

## Dynamics of Model Error: The Role of the Boundary Conditions

C. NICOLIS

*Institut Royal Météorologique de Belgique, Brussels, Belgium*

(Manuscript received 6 May 2005, in final form 13 March 2006)

### ABSTRACT

The different modes of the early stages of the response of a forecasting model to a small error in the boundary conditions are analyzed. A general formulation of the problem based on the use of Green's functions is developed and implemented on systems in which the operators acting on the spatial coordinates of the fields involved are diffusion-like and advection-like. It is shown that the generic behavior displays a nonanalytic structure not reducible to a power of time for short times.

### 1. Introduction

The fundamental laws governing the atmosphere and the climate are those of fluid dynamics in a rotating frame, of chemical kinetics and of thermodynamics. They are expressed in the form of a set of coupled nonlinear partial differential equations:

$$\frac{\partial \mathbf{X}(\mathbf{r}, t)}{\partial t} = \mathbf{F}\{\mathbf{X}(\mathbf{r}, t), [\nabla^k \mathbf{X}(\mathbf{r}, t)], \lambda\}, \quad \mathbf{r} \in \Omega. \quad (1)$$

Here  $\Omega$  is the spatial region of interest,  $\mathbf{X} = (X_1, \dots, X_r)$  is the set of the unknown fields,  $\mathbf{F} = (F_1, \dots, F_r)$  the corresponding evolution laws,  $\nabla^k$ ,  $k = 1, \dots, f$  is the  $k$ th power of the space derivative operator, and  $\lambda$  stands for a set of parameters. In most cases of interest these parameters are entering through phenomenological relations complementing (1) and providing information on quantities not expressible directly in terms of  $\mathbf{X}$  such as dissipation rates, the effect of unresolved scales, and so forth.

It is well known that for Eqs. (1) to constitute a well-posed problem it is necessary to prescribe appropriate initial and boundary conditions:

$$\mathbf{X}(\mathbf{r}, 0) = \mathbf{f}(\mathbf{r}), \quad (2a)$$

$$\mathbf{B}[\mathbf{X}, (\nabla^k \mathbf{X}), t] = 0, \quad \mathbf{r} \in \partial\Omega, \quad (2b)$$

where  $\partial\Omega$  denotes the boundary of  $\Omega$  and  $\mathbf{B} = (B_1, \dots, B_\ell)$  is a set of relations linking the variables and their spatial derivatives on the boundaries.

It is by now established that in most of the problems of interest Eqs. (1)–(2) generate complex solutions, notably in the form of spatiotemporal chaos and of a multiplicity of regimes for given parameters values (Nicolis and Nicolis 1987). Two properties of these solutions that have attracted considerable attention is their sensitivity to the initial conditions (Lorenz 1984) and to the parameters or to the way the unresolved scales are accounted for (special issue of *Quart. J. Roy. Meteor. Soc.*, 2000, Vol. 126, No. 567). Since these two kinds of data are subjected to irreducible uncertainties, such sensitivity properties have momentous consequences. In particular small errors arising from uncertainties in the parameters or in the treatment of unresolved scales, to which we will subsequently refer as “classical model errors” are bound to be amplified (Reynolds et al. 1994) thereby imposing limits in the skill of a forecasting model. There is currently a vivid discussion in the literature on the extent to which one can counteract this trend by applying appropriate extra forcing terms in the equations (Tribbia and Baumhefner 1988; Buizza et al. 1999; Palmer 2001; Chu et al. 2002) but the question is far from being fully settled.

Contrary to initial and to classical model errors, the dynamics of growth of errors in the boundary conditions and their effect on predictability have received much less attention. One of the main reasons is that most of the weather forecasting and general circulation models are global. Furthermore, they are expressed in a discrete form in space, notably by expanding the me-

*Corresponding author address:* Catherine Nicolis, Institut Royal Météorologique de Belgique, Avenue Circulaire 3, Brussels B-1180, Belgium.  
E-mail: cnicolis@oma.be

teorological and climatic fields in a basis of functions (modes) truncated to a resolution allowed by present-day computer technology. In this setting boundary conditions are in a sense losing their specificity, as they are incorporated in one way or the other in the resulting ordinary differential equations governing the evolution of the different modes. Put differently, by this procedure boundary error becomes mixed with classical model error.

