Sensitivity of Perturbation Variance and Fluxes in Turbulent Jets to Changes in the Mean Jet

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ABSTRACT

Synoptic-scale eddy variance and fluxes of heat and momentum in midlatitude jets are sensitive to small changes in mean jet velocity, dissipation, and static stability. In this work the change in the jet producing the greatest increase in variance or flux is determined. Remarkably, a single jet structure change completely characterizes the sensitivity of a chosen quadratic statistical quantity to modification of the mean jet in the sense that an arbitrary change in the jet influences a chosen statistical quantity in proportion to the projection of the change on this single optimal structure. The method used extends previous work in which storm track statistics were obtained using a stochastic model of jet turbulence.

1. Introduction

Recognition of the role of nonnormality in the linear stability theory of physical systems led to the development of Generalized Stability Theory (GST; Farrell 1982, 1988; Farrell and Ioannou 1996a, hereafter FI96a). Compared to the more traditional method of modes, GST allows a far wider class of stability problems to be addressed including stability of aperiodic operators (Farrell and Ioannou 1996b, hereafter FI96b; Farrell and Ioannou 1999, hereafter FI99). In addition to addressing evolution of sure initial perturbations, GST also provides methods for calculating stability of uncertain and stochastic systems (Farrell and Ioannou 2002b) and statistically steady distributions of variance and fluxes in stochastically forced systems (Farrell and Ioannou 1993a,b), which led to a new mechanistic model of geophysical and laboratory turbulence (Farrell and Ioannou 1994, 1995, 1998; DelSole 1996, 2001b). This theory has been extensively verified in recent studies of storm track statistics (Whitaker and Sardeshmukh 1998; Zhang and Held 1999; DelSole 2001a).

In this work we extend this stochastic turbulence theory to study the sensitivity of storm track statistics to changes in mean jet structure and parameters. Such an extension of stability theory to address the sensitivity of statistically steady quadratic quantities is natural to the point of view taken in GST, which directs attention to transient growth supported by the nonnormality of the jet dynamics and therefore allows a physical theory to be constructed for the statistical steady state of a system characterized by robust energy transfer between the mean state and the perturbations, such as the turbulent meridional jet to be modeled. This study is motivated by the observation that relatively small changes in jet structure, as occur for example in association with El Niño, at times produce large changes in storm track statistics while at other times apparently similar alterations of jet structure result in modest changes in the statistics (Chang et al. 2002). We show that this observation can be understood from the fact that a single optimal structure determines the sensitivity of a quadratic statistical quantity to jet structure change.

The problem is framed as an optimization of the response to stochastic forcing of the dynamical operator associated with the jet, the optimization being done over structured changes in the operator. The term structured change refers to a change such as would result from variation of jet velocity as distinct from arbitrary changes in the operator which may have no direct physical interpretation. The quantity to be optimized is quadratic and may be a variance or a flux such as that of momentum or heat. The optimization can be generalized to target a particular region.

We first obtain the change of the covariance matrix resulting from a complete set of structured changes in

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the dynamical operator; then the sensitivity is found by optimizing over this set of structured operator changes. In a study related to the first of these objectives Penland and Sardeshmukh (1995) examined the sensitivity of eigenvalues and eigenvectors of dynamical operators and their associated covariance matrices to specified small operator changes.

2. Method for obtaining the sensitivity of variance and fluxes to structured operator changes

Consider a perturbation vector $\boldsymbol{\psi}$ governed by the equation

$$\frac{d\boldsymbol{\psi}}{dt} = \mathbf{A}\boldsymbol{\psi} + \mathbf{F}\boldsymbol{\eta}(t), \tag{1}$$

in which $\mathbf{A}$ is the time independent linear dynamical operator, $\boldsymbol{\eta}$ is a vector noise process white in time with uncorrelated components, and $\mathbf{F}$ is a structure matrix that determines the covariance matrix of the stochastic forcing $\mathbf{Q} = \mathbf{F}^\dagger\mathbf{F}$ (superscript † indicates Hermitian transpose). The white noise process is assumed to be Gaussian and is fully specified by its ensemble average statistics:

$$\langle \boldsymbol{\eta}(t) \rangle = 0, \qquad \langle \boldsymbol{\eta}_i(t)\boldsymbol{\eta}_j(t') \rangle = \delta_{ij}\delta(t - t'), \tag{2}$$

in which ensemble averages are denoted by $\langle \cdot \rangle$. If $\mathbf{A}$ is asymptotically stable, a statistical steady state is reached in which the ensemble average statistics are equal to the time averages.[1]

It was shown by Farrell and Ioannou (1993, 1995, 1996a) that with the operator $\mathbf{A}$ a linearization of the jet dynamics and with appropriate dissipation and forcing, (1) accurately models jet turbulence, a result that has been extensively verified (Whitaker and Sardeshmukh 1998; Zhang and Held 1999; DelSole 2001a). Equilibrium perturbation variance and fluxes are obtained in this turbulence model from the ensemble average correlation matrix $\mathbf{C} = \langle \boldsymbol{\psi}^\dagger\boldsymbol{\psi} \rangle$, which is in turn obtained from the Lyapunov equation (FI96a)

$$\mathbf{AC} + \mathbf{CA}^\dagger = -\mathbf{Q}. \tag{3}$$

In this way quadratic statistical quantities such as variance and fluxes are determined directly from the operator $\mathbf{A}$ and the forcing structure $\mathbf{Q}$. The operator $\mathbf{A}$ in physical applications is a function of the background state of the system including the jet velocity and the distribution of damping and static stability. The stability issue examined in this work is the sensitivity of quadratic statistical quantities to physically relevant changes in jet structure. Such physical changes in the jet are reflected in specific structured changes of the operator $\mathbf{A}$. As an example, we may wish to determine the unit norm change in background jet velocity resulting in the greatest increase in momentum flux convergence. In addition we may wish to target the optimization to concentrate on specific regions of the jet, for instance to maximize momentum flux into the jet.

