7).

2. Generalized potential vorticity and its flux vector

The momentum equation for three-dimensional, rotating, compressible flow of a single-velocity, multi-component fluid is

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi + \mathbf{F}, \quad (9)$$

where $\boldsymbol{\Omega}$ is the rotation rate, p is the pressure, Φ is the geopotential, and \mathbf{F} is the frictional force per unit mass. Following Pedlosky (1986), we take the curl of (9), use continuity in the form

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = 0 \quad (10)$$

to eliminate the velocity divergence, and combine the result with the gradient of the conservation equation (5) for the arbitrary scalar function λ . The result is the general potential vorticity equation

$$\begin{aligned} \frac{DQ_\lambda}{Dt} = & \frac{\boldsymbol{\omega}}{\rho} \cdot \nabla\lambda + \frac{\nabla\lambda}{\rho} \cdot \mathbf{B} \\ & + \frac{\nabla\lambda}{\rho} \cdot [\nabla \times (-\nabla\Phi + \mathbf{F})], \end{aligned} \quad (11)$$

where $\boldsymbol{\omega} = 2\boldsymbol{\Omega} + \nabla \times \mathbf{u}$ is the absolute vorticity. We retain the term associated with the curl of the gradient of the geopotential for reasons to be explained below. The baroclinity vector

$$\mathbf{B} \equiv \nabla \times \left(-\frac{1}{\rho}\nabla p \right) = \frac{1}{\rho^2}\nabla\rho \times \nabla p \quad (12)$$

is the curl of the pressure gradient force per unit mass. Following Obukhov (1963), the sources and sinks of potential vorticity may be written in terms of the divergence of a vector \mathbf{O} :

$$\frac{DQ_\lambda}{Dt} = \frac{1}{\rho}\nabla \cdot \mathbf{O}, \quad (13)$$

where

$$\mathbf{O} = \dot{\lambda}\boldsymbol{\omega} + \lambda\nabla \times (-\nabla\Phi + \mathbf{F}) + \lambda\mathbf{B}. \quad (14)$$

This result follows from the vector identity for the divergence of the product of a scalar and a vector and the fact that the divergence of a curl is identically zero. It is noted that there is no need for λ to be chosen so that its gradient annihilates the baroclinity vector \mathbf{B} when forming a generalized potential vorticity equation.

It is straightforward to show, following Haynes and McIntyre (1987), that (13) may be written in flux form as (2) with flux vector given by (4) such that the impermeability theorem (3) holds. In addition, it is readily shown that (4) shares the uniqueness property of Bretherton and Schär (1993) for a flux vector that is linear in the net force \mathbf{F}_{net} . With this choice of \mathbf{F}_λ , the flux vector \mathbf{J}_λ has the convenient property of vanishing for an atmosphere at rest. This desirable property motivated the retention of the gradient of the geopotential in (11) and (14).

3. Kinetic energy and the potential vorticity flux vector

The momentum equation (9) may be rewritten in the form

$$\frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) = -\boldsymbol{\omega} \times \mathbf{u} - \frac{1}{\rho}\nabla p - \nabla\Phi + \mathbf{F}. \quad (15)$$

Following Schär (1993), we take the vector cross product of the gradient of λ with this equation. Using the ‘‘bac-cab’’ rule for the vector triple product term yields

$$\begin{aligned} \nabla\lambda \times \left[\frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) \right] \\ = (\boldsymbol{\omega} \cdot \nabla\lambda)\mathbf{u} - (\mathbf{u} \cdot \nabla\lambda)\boldsymbol{\omega} \\ + \nabla\lambda \times \left(-\frac{1}{\rho}\nabla p - \nabla\Phi + \mathbf{F} \right). \end{aligned} \quad (16)$$