The increasing demand for developing regional models (see, e.g., Kalnay 2003 and references therein) in which a high-resolution, limited-area model is embedded into a coarser, global one is bound to drastically change this state of affairs. To begin with, boundary conditions are now manifested in an “irreducible” manner, as they prescribe how the limited area of interest communicates with the external world in the region where it is merging with its global counterpart. Furthermore, they are subjected to a variety of errors arising, for instance, from the way these two systems are matched in the boundaries, from the frequency at which they are updated, or from the quality of resolution and parameterization performed in the global area. Interesting numerical experiments on the influence of lateral boundary conditions in limited-area models have been reported in the literature (see, e.g., Warner et al. 1997; Nutter et al. 2004). The objective of the present work is to propose a general formulation of the dynamics of boundary errors in a limited-area model and to identify some generic features. The analysis is carried out in the spirit of recent work on model error (Nicolis 2003, 2004, 2005). To identify those aspects that are specific to boundary errors it will be assumed that other sources of error such as errors due to the initial conditions or to the form of the model equations themselves are absent.

In section 2, a linearized set of equations obeyed by small boundary errors is deduced from Eqs. (1)–(2). Its formal solution, expressed in terms of a multivariable generalization of Green’s functions (Morse and Feshbach 1953), allows one to explicitly link the error in the boundary conditions to the error induced at the level of the variables  $\mathbf{X}$  themselves. Sections 3 and 4 are devoted to the implementation of this general formulation on two representative cases in which the spatial effects due to boundary error lead, successively, to diffusive and advective behavior. The main conclusions are summarized in section 5.

## 2. Formulation

Similar to most studies concerned with the sensitivity of the response to initial and model errors, we consider

a reference system described by Eqs. (1)–(2) assumed to adequately represent the phenomenon of interest, to which we refer from now on as “nature.” We assume that contrary to this (unknown) system, in the model that we actually have at our disposal the boundary conditions in (2b) are known only approximately. The model equations then have the following form:

$$\frac{\partial \mathbf{X}_M(\mathbf{r}, t)}{\partial t} = \mathbf{F}\{\mathbf{X}_M(\mathbf{r}, t), [\nabla^k \mathbf{X}_M(\mathbf{r}, t)], \lambda\}, \quad \mathbf{r} \in \Omega, \tag{3}$$

$$\mathbf{X}_M(\mathbf{r}, 0) = \mathbf{f}(\mathbf{r}), \tag{4a}$$

$$\mathbf{B}_M(\mathbf{X}, \nabla^k \mathbf{X}, t) = 0 \quad \mathbf{r} \in \partial\Omega, \tag{4b}$$

where the subscript  $M$  stands for the model variables and it is understood that  $\mathbf{B}_M$  differs (in norm) from  $\mathbf{B}$  in Eq. (2b) by a small quantity. In writing these equations we took the point of view developed in the introduction, namely, that all other sources of error with the exception of the boundary error are absent. Setting

$$\mathbf{u} = \mathbf{X}_M - \mathbf{X} \tag{5}$$

subtracting (1)–(2) from (3)–(4) and expanding to the first nontrivial order in  $\mathbf{u}$  and in  $\mathbf{b} = \mathbf{B}_M - \mathbf{B}$  one obtains

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{F}}{\partial \mathbf{X}} \cdot \mathbf{u} + \sum_{k=1}^f \frac{\partial \mathbf{F}}{\partial (\nabla^k \mathbf{X})} \cdot \nabla^k \mathbf{u}, \quad \mathbf{r} \in \Omega, \tag{6}$$

$$\mathbf{u}(\mathbf{r}, 0) = 0, \tag{7a}$$

$$\mathbf{b}(\mathbf{u}, \nabla^k \mathbf{u}) = \mathbf{b}_0(t) + \frac{\partial \mathbf{B}}{\partial \mathbf{X}} \cdot \mathbf{u} + \sum_{k=1}^f \frac{\partial \mathbf{B}}{\partial (\nabla^k \mathbf{X})} \cdot \nabla^k \mathbf{u} = 0 \tag{7b}$$

$$\mathbf{r} \in \partial\Omega,$$

where  $\mathbf{b}_0$  is a  $\mathbf{u}$ -independent term that generally depends explicitly on time.