Before addressing sensitivity it is necessary to choose a norm to measure operator magnitude and a linear form to measure the quadratic quantity being optimized. One convenient choice for operator norm is the Frobenius norm, which is the square root of the sum of the squares of the elements of $\mathbf{A}$:

$$\|\mathbf{A}\|_F^2 = \sum_i \sum_j A_{ij}^* A_{ij} = \mathrm{trace}(\mathbf{A}^\dagger\mathbf{A}). \tag{4}$$

In addition other more targeted norms may be introduced. For example, if we wish to investigate operator changes caused by changes in the mean wind then a norm on velocity change of the form

$$\|U\|^2 = \int (U^2 + U'^2)\, dy, \tag{5}$$

where U' is the shear, may be more appropriate. In operators obtained by discretization of continuous systems, such as occur in the formulation of problems in hydrodynamics, care must be taken that the optimal operator converges as resolution increases. This is not necessarily ensured as, for instance, the magnitude of the derivatives available for the optimum change in the mean jet may increase with increasing resolution. In such cases the choice of a norm such as (5) may be important in ensuring convergence as the continuous limit is approached.

The quadratic quantity being optimized is obtained from the statistical steady covariance matrix and is measured by a linear form $\mathcal{T}$ that maps the covariance matrix to the positive real scalars. For example, the operator change leading to the maximum increase in total eddy variance is found by choosing $\mathcal{T}$ to be the trace linear form.

Let the operator $\mathbf{A}$ be changed to $\mathbf{A}(\epsilon)$, where ϵ is the scalar magnitude of the perturbation. The covariance $\mathbf{C}(\epsilon)$ maintained at equilibrium by $\mathbf{A}(\epsilon)$ satisfies the Lyapunov equation

$$\mathbf{A}(\epsilon)\mathbf{C}(\epsilon) + \mathbf{C}(\epsilon)\mathbf{A}(\epsilon)^\dagger = -\mathbf{Q}. \tag{6}$$

It follows then that the covariance tendency

$$\mathbf{C}' = \left.\frac{d\mathbf{C}}{d\epsilon}\right|_{\epsilon=0} \tag{7}$$

can be determined by solving the equation

$$\mathbf{AC}' + \mathbf{C}'\mathbf{A}^\dagger = -(\mathbf{A}'\mathbf{C} + \mathbf{CA}'^\dagger), \tag{8}$$

in which

$$\mathbf{A}' = \left.\frac{d\mathbf{A}}{d\epsilon}\right|_{\epsilon=0} \tag{9}$$

is the mean operator tendency and the covariance $\mathbf{C}$ is obtained by solving the Lyapunov equation (3).

[1] For a discussion of this interpretation of the ensemble average see Farrell and Ioannou (2002a, 2003).

Assume that the total structured operator tendency $\mathbf{A}'$ is spanned by a basis of structured tendencies $\mathbf{A}'_i$; that is,

$$\mathbf{A}' = \sum_i \alpha_i \mathbf{A}'_i. \tag{10}$$

If $\mathbf{C}'_i$ is the covariance tendency induced by $\mathbf{A}'_i$, then the total covariance tendency is

$$\mathbf{C}' = \sum_i \alpha_i \mathbf{C}'_i. \tag{11}$$

We seek the total structured operator tendency $\mathbf{A}'$ that maximizes $\mathcal{T}\mathbf{C}'$. This operator tendency will be called the optimal operator tendency. In order to rationalize the maximization we must normalize the individual operator tendencies $\mathbf{A}'_i$. We seek the operator tendency, specified by the vector $\boldsymbol{\alpha}$ through (10), that maximizes the quotient:

$$Q = \frac{\mathcal{T}\mathbf{C}'}{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{W}\boldsymbol{\alpha}}. \tag{12}$$

The matrix $\mathbf{W}$ is the inner product that measures and normalizes the total operator tendency $\mathbf{A}'$. For example if the Frobenius inner product is chosen then the entries of the matrix $\mathbf{W}$ are

$$W_{ij} = (\mathbf{A}'_i, \mathbf{A}'_j) = \sum_a \sum_b A'^{*}_{i,ab} A'_{j,ab}. \tag{13}$$

Maximization of (12) maximizes the quadratic quantity tendency measured by $\mathcal{T}$ over combined operator tendencies (10) measured in the metric induced by $\mathbf{W}$. The optimum operator tendency is found by noting that using the column vector $\boldsymbol{\lambda}$ with elements

$$\boldsymbol{\lambda}_i = \mathcal{T}\mathbf{C}'_i, \tag{14}$$

the quotient (12) can be written as

$$Q = \frac{\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{\alpha}}{\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{W}\boldsymbol{\alpha}}. \tag{15}$$

The maximum of Q is obtained for

$$\boldsymbol{\alpha}_{\mathrm{opt}} = \frac{\mathbf{W}^{-1}\boldsymbol{\lambda}}{\sqrt{\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{W}^{-2}\boldsymbol{\lambda}}}. \tag{16}$$

This is the method for determining the normalized structured operator change leading to maximum increase in a specified quadratic quantity. This operator change will be called the optimal structure change.

It is remarkable that a single operator change fully characterizes any quadratic quantity tendency. If an arbitrary operator change is performed the quadratic tendency is immediately obtained by projecting the operator change on this single optimal structure change. In the next section this remarkable result will be examined further.

Having obtained the normed operator change leading to the maximum tendency in the chosen integrated quadratic quantity, as specified by the linear form $\mathcal{T}$, we can define a fractional optimal operator sensitivity measure:

$$s = \frac{d \ln E_{\mathrm{opt}}(0)}{d\epsilon} = \frac{\mathcal{T}\mathbf{C}'_{\mathrm{opt}}}{\mathcal{T}\mathbf{C}}, \tag{17}$$

where $E(\epsilon) = \mathcal{T}\mathbf{C}(\epsilon)$ and $\mathbf{C}'_{\mathrm{opt}}$ is the optimum covariance tendency.