The first term on the right-hand side can be written in terms of the transport of potential vorticity and the second may be rewritten using the λ equation (5). One finds

$$\begin{aligned} \nabla\lambda \times \left[\frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) \right] - \frac{\partial\lambda}{\partial t}\boldsymbol{\omega} \\ = \rho Q_\lambda \mathbf{u} - \dot{\lambda}\boldsymbol{\omega} + \nabla\lambda \times \left(-\frac{1}{\rho}\nabla p - \nabla\Phi + \mathbf{F} \right). \end{aligned} \quad (17)$$

Then, because \mathbf{J}_λ is defined by (4), we have the result (7). In the steady state, (7) reduces to

$$\mathbf{J}_\lambda = \nabla\lambda \times \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right), \quad (18)$$

where the quantity in parentheses is the specific kinetic energy. Because the divergence of (18) is identically zero, the divergences of the convective and nonconvective components of the flux vector must cancel. Unlike the choice of Davies-Jones (2003), the flux vector (18) does not, in general, vanish in the steady state. However, like the vector of Davies-Jones, our choice (4) with (6) for arbitrary λ does not reduce to the pure convective potential vorticity flux vector in the absence of frictional forces, and thus it violates one of the stipulations of Bretherton and Schär (1993).

4. Comparison with Schär (1993)

Schär (1993) derives an alternative expression for the special case where the scalar function is the dry potential temperature, $\lambda = \theta$ and $\dot{\lambda} = \dot{\theta}$. Specifically, Schär chooses a flux vector of the form

$$\mathbf{J}_s = \rho Q_\theta \mathbf{u} - \dot{\theta}\boldsymbol{\omega} + \nabla\theta \times \mathbf{F}, \quad (19)$$

which satisfies (2) and (3) but does not contain the pressure gradient contribution. He shows that

$$\mathbf{J}_s = \nabla\theta \times \left(\frac{\partial\mathbf{u}}{\partial t} + \nabla B \right) - \frac{\partial\theta}{\partial t}\boldsymbol{\omega}, \quad (20)$$

where the Bernoulli function is

$$B \equiv \frac{1}{2}|\mathbf{u}|^2 + h + \Phi, \quad (21)$$

and h is the enthalpy. In his derivation he invokes the thermodynamic identity for a homogeneous fluid to express the pressure gradient in terms of gradients of ther-

modynamic state variables. The general form of this identity for a multicomponent fluid is

$$T\nabla s = \nabla h - \frac{1}{\rho}\nabla p - \sum_{i=1}^l \mu_i \nabla \chi_i, \quad (22)$$

where s is the entropy, and the summation is over the chemical potentials μ_i of the mixture with the concentrations χ_i . Thus, in contrast to (20), the present expression (7) for the flux vector is more general and is valid for multicomponent fluids. For example, λ can be the water mixing ratio of a moist atmosphere or the salinity of the ocean.

In addition, the present approach also has analogous results to those of Schär (1993) for the isentropic coordinate version. In particular his equations (26), (27), and (29) may be written as

$$\frac{\partial\mathbf{v}}{\partial t} + \mathbf{k} \times \mathbf{J}_\theta = -\nabla\left(\frac{1}{2}|\mathbf{v}|^2\right), \quad (23)$$

where

$$\mathbf{J}_\theta = \zeta_\theta \mathbf{v} + \mathbf{k} \times (-\nabla h - \nabla\Phi + \mathbf{F}) - \mathbf{k} \times \dot{\theta} \frac{\partial\mathbf{v}}{\partial\theta}, \quad (24)$$

or

$$\mathbf{J}_\theta = \mathbf{k} \times \left[\frac{\partial\mathbf{v}}{\partial t} + \nabla\left(\frac{1}{2}|\mathbf{v}|^2\right) \right], \quad (25)$$

respectively. Again the kinetic energy replaces the Bernoulli function in the formulation.