The right-hand side of Eq. (6) displays two parts of a very different nature. The first, containing  $\mathbf{u}$  alone, is a “source” term accounting for processes of intrinsic origin occurring within  $\Omega$ . In the linearized approximation adopted it reduces to the Jacobian matrix of  $\mathbf{F}$  with respect to  $\mathbf{X}$  acting on the vector  $\mathbf{u}$ . Of crucial interest for our purposes is, however, the second part, containing the action of operators on the spatial dependence of  $\mathbf{u}$ , as it provides the mechanism of transfer of the boundary error to the interior of the system. We denote it as

$$\mathcal{L}_{\mathbf{r}} \cdot \mathbf{u} = \sum_{k=1}^f \frac{\partial \mathbf{F}}{\partial (\nabla^k \mathbf{X})} \cdot \nabla^k \mathbf{u} \tag{8}$$

and, for subsequent use, separate it from the first part. Furthermore, to avoid heavy notation we consider from now on the case of Dirichlet boundary conditions on  $\partial\Omega$  (the extension to the most general case being straightforward). Equations (6)–(7) can now be written in the compact form:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathcal{L}_{\mathbf{r}} \cdot \mathbf{u} = \frac{\partial \mathbf{F}}{\partial \mathbf{X}} \cdot \mathbf{u}, \quad \mathbf{r} \in \Omega, \quad (9)$$

$$\mathbf{u}(\mathbf{r}, 0) = 0, \quad (10a)$$

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{a}(\mathbf{r}, t), \quad \mathbf{r} \in \partial\Omega, \quad (10b)$$

where  $\mathbf{a}(t)$  is a prescribed function of time. Care should be taken to ensure that (10b) is compatible with the structure of the operator  $\mathcal{L}$ , particularly when this operator is purely hyperbolic.

The study of the short time behavior of Eqs. (9)–(10) is best carried out by converting the problem to an integral equation. We first illustrate the procedure in the case of a single unknown field  $u(\mathbf{r}, t)$  and an operator  $\mathcal{L}$  containing derivatives of even order only. This includes, in particular, the case where the evolution equations belong to the class of partial differential equations of the parabolic type. We define Green's function associated to the operator in the left-hand side of (9) by

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\mathbf{r}}\right)G(\mathbf{r}, t; \mathbf{r}', t') = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \quad \mathbf{r}, \mathbf{r}' \in \Omega, \quad (11)$$

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0 \quad t < t', \quad (12a)$$

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0 \quad \mathbf{r}, \mathbf{r}' \in \partial\Omega. \quad (12b)$$

This function describes the response to an impulsive point source placed on  $\mathbf{r}'$  at  $t = t'$ , under homogeneous initial and boundary conditions. It is subjected to the reciprocity condition:

$$G(\mathbf{r}, t; \mathbf{r}', t') = G(\mathbf{r}', -t'; \mathbf{r}, -t), \quad (13)$$

entailing that it also satisfies a modified form of Eq. (11) in which the time and space derivatives act on  $t'$  and  $\mathbf{r}'$  and the sign of the time derivative is reversed. The point is that Eqs. (11)–(12) are in principle solvable, contrary to the general problem of Eqs. (9)–(10), where the elements of the Jacobian matrix multiplying  $\mathbf{u}$  may have intricate space and time dependencies.