3. Example: The operator tendency leading to the greatest variance increase in a barotropic jet

Consider a zonal jet with streamfunction perturbations $\boldsymbol{\psi}(t)e^{ikx}$, where x is the coordinate in the zonal direction and y, the coordinate in the meridional direction. The perturbation dynamics is governed by the barotropic vorticity equation

$$\frac{\partial \mathbf{D}^2\boldsymbol{\psi}}{\partial t} = -ik\mathbf{U}\mathbf{D}^2\boldsymbol{\psi} - ik(\beta - \mathbf{U}'')\boldsymbol{\psi} - r\mathbf{D}^2\boldsymbol{\psi} + \nu\mathbf{D}^4\boldsymbol{\psi} + \mathbf{F}\boldsymbol{\eta}(t), \tag{18}$$

where $\mathbf{U}$ is the diagonal matrix of mean flow, $\mathbf{U}''$ is its curvature, and the jet is excited stochastically by $\boldsymbol{\eta}(t)$ assumed to be Gaussian and delta correlated in time. The operators $\mathbf{D}^2$ and $\mathbf{D}^4$ in the above equation are discretizations of

$$\mathbf{D}^2 = \frac{d^2}{dy^2} - k^2, \qquad \mathbf{D}^4 \equiv \left(\frac{d^2}{dy^2} - k^2\right)^2. \tag{19}$$

The flow is confined to the channel $-1 \le y \le 1$, with boundary conditions of vanishing streamfunction perturbation at the channel walls. The constant damping rate is r, and the diffusion coefficient ν is taken nonzero in some examples in order to damp structures on the discretization scale. Distances are nondimensionalized by $L = 10^4$ km^{-1}, velocities by $U = 20$ m s^{-1}, and time by $T = L/U$; with these choices, a time unit corresponds to 1.8 days, $\beta = 8$ corresponds to the planetary vorticity gradient at 45° latitude, a linear damping coefficient of $r = 0.18$ corresponds to a damping rate of 1/10 day^{-1}, and ν with nondimensional value 10^{-4} damps perturbations of 100-km scale at the rate of 1/10 day^{-1}.

Consider the mean zonal jet:

$$U(y) = \mathrm{sech}^2(4y). \tag{20}$$

We seek the change in $U(y)$ resulting in the greatest increase in the mean energy density of the perturbation field:

$$E \propto \langle \boldsymbol{\psi}^{\dagger}\mathbf{M}\boldsymbol{\psi} \rangle = \mathrm{trace}(\mathbf{C}), \tag{21}$$

where $\mathbf{M}$ is the matrix representation of the discretized operator $-\mathbf{D}^2$ so that E is proportional to the kinetic energy of the perturbation field. In (21) $\boldsymbol{\psi}$ is the column vector of the streamfunction at the discretization levels and $\mathbf{C}$ is the steady-state covariance. For simplicity, the stochastic forcing is assumed to be distributed uniformly across the channel; that is, $\mathbf{F}$ in (18) is chosen to be the identity matrix.

The sensitivity of the perturbation energy to changes

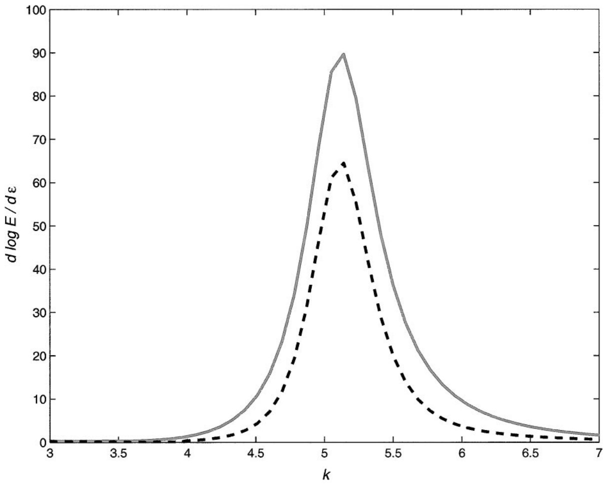

FIG. 1. For the barotropic jet $U(y) = \text{sech}^2(4y)$: sensitivity of eddy variance to the optimal structure for mean velocity change as a function of zonal wavenumber k of the forcing eddy field (continuous line). Also plotted is the sensitivity obtained for the equivalent normal system in which the variance is half the sum of the inverses of the decay rates of the mean operator modes (dashed line). The coefficient of linear damping is $r = 0.15$ and the diffusion coefficient is $\nu = 6.8 \times 10^{-4}$.

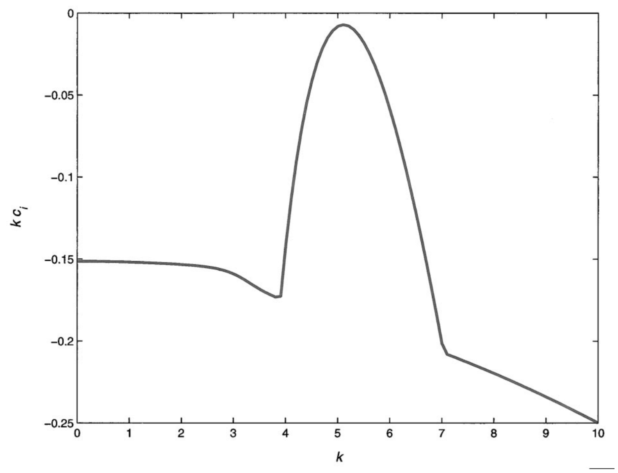

FIG. 2. For the barotropic jet $U(y) = \text{sech}^2(4y)$: growth rate of the least damped perturbation as a function of zonal wavenumber k of the eddy field. Other parameters as in Fig. 1.