We note that, because of the arbitrariness of the flux vector, one may choose an alternative definition of the flux vector that enables it to be written using the Bernoulli function. Specifically, the choice

$$\begin{aligned} \mathbf{J}_\lambda^* &= \mathbf{J}_\lambda + \nabla\lambda \times (\nabla h + \nabla\Phi) \\ &= \nabla\lambda \times \left(\frac{\partial\mathbf{u}}{\partial t} + \nabla B \right) - \frac{\partial\lambda}{\partial t}\boldsymbol{\omega} \\ &= \rho Q_\lambda \mathbf{u} - \dot{\lambda}\boldsymbol{\omega} + \nabla\lambda \times \left(\nabla h - \frac{1}{\rho}\nabla p \right) \end{aligned} \quad (26)$$

satisfies (2) and (3) and may be more useful in certain circumstances. But (26) does not follow directly from the momentum equation and, in the case of no motion, there can be large do-nothing fluxes. Occam's razor suggests that the choice (7) is preferable.

5. Potential vorticity flux vector for a salty ocean

In the case of a single-velocity binary fluid like the ocean that contains dissolved sea salts, a material-conserved potential vorticity does not exist (Müller 1995) even in the absence of viscous, conductive, and diffusive properties because it is not possible to annihilate the baroclinity vector in the potential vorticity equation (11). However, the flux form (2) of the potential vorticity may always be constructed such that it also satisfies the

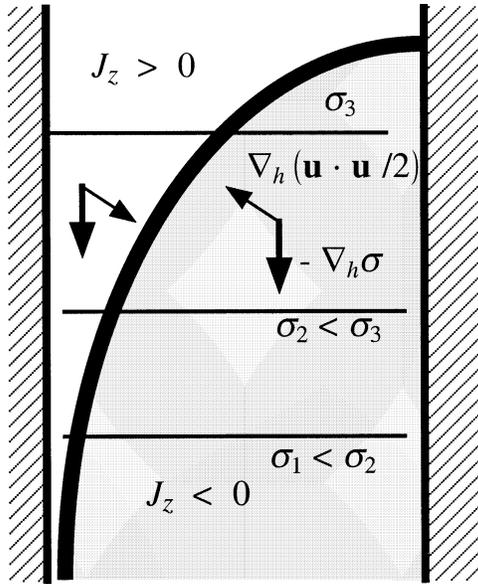


FIG. 1. Schematic illustration of the North Atlantic subtropical basin bounded by the continents (denoted by single hatching). Three surface contours of potential density σ are labeled. The heavy solid curve is the axis of maximum kinetic energy of the Gulf Stream that roughly separates the regions of downward potential vorticity flux ($J_z < 0$; shaded) from the regions of upward flux ($J_z > 0$; clear). Here, J_z is the vertical component of \mathbf{J}_σ such that $J_z = \mathbf{k} \cdot [-\nabla_h \sigma \times \nabla_h(\mathbf{u} \cdot \mathbf{u}/2)]$, where ∇_h is the horizontal gradient operator. The sketch assumes that cross-stream gradients in kinetic energy dominate the long-stream gradients.

impermeability condition (3). Then the current results are directly applicable to the ocean. As a specific example we take the scalar function to be the negative of the potential density $\lambda = -\sigma$ with source $\lambda = -\dot{\sigma}$. Then one finds

$$Q_\sigma = -\frac{\boldsymbol{\omega} \cdot \nabla \sigma}{\rho}, \quad \frac{D\sigma}{Dt} = \dot{\sigma}, \quad \text{and} \quad (27)$$

$$\mathbf{J}_\sigma = \rho Q_\sigma \mathbf{u} + \dot{\sigma} \boldsymbol{\omega} - \nabla \sigma \times \mathbf{F}_\sigma, \quad \text{or} \quad (28)$$

$$\mathbf{J}_\sigma = -\nabla \sigma \times \left[\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right) \right] + \frac{\partial \sigma}{\partial t} \boldsymbol{\omega}, \quad (29)$$

where

$$\mathbf{F}_\sigma = -\frac{1}{\rho} \nabla p - \nabla \Phi + \mathbf{F}. \quad (30)$$

Then potential vorticity diagnostics of the ocean circulation (cf. Marshall et al. 2001) may be more easily accomplished using the kinetic energy rather than the Bernoulli function. Figure 1 provides an illustrative example delineating the regions of potential vorticity flux through the sea surface in the steady state.