Multiplying Eq. (9) expressed in terms of  $t'$  and  $\mathbf{r}'$  by  $G$  and the aforementioned equation for  $G$  by  $u$ , subtracting and integrating over space  $\mathbf{r}'$  and over time  $t'$  from 0 to  $t^+ = t + \varepsilon$  ( $0 < \varepsilon \ll 1$ ) we obtain, keeping also in mind Eqs. (10a) and (12a):

$$u(\mathbf{r}, t) = \int_0^{t^+} dt' \int d\mathbf{r}' G(\mathbf{r}, t; \mathbf{r}', t') \left(\frac{\partial \mathbf{F}}{\partial \mathbf{X}}\right)_{\mathbf{r}', t'} u(\mathbf{r}', t') + \int_0^{t^+} dt' \int d\mathbf{r}' (G \mathcal{L}_{\mathbf{r}'} u - u \mathcal{L}_{\mathbf{r}'} G). \quad (14)$$

This equation constitutes our first important result. It expresses the response to a boundary error as a sum of two terms. The first is a bulk term accounting for the cumulative effect of all previous values of  $u$  all over space as propagated by  $G$ , on its present value at location  $\mathbf{r}$ . But the main point is that if, as stipulated above,  $\mathcal{L}$  contains only derivatives of an even order, the second term in Eq. (14) will reduce upon partial integrations to a boundary term explicitly displaying the boundary error  $a(t)$  [Eq. (10b)]. Furthermore, since the initial error vanishes one expects that this latter term will provide the dominant contribution to the response for short times. It should also be realized that the mere fact to integrate up to  $t^+$  in both terms does not guarantee that  $|u|$  displays a linear dependence on  $t$  for short times. Indeed, in many cases of interest  $G$  is a nonanalytic function of  $t$ , entailing that a Taylor series expansion in its argument diverges. Specific examples will be considered in section 3. The case of operators involving derivatives of odd order and, in particular, of hyperbolic operators is somewhat subtler and will be considered in section 4 on a prototypical example.

We close this section by outlining the generalization of Eq. (14) to an arbitrary number of unknown fields. To this end we write Eq. (9) in the abstract operator form

$$\boldsymbol{\phi} \cdot \mathbf{u} = \frac{\partial \mathbf{F}}{\partial \mathbf{X}} \cdot \mathbf{u}, \quad (15)$$

and introduce the *Green operator*  $\mathbf{G}$  through

$$\boldsymbol{\phi} \cdot \mathbf{G} = \mathbf{J}, \quad (16)$$

where  $\mathbf{J}$  is the idemfactor:

$$\mathbf{J} = \int d\mathbf{x} \mathbf{e}(\mathbf{x}) \mathbf{e}^+(\mathbf{x}). \quad (17)$$

Here  $\mathbf{x} = (\mathbf{r}, t)$ ,  $\{\mathbf{e}(\mathbf{x})\}$  are the eigenvectors of  $\boldsymbol{\phi}$  and  $\mathbf{e}^+(\mathbf{x})$  their adjoints. We notice that the components of  $\mathbf{J}$  along  $\mathbf{e}(\mathbf{r})$  are just delta functions  $\delta(\mathbf{x} - \mathbf{x}_0) = \delta(\mathbf{r} - \mathbf{r}_0) \delta(t - t_0)$ . One can now adapt in this abstract space the procedure leading to Eq. (14), to get the formal relation:

$$\mathbf{u} = \int d\mathbf{x}' \mathbf{G}(\mathbf{x}, \mathbf{x}') \left[ \left(\frac{\partial \mathbf{F}}{\partial \mathbf{X}}\right)_{\mathbf{x}'} \cdot \mathbf{u}(\mathbf{x}') \right] + \int d\mathbf{x}' [\mathbf{G} \cdot (\boldsymbol{\phi}_{\mathbf{x}'} \cdot \mathbf{u}) - \mathbf{u} \cdot (\boldsymbol{\phi}_{\mathbf{x}'} \cdot \mathbf{G})]. \quad (18)$$

As in Eq. (14), under certain conditions the second term can be transformed to a boundary term by introducing the adjoint operator  $\Phi^+$ , thereby allowing one to track the role of the boundary error  $\mathbf{a}(t)$  on the system's response  $\mathbf{u}$ .

**3. Spatiotemporal dynamics of boundary error: Diffusive behavior**

In this section we focus on the novel features underlying the dynamics of the boundary error as compared to classical model errors in the case where the operator  $\mathcal{L}$  contains solely spatial derivatives of an even order, as it happens with operators of the parabolic type.