in mean zonal velocity is determined by expanding the mean velocity change in the trigonometric basis:

$$\delta U = \sum_{j=1}^{n/2} \alpha_{2j-1} \cos\left[\frac{(2j-1)\pi y}{2}\right] + \alpha_{2j} \sin(j\pi y)$$
$$\equiv \sum_{i=1}^{n} \alpha_i U_i(y), \tag{22}$$

where n is the number of discretization levels; forming the perturbation operator associated with each discretized sinusoidal velocity perturbation $\mathbf{U}$

$$\mathbf{A}_i' = -ik(\mathbf{D}^2)^{-1}[\mathbf{U}\mathbf{D}^2 - \mathbf{U}_i''], \tag{23}$$

and using the Lyapunov equation (8) the covariance tendency $\mathbf{C}_i'$ produced by each of the elements of the basis $\mathbf{A}_i'$. The perturbation operator is normalized in the Frobenius norm (4); other reasonable choices of norm lead to qualitatively similar results. The sensitivity shown as a function of wavenumber in Fig. 1 is greatest for wavenumbers for which the spectrum of the stable mean operator $\mathbf{A}$ is closest to neutrality as may be verified by inspection of the (negative) growth rate of the least damped eigenfunction of the mean operator $\mathbf{A}$ shown in Fig. 2. This dependence is expected in normal systems; for example, in a one-dimensional system with growth rate $\lambda(\lambda < 0)$, the perturbation variance maintained by unit variance white noise is

$$E = -\frac{1}{2\lambda}, \tag{24}$$

and the sensitivity of the maintained variance to changes in the damping rate

$$\frac{d \ln E}{d\lambda} = -\frac{1}{\lambda} \tag{25}$$

is inversely proportional to the distance of the system from neutrality. In dimensions greater than one, for which nonnormality of $\mathbf{A}$ is possible, the sensitivity is maximized at the wavenumber for which the combined influence of distance from neutrality and nonnormality is maximum. The sensitivity obtained for the equivalent normal system can be used in order to quantify the importance of operator nonnormality. In a direct extension of the result for a one-dimensional system the variance maintained in higher-dimensional equivalent normal systems is half the sum of the inverses of the decay rates of the modes of the mean operator. This equivalent normal sensitivity is shown in Fig. 1 (dashed line). The sensitivity of the equivalent normal system underestimates the sensitivity of the nonnormal operator.

A velocity change produces greatest variance increase for stochastic forcing with zonal wavenumber $k = 5.14$. This mean flow modification is an increase in shear in the neighborhood of the inflection point of the absolute vorticity of the jet, that is, the latitude at which $U'' = \beta$, which is also the region of primary support of the stochastic optimal (as shown in Fig. 3). The stochastic optimal is the forcing structure $\mathbf{F}_{\text{SO}}$ that when excited uncorrelated in time maintains the greatest variance (cf. Farrell and Ioannou 1996a; Kleeman and Moore 1997). It can be inferred from (8) and the solution of the asymptotic Lyapunov equation that the greatest sensitivity is biased toward operator tendencies $\mathbf{A}'$ that lead the matrix $\mathbf{A}'\ \mathbf{C} + \mathbf{C}\mathbf{A}'^{\dagger}$ to project most strongly on the covariance of the stochastic optimal $\mathbf{Q}_{\text{SO}} = \mathbf{F}_{\text{SO}}\mathbf{F}_{\text{SO}}^{\dagger}$. It is natural then to expect optimal operator modification to be located near the region of the flow where the stochastic optimal is concentrated and this is verified, for example, for $k = 5.18$ (Fig. 3). This tendency for

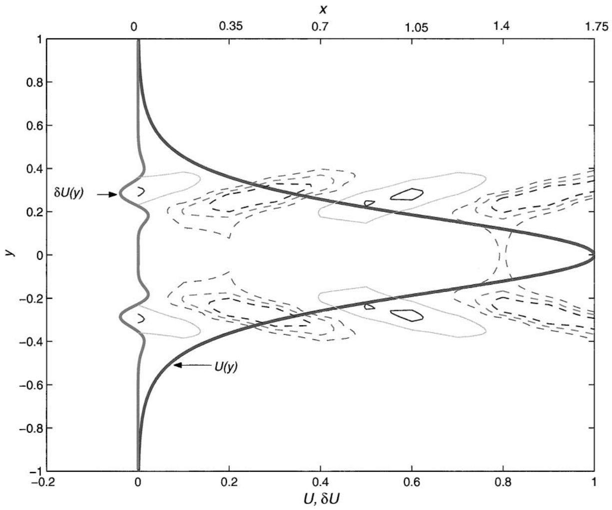

FIG. 3. Optimal structure $\delta U(y)$, which is the structure of the mean flow change resulting in maximum tendency in perturbation energy for the barotropic jet. Also shown is the background mean jet that is changed and a contour plot of the structure of the stochastic optimal associated with the mean jet (contour values arbitrary). The optimal structure change is centered in the flow region, where the stochastic optimal is concentrated. The eddy field has zonal wavenumber $k = 5.14$. The other parameters are as in Fig. 1.

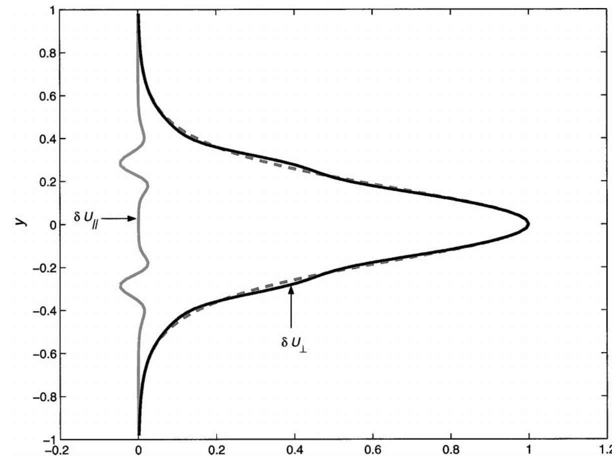

FIG. 4. Decomposition of the mean velocity profile of the barotropic jet $U(y)$ (dashed) into a velocity change in the direction of the optimal structure of velocity change $\delta U_{\parallel}$ and a velocity change perpendicular to the structure of the optimal $\delta U_{\perp}$. The inner product used for this orthogonal decomposition is the Frobenius. All the integrated variance tendency is produced by $\delta U_{\parallel}$ while the velocity change $\delta U_{\perp}$ produces 0 integrated variance tendency. As a result, while the optimal sensitivity is about 89, imposing a velocity perturbation proportional to the shape of the barotropic jet itself produces a sensitivity in total variance of only 3.5.

concentration of the optimal modification in the region of the stochastic optimal is commonly seen in examples and is useful for first-order understanding, however, this result is contingent on the choice of norm and on the structured changes of the jet that are allowed. Clearly if we allowed only changes in the region of the jet maximum the optimal modification would not be concentrated in the region of the first stochastic optimal.