6. Potential vorticity flux vector for a cloudy atmosphere

Bannon (2002) derived equations suitable for multiphase, multivelocity, multitemperature cloudy air in

which the hydrometeors can move relative to the moist air and exert a hydrometeor drag on the flow. Following the approach of section 2, the potential vorticity equation for the moist air can be written as

$$\frac{\partial \rho_m Q_{\theta_v}}{\partial t} + \nabla \cdot \mathbf{J}_{\theta_v} = 0, \quad (31)$$

where the moist potential vorticity is

$$Q_{\theta_v} \equiv \frac{\boldsymbol{\omega} \cdot \nabla \theta_v}{\rho_m}, \quad (32)$$

and here,

$$\mathbf{F} = \frac{1}{\rho_m} \nabla \cdot \boldsymbol{\sigma} + \frac{\rho_a}{\rho_m} \dot{\mathbf{u}} \equiv \mathbf{F}_m \quad (33)$$

is the sum of the frictional force and the total momentum forcing due to the hydrometeors (see Bannon 2002 for details). The virtual potential temperature is

$$\theta_v \equiv T_v \left(\frac{p_{00}}{p} \right)^{R/c_{pa}} = \frac{p}{\rho_m R} \left(\frac{p_{00}}{p} \right)^{R/c_{pa}}, \quad (34)$$

where T_v is the virtual temperature, $p_{00} = 1000$ mb, and $\rho_m = \rho_a(1 + r_v)$ is the density of moist air. Here, r_v is the water vapor mixing ratio. The flux vector that is consistent with the general flux vector (4) is

$$\begin{aligned} \mathbf{J}_{\theta_v} &= \rho_m Q_{\theta_v} \mathbf{u} - \dot{\theta}_v \boldsymbol{\omega} + \nabla \theta_v \times \mathbf{F}_{\theta_v} \\ &= \nabla \theta_v \times \left[\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right) \right] - \frac{\partial \theta_v}{\partial t} \boldsymbol{\omega}, \end{aligned} \quad (35)$$

where the net force is

$$\mathbf{F}_{\theta_v} = -\frac{1}{\rho_m} \nabla p - \nabla \Phi + \mathbf{F}_m. \quad (36)$$

Note that the impermeability condition (3) holds for the moist potential vorticity.

Because the virtual potential temperature is only a function of the total pressure and the density of moist air, the baroclinic term vanishes identically in the vorticity equation (8.3) of Bannon (2002). Then the moist potential vorticity equation may also be written in terms of the flux vector:

$$\begin{aligned} \mathbf{J}'_{\theta_v} &= \mathbf{J}_{\theta_v} + \nabla \theta_v \times \left(\frac{1}{\rho_m} \nabla p + \nabla \Phi \right) \\ &= \rho_m Q_{\theta_v} \mathbf{u} - \dot{\theta}_v \boldsymbol{\omega} + \nabla \theta_v \times \mathbf{F}_m. \end{aligned} \quad (37)$$

This result is the extension of that of Haynes and McIntyre (1987) to a cloudy atmosphere. An analogous result for a cloudy anelastic atmosphere may be derived from the results in appendix B of Bannon (2002).

7. Conclusions

This analysis has explored the definition of the flux vector for a generalized potential vorticity function. The

results indicate that the general flux vector contains a contribution due to gravity and the pressure gradient force but that this contribution does not prohibit the utility of the conservation of the potential vorticity or the existence of an impermeability theorem. In addition, the analysis presents convenient expressions for the flux vector in terms of the kinetic energy of the flow rather than a Bernoulli function. The two versions of the theorem are physically equivalent for single-component flows, but the revised version using the kinetic energy easily allows for multicomponent fluids such as cloudy air and salty water.

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