Much of the novelty in question becomes already apparent in the limit where Eqs. (1)–(2) reduce to the diffusion equation in one dimension:

$$\begin{aligned} \frac{\partial X}{\partial t} &= D \frac{\partial^2 X}{\partial r^2} \quad 0 \leq r \leq \ell \\ X(r, 0) &= f(r) \\ X(0, t) = X(\ell, t) &= B_0, \end{aligned} \tag{19}$$

where  $B_0$  is a constant. The corresponding equation for the boundary error is [cf. Eqs. (6)–(7)]:

$$\begin{aligned} \frac{\partial u}{\partial t} &= D \frac{\partial^2 u}{\partial r^2} \quad 0 \leq r \leq \ell \\ u(r, 0) &= 0 \\ u(0, t) = u(\ell, t) &= a. \end{aligned} \tag{20}$$

In the notation of the second part of section 2 the operator  $\mathcal{L}_*$  reduces to  $D$  multiplying the second derivative, and the Jacobian  $\partial F/\partial X$  is identically zero. Therefore, to evaluate the linear response using Eq. (14) one merely needs Green's function of the diffusion operator—a classical problem of analysis (Morse and Feshbach 1953). To sort out the essential points as straightforwardly as possible we first limit ourselves to the short time behavior near one of the boundaries, say  $r = 0$ . The area of interest can then be assimilated to a semi-infinite system for which Green's function can be evaluated explicitly by Laplace transform or by the method of images, yielding

$$\begin{aligned} G(r, t; r', t') &= \sqrt{\frac{D}{2\pi(t-t')}} \left\{ \exp\left[-\frac{(r-r')^2}{4D(t-t')}\right] \right. \\ &\quad \left. - \exp\left[-\frac{(r+r')^2}{4D(t-t')}\right] \right\}. \end{aligned} \tag{21}$$

Substituting into Eq. (14) leads to

$$\begin{aligned} u(r, t) &= \frac{ar}{2\sqrt{\pi D}} \int_0^t dt' \frac{1}{(t-t')^{3/2}} \exp\left(-\frac{r^2}{4D(t-t')}\right) \\ &= a \operatorname{erfc}\left(\frac{r}{2\sqrt{Dt}}\right), \end{aligned} \tag{22}$$

where  $\operatorname{erfc}$  denotes the complementary error function.

The short time behavior of  $u(r, t)$  corresponds to the argument  $r/2\sqrt{Dt}$  going to infinity in Eq. (22), as long as  $r$  (which was assumed before to be small) is compatible with the inequality:

$$t \ll \frac{r^2}{4D}. \tag{23}$$

Utilizing the well-known asymptotic expansion of  $\operatorname{erfc}$  this leads to

$$u(r, t) \approx 2a \sqrt{\frac{D}{\pi}} \exp\left(-\frac{r^2}{4Dt}\right) \frac{t^{1/2}}{r}. \tag{24}$$

We observe a highly nonanalytic dependence of  $u$  on  $t$ , as anticipated in the comment following Eq. (14). This is in sharp contrast to the response characteristic of mean quadratic classical model error which, as well known, displays a  $\sim t^2$  dependence for short times (Nicolis 2003, 2004). In view of the general structure of Eqs. (14) and (18) and of the initial condition of vanishing error, one is entitled to assert that this behavior is generic, encompassing the entire class of systems in which (i) the operator  $\mathcal{L}$  contains only even space derivatives (or more generally is parabolic), and (ii) the dependence of the evolution law  $F$  on  $X$  is smooth.

In the remaining part of this section we outline some extensions of the result of Eq. (24) corroborating these assertions.