The optimal structure of velocity change δU_{opt} shown in Fig. 3 characterizes fully the sensitivity of the flow in the sense that the tendency in total variance results only from the part of an arbitrary δU that projects on δU_{opt}, which we refer to as $\delta U_{\parallel}$. This single structure characterizes the quadratic tendency, and the orthogonal complement to δU_{opt}, which we denote $\delta U_{\perp}$, produces 0 total variance tendency. It is remarkable that if the total mean state dimension is n, the space of velocity changes leading to 0 total variance tendencies has dimension $n - 1$. That there is such a large subspace producing 0 tendency indicates the difficulty of obtaining large tendencies in quadratic statistical quantities by arbitrarily changing velocity profiles. Large quadratic quantity tendencies occur only where there is a substantial projection on the optimal structure. Consider for example a velocity change with the structure of the mean profile $U(y)$ itself. At the perturbation wavenumber $k = 5.14$ at which the maximum tendency occurs we find that this change leads to a tendency of only 3.5, which is far smaller than the tendency of 89 that occurs if the optimal structure velocity change is imposed. The decomposition of such a velocity change into its $\delta U_{\parallel}$ and $\delta U_{\perp}$ components is shown in Fig. 4. Only a small part of this velocity change projects on the optimal structure velocity change. Most of the velocity change is in the direction perpendicular to δU_{opt} and this velocity change makes zero contribution to the tendency in total variance.

The results obtained are converged both as the resolution is increased and also as the number of spectral coefficients a_i is increased. This convergence is facilitated by the choice of norm and the inclusion of diffusion to damp the largest wavenumbers.

Consider the sensitivity of the variance to the friction parameter now taken to be a function of y, $r(y)$. The change in friction parameter is expanded as

$$\delta r(y) = \sum_{j=1}^{n/2} \alpha_{2j-1} \cos\left[\frac{(2j-1)\pi y}{2}\right] + \alpha_{2j} \sin(j\pi y)$$

$$\equiv \sum_{i=1}^{n} \alpha_i r_i(y). \tag{26}$$

For each element of the basis, $r_i(y)$, form the diagonal discretized friction $\mathbf{R}_i$ and derivative $\mathbf{R}_i'$ and from these the modified operators

$$\mathbf{A}_i' = -(\mathbf{D}^2)^{-1}[R_i'\mathbf{D}_y + R_i\mathbf{D}^2], \tag{27}$$

with $\mathbf{D}_y$ the matrix y derivative. The covariance tendencies $\mathbf{C}_i'$ are then obtained as before. The operator tendencies are again normalized in the Frobenius norm. The sensitivity as a function of wavenumber k is shown in Fig. 5 has the same form as that obtained for velocity changes. The structure of the optimal change is shown in Fig. 6 for wavenumber $k = 5.14$.

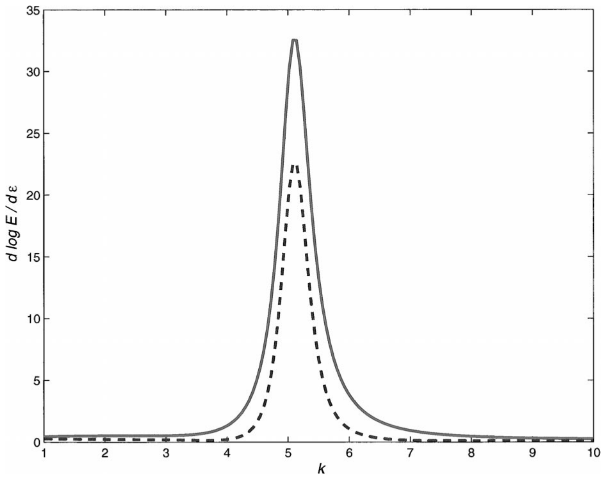

FIG. 5. For the barotropic jet: optimal sensitivity of eddy variance to change in linear damping as a function of zonal wavenumber k of the eddy field (continuous line). Also shown is the optimal sensitivity obtained in the equivalent normal system for which the variance is half the sum of the inverses of the decay rates of the modes of the mean operator (dashed line).

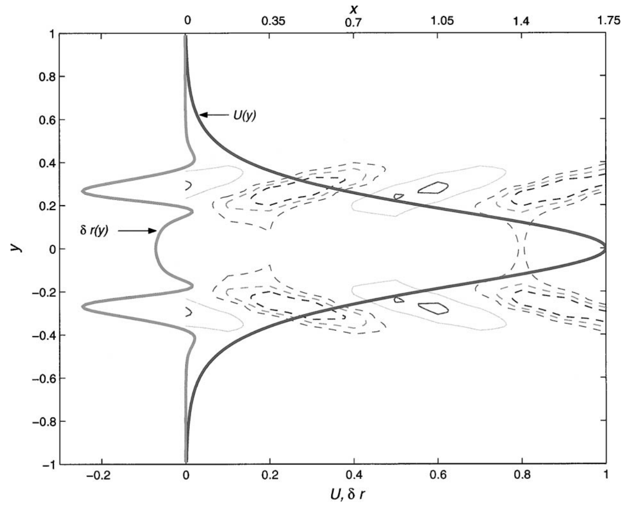

FIG. 6. For the barotropic jet: optimal structure of the linear damping change resulting in greatest tendency in perturbation energy. Also shown is the background jet and a contour plot of the structure of the stochastic optimal associated with the mean jet (contour values arbitrary). The optimal structure of the jet is centered in the flow region where the stochastic optimal is concentrated. The eddy field has zonal wavenumber $k = 5.14$. The other parameters are as in Fig. 1.

4. Example: The operator tendency leading to the greatest heat flux increase in a baroclinic jet

We wish to determine the sensitivity of heat flux to jet structure changes in a baroclinic model and choose the Green problem as an example. This model is Boussinesq with constant stratification on a β plane and has periodic boundary conditions in the zonal x direction; solid walls in the meridional y direction and a solid lid at height $z = H$. The zonal flow is $U(z) = z$. Horizontal scales are nondimensionalized by $L = 1000$ km; vertical scales by $H = fL/N = 10$ km; velocity by $U_0 = 30$ m s^{-1}, and time by $T = L/U_0$, so that a nondimensional time unit corresponds to approximately 9.25 h. Midlatitude values are chosen for the Brunt–Väisälä frequency, $N = 10^{-2}$ s^{-1}, and Coriolis parameter, $f = 10^{-4}$ s^{-1}. The nondimensional planetary vorticity gradient at 45° latitude is $\beta = 0.53$.

The nondimensional linearized equation governing streamfunction perturbations $\boldsymbol{\psi}$ collocated on discretization levels in z is

$$\frac{\partial \mathbf{D}^2\boldsymbol{\psi}}{\partial t} = -ik\mathbf{U}\mathbf{D}^2\boldsymbol{\psi} - ik\beta\boldsymbol{\psi} - r\mathbf{D}^2\boldsymbol{\psi} + \mathbf{F}\boldsymbol{\eta}(t), \qquad (28)$$

in which the perturbation is assumed to be of the form $\boldsymbol{\psi}(z, t)\, e^{ikx+ily}$, where k is the zonal and l the meridional wavenumber. The perturbation potential vorticity is $\mathbf{D}^2\boldsymbol{\psi}$, with $\mathbf{D}^2 \equiv \partial^2/\partial z^2 - k^2 - l^2$ and the perturbation potential vorticity damping rate is r, which is selected to have the value $r = 0.2$.

Conservation of potential temperature at the ground, $z = 0$, and tropopause, $z = 1$, provides the boundary conditions

$$\frac{\partial^2\boldsymbol{\psi}}{\partial t\partial z} = -ikU(0)\frac{\partial\boldsymbol{\psi}}{\partial z} + ikU'(0)\boldsymbol{\psi} - r\frac{\partial\boldsymbol{\psi}}{\partial z} + \Gamma_g(k^2 + l^2)\boldsymbol{\psi} \qquad \text{at } z = 0, \qquad (29)$$

$$\frac{\partial^2\boldsymbol{\psi}}{\partial t\partial z} = -ikU(1)\frac{\partial\boldsymbol{\psi}}{\partial z} + ikU'(1)\boldsymbol{\psi} - r\frac{\partial\boldsymbol{\psi}}{\partial z} \quad \text{at } z = 1, \qquad (30)$$

where $U'(0)$ and $U'(1)$ denote the velocity shear at $z = 0$ and $z = 1$, respectively. The coefficient of Ekman damping $\Gamma_g \equiv (N/U_0)\sqrt{\nu/2f}$ has value $\Gamma_g = 0.105$ corresponding to an effective vertical eddy momentum diffusion coefficient $\nu = 20$ m^2 s^{-1} in the boundary layer.

We seek the modification of the zonal jet producing the greatest increase in the time mean vertically integrated northward heat flux by the perturbation field. The heat flux is obtained from the ensemble mean covariance $\mathbf{C} = \langle\boldsymbol{\psi}\boldsymbol{\psi}^\dagger\rangle$:

$$H \propto \langle vT\rangle = \frac{k}{2}\,\text{trace}\{\text{Im}[\text{diag}(\mathbf{D}_z\mathbf{C})]\}, \qquad (31)$$

where diag is the matrix diagonal and $\mathbf{D}_z$ is the matrix representation of the vertical derivative. For simplicity the stochastic forcing is distributed uniformly.

Expanding the mean velocity change in the trigonometric basis,

$$\delta U = \sum_{j=1}^{n/2} \alpha_{2j-1}\cos j\pi z + \alpha_{2j}\sin j\pi z \equiv \sum_{i=1}^{n} \alpha_i U_i(z), \qquad (32)$$

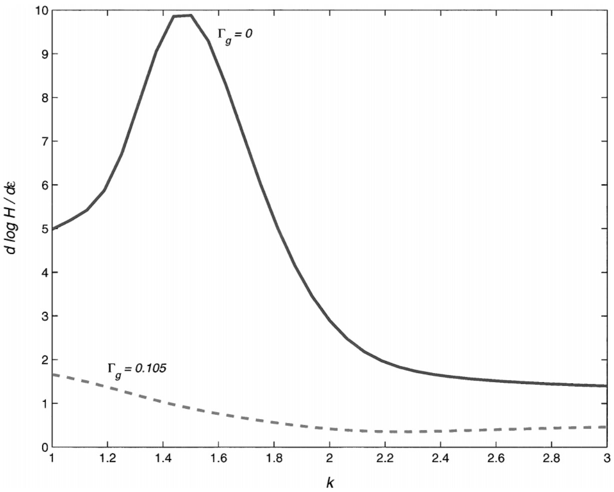

FIG. 7. For the baroclinic problem: optimal sensitivity of eddy heat flux to mean velocity change as a function of zonal wavenumber k of the eddy perturbation field in the case of no Ekman damping ($\Gamma_g = 0$) (continuous line). Optimal sensitivity obtained when Ekman damping is included at the lower boundary ($\Gamma_g = 0.105$) (dashed line). The meridional wavenumber of the eddies is $l = \pi/2$ and the coefficient of linear damping is $r = 0.2$.