*a. Finite system, constant boundary conditions*

Green's function associated to Eq. (20) is now best constructed by eigenfunction expansion, leading to

$$G(r, t; r', t') = \sum_{n \text{ odd}} \exp\left(-\frac{n^2 \pi^2 Dt}{\ell^2}\right) \sin \frac{n\pi r}{\ell} \sin \frac{n\pi r'}{\ell}, \tag{25}$$

where  $\ell$  is the system's size. Equation (14) becomes

$$\begin{aligned} u(r, t) &= a \left\{ 1 - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \exp\left[-\frac{(2n-1)^2 \pi^2 Dt}{\ell^2}\right] \right. \\ &\quad \left. \sin \frac{(2n-1)\pi r}{\ell} \right\}. \end{aligned} \tag{26}$$

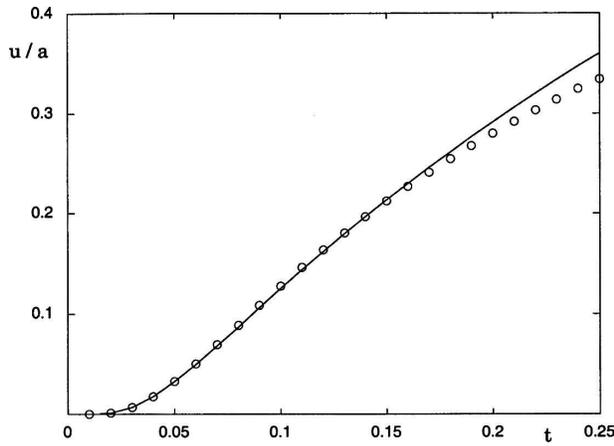


FIG. 1. Short time error evolution normalized by a constant error at the boundary as obtained numerically from the diffusion equation (circles) and analytically [Eq. (27)] (full line). Parameter values are  $D = 0.1$ ,  $r = 0.2$ ,  $\ell = 1$ , and  $a = 0.01$ .

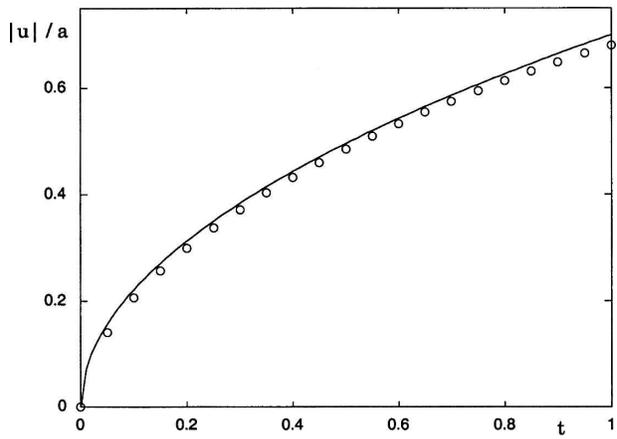


FIG. 2. Same as in Fig. 1 but for the global error [Eq. (29)].

$$|u|_t \approx 4a \left( \frac{Dt}{\pi} \right)^{1/2}. \tag{29}$$

This expression illustrates why a straightforward expansion in powers of  $t$  is not adequate to determine the short time behavior of the response  $u(r, t)$ : all coefficients of the resulting expansion would diverge, and truncating the series to a (finite) maximum value of  $n$  would lead to a result that is both resolution dependent and lacks any convergence property. To address the short time behavior in a consistent manner one may resort to the Euler–MacLaurin summation formula (Abramowitz and Stegun 1972) for the infinite series in  $n$ . Alternatively, one may switch to Laplace space and use Tauberian-type theorems (Widder 1941). One then obtains the following dominant contribution for the short time behavior:

$$u(r, t) = 2a \left( \frac{Dt}{\pi} \right)^{1/2} \left\{ \frac{1}{r} \exp \left( - \frac{r^2}{4Dt} \right) + \frac{1}{\ell - r} \exp \left[ - \frac{(\ell - r)^2}{4Dt} \right] \right\}, \tag{27}$$

provided that  $r$  is not too close to the boundaries, in the sense of relation (23). As expected the response is minimum at the farthest point from the boundaries  $r = \ell/2$ , and increases as the boundaries are approached.

One may also compute the global error from Eq. (26):

$$|u|_t = \frac{1}{\ell} \int_0^\ell dr u(r, t). \tag{28}$$

Again using the Euler–MacLaurin formula or switching to the Laplace space one obtains to the dominant order:

Figure 1 depicts the analytic relation of Eq. (27) versus time for  $r = 0.2$ , in perfect agreement with the numerically fully solved Eq. (20) for  $a = 0.01$ ,  $\ell = 1$ , and  $D = 0.1$ . The agreement extends to the global error as well, as seen in Fig. 2. It is noteworthy that the evolution in Fig. 1 goes through an inflection point (around which the behavior is quasi-linear), signaling the transition from a stage of slow growth to a more rapid one preceding saturation. This bears some resemblance with the behavior of error associated with sensitivity to initial conditions.

*b. Higher-order dissipation*

In many numerical forecasting models it is customary to introduce dissipative terms beyond the second-order space derivative, to account for the effect of short scales and to enhance numerical stability. To see the effect of such terms on boundary error we consider the extended form of Eq. (20):

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= -K \frac{\partial^4 u}{\partial r^4} & 0 \leq r \leq \ell \\ u(r, 0) &= 0 \\ u(0, t) = u(\ell, t) &= a, & \frac{\partial^2 u}{\partial r^2}(0, t) = \frac{\partial^2 u}{\partial r^2}(\ell, t) = 0 \end{aligned} \right\}, \tag{30}$$

where  $K > 0$  and the minus sign in the right-hand side guarantees the dissipative character and the stability of the process.

Owing to the boundary conditions, Green’s function has the same spatial structure as in Eq. (25), but differs in its time dependence in the replacement of the second

powers by fourth ones in the exponent. Using the Euler–MacLaurin summation formula and proceeding as in section 3a one arrives at an expression involving the incomplete gamma function. An expansion for short times leads to the dominant order to the following formula extending Eq. (29):

$$|u|_t \approx \frac{a}{\pi} \Gamma\left(\frac{3}{4}\right) (Kt)^{1/4}. \quad (31)$$

As can be seen, the main difference with the behavior corresponding to ordinary diffusion is the replacement of the square root dependence on  $t$  by a weaker one, entailing a faster take off of the error as the system is removed from its initial state.

*c. Linear source term*

We next consider an example closer to the general setting of Eqs. (14) or (18), in that the evolution law  $F$  has a nontrivial dependence on  $X$  itself. In terms of the boundary error, a minimal model of this instance is

$$\begin{aligned} \frac{\partial u}{\partial t} &= -ku + D \frac{\partial^2 u}{\partial r^2} \\ u(r, 0) &= 0 \\ u(0, t) = u(\ell, t) &= a, \end{aligned} \quad (32)$$

where  $k$  is a positive constant.

The solution of Eq. (32) is given by (Crank 1956)

$$u(r, t) = \int_0^t dt' \bar{u}(r, t') e^{-kt'} + \bar{u}(r, t) e^{-kt}, \quad (33)$$

where  $\bar{u}(r, t)$  is given by Eq. (26). This expression can be evaluated exactly in the short time regime, utilizing the appropriate asymptotic expansions as in the sections 3a, b. The result reduces to

$$\begin{aligned} u(r, t) \approx & a e^{-kt} \left[ \sqrt{\frac{\beta t}{\beta - kt^2}} \exp\left(-\frac{\beta}{t}\right) \right. \\ & \left. + \sqrt{\frac{\alpha t}{\alpha - kt^2}} \exp\left(-\frac{\alpha}{t}\right) \right], \end{aligned} \quad (34)$$

where we set, for compactness,  $\beta = (\ell - r)^2/4D$ ,  $\alpha = r^2/4D$ .

Upon expanding further  $e^{-kt}$  and the denominators in the bracket in powers of  $t$  one immediately sees that the overall effect of the linear source term is a higher order contribution starting as  $kt^{3/2} e^{-1/t}$ :

$$\begin{aligned} u(r, t) \approx & \text{dominant term coming from the diffusion part} \\ & + O(kt^{3/2} e^{-1/t}). \end{aligned} \quad (35)$$

Figure 3 illustrates the short time behavior of  $u$  as obtained from numerical