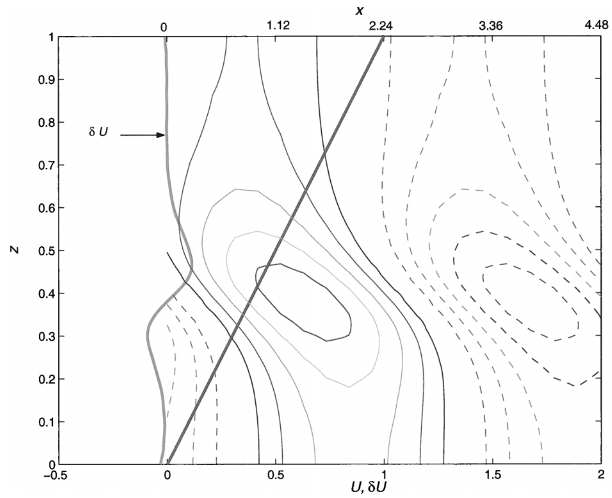

FIG. 9. Optimal structure leading to maximum tendency in northward heat flux. The straight line is the background mean jet that is being changed. Also shown is a contour plot of the structure of the stochastic optimal associated with the mean jet (contour values arbitrary). The most sensitive region for jet change is where the stochastic optimal is concentrated. The eddy field has zonal wavenumber $k = 1.4$ and there is no Ekman damping. The other parameters are as in Fig. 7.

and forming the perturbation operator associated with each sinusoidal velocity change $U_i(z)$,

$$\mathbf{A}'_i = -ik(\mathbf{D}^2)^{-1}(\mathbf{U}_i\mathbf{D}^2 - \mathbf{U}''_i), \qquad (33)$$

allows, using the Lyapunov equation (8), calculation of the covariance tendency $\mathbf{C}'_i$ produced by each of the elements of the basis, $\mathbf{A}'_i$ from which the associated heat flux tendency can be found using (31). The perturbation operator is normalized in the Frobenius norm (4); other reasonable choices of norm lead to qualitatively similar results.

The optimal sensitivity (17) for total heat flux tendency is shown in Fig. 7 as a function of the zonal wavenumber of the stochastic forcing. The heat flux sensitivity is greatest for wavenumbers for which the spectrum of the stable mean operator is closest to neutrality (cf. the growth rate of the least damped eigenfunction of the mean operator $\mathbf{A}$ is shown in Fig. 8).

The jet modification producing the greatest heat flux tendency for stochastic forcing with zonal wavenumber $k = 1.4$ is shown in Fig. 9, and in Fig. 10 for $k = 3$. In both cases, there is no Ekman damping and the sto-

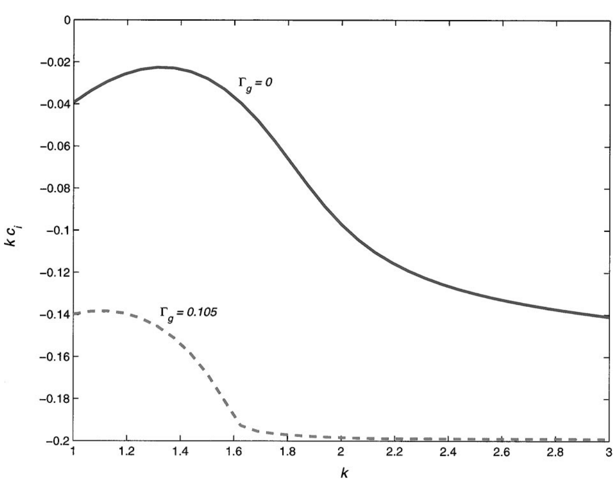

FIG. 8. For the baroclinic problem: growth rate of the least damped eigenfunction as a function of zonal wavenumber k. The meridional wavenumber of the eddies is $l = \pi/2$. The continuous curve is with no Ekman damping, and the dashed curve is with Ekman damping $\Gamma_g = 0.105$. The coefficient of linear damping is $r = 0.2$.

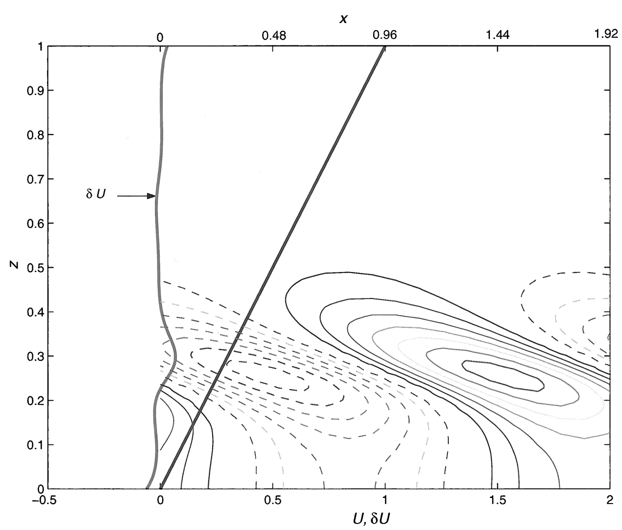

FIG. 10. As in Fig. 9, except that the zonal wavenumber of the eddy field is $k = 3$.

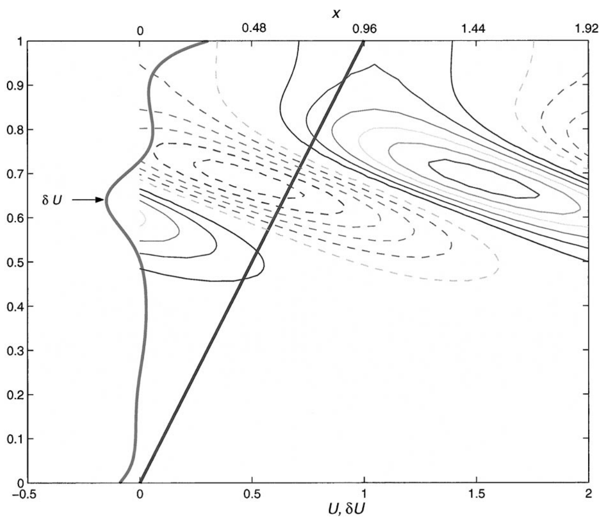

FIG. 11. As in Fig. 10, except that Ekman damping with $\Gamma_g = 0.105$ has been introduced at the lower boundary. Note that the optimal structure has moved to upper levels, where the support of the stochastic optimal is also concentrated.

chastic optimal is located in the lower troposphere. The stochastic optimal for $k = 1.4$, which is the wavenumber associated with the greatest tendency is located near the steering level of the least stable mode. The stochastic optimal for $k = 3$ for which the tendency is low is located in the lower troposphere at a location that would allow growth of the optimal perturbations as they propagate upward, in the manner described by Tung (1983). The presence of Ekman damping at the lower boundary results in the optimal change of the mean flow moving to upper levels where the stochastic optimal is located (Fig. 11).

While we have optimized heat flux tendency in this baroclinic jet example, similar jet modifications also produce maximum increase in perturbation energy tendency.

5. Conclusions

This work extends GST to obtain a method for calculating the sensitivity of quadratic quantities such as variance, energy, and fluxes of heat and momentum, to change in jet structure. The jet structure change could include velocity, dissipation, and other dynamical variables, and these jet structure changes, as well as region of the response optimization, can be localized in the jet. The unique jet structure change producing the greatest change in a chosen quadratic quantity also completely characterizes the sensitivity of the quadratic quantity to jet change in the sense that an arbitrary jet change increases the quadratic quantity in proportion to its projection on this optimal structure change. This result provides an explanation for observations that substantial differences in quadratic storm track quantities such as variance occur in response to apparently similar influences such as comparable SST changes and moreover provides a method for obtaining the optimal structural change. The mechanism underlying the optimal jet change can in many instances be understood to be facilitating growth of the stochastic optimal for the jet by increasing shear or decreasing damping in the region of primary support of the stochastic optimal.

It should be noted that initial jet changes resulting from an externally applied forcing produced, for example, by sea surface temperature changes, may induce yet further changes in the jet through modification of the divergence of heat and momentum fluxes and these secondary effects are observed to be substantial in some situations (Held et al. 1989). Complete description of the equilibrium jet configuration requires accounting self-consistently for these secondary changes in the jet produced by the induced fluxes of heat and momentum. Such a study coupling the mean jet dynamics and the statistical flux model has been carried out and steady, periodic and aperiodic self-consistent solutions were found in this nonlinear stochastic system (Farrell and Ioannou 2003).

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REFERENCES

Chang, E. K. M., S. Lee, and K. L. Swanson, 2002: Storm track dynamics. *J. Climate,* **15,** 2163–2183.
DelSole, T., 1996: Can quasigeostrophic turbulence be modeled stochastically? *J. Atmos. Sci.,* **53,** 1617–1633.
——, 2001a: A simple model for transient eddy momentum fluxes in the upper troposphere. *J. Atmos. Sci.,* **58,** 3019–3035.
——, 2001b: A theory for the forcing and dissipation in stochastic turbulence models. *J. Atmos. Sci.,* **58,** 3762–3775.
Farrell, B. F., 1982: The initial growth of disturbances in a baroclinic flow. *J. Atmos. Sci.,* **39,** 1663–1686.
——, 1988: Optimal excitation of neutral Rossby waves. *J. Atmos. Sci.,* **45,** 163–172.
——, and P. J. Ioannou, 1993a: Stochastic dynamics of baroclinic waves. *J. Atmos. Sci.,* **50,** 4044–4057.
——, and ——, 1993b: Stochastic forcing of the linearized Navier–Stokes equations. *Phys. Fluids,* **A5,** 2600–2609.
——, and ——, 1994: A theory for the statistical equilibrium energy and heat flux produced by transient baroclinic waves. *J. Atmos. Sci.,* **51,** 2685–2698.
——, and ——, 1995: Stochastic dynamics of the midlatitude atmospheric jet. *J. Atmos. Sci.,* **52,** 1642–1656.
——, and ——, 1996a: Generalized stability theory. Part I: Autonomous operators. *J. Atmos. Sci.,* **53,** 2025–2040.
——, and ——, 1996b: Generalized stability theory. Part II: Nonautonomous operators. *J. Atmos. Sci.,* **53,** 2041–2053.
——, and ——, 1998: Perturbation structure and spectra in turbulent channel flow. *Theor. Comput. Fluid Dyn.,* **11,** 215–227.
——, and ——, 1999: Perturbation growth and structure in time-dependent flows. *J. Atmos. Sci.,* **56,** 3622–3639.
——, and ——, 2002a: Perturbation growth and structure in uncertain flows. Part I. *J. Atmos. Sci.,* **59,** 2629–2646.
——, and ——, 2002b: Perturbation growth and structure in uncertain flows. Part II. *J. Atmos. Sci.,* **59,** 2647–2664.
——, and ——, 2003: Structural stability of turbulent jets. *J. Atmos. Sci.,* **60,** 2101–2118.

Held, I. M., S. W. Lyons, and S. Nigam, 1989: Transients and the extratropical response to El Niño. *J. Atmos. Sci.,* **46,** 163–174.

Kleeman, R., and A. M. Moore, 1997: A theory for the limitation of ENSO predictability due to stochastic atmospheric transients. *J. Atmos. Sci.,* **54,** 753–767.

Penland, C., and P. D. Sardeshmukh, 1995: Error and sensitivity analysis of geophysical eigensystems. *J. Climate,* **8,** 1988–1998.

Tung, K. K., 1983: Initial value problems for Rossby waves in a sheared flow with critical level. *J. Fluid Mech.,* **133,** 443–469.

Whitaker, J. S., and P. D. Sardeshmukh, 1998: A linear theory of extratropical synoptic eddy statistics. *J. Atmos. Sci.,* **55,** 237–258.

Zhang, Y., and I. Held, 1999: A linear stochastic model of a GCM's midlatitude storm tracks. *J. Atmos. Sci.,* **56,** 3416–3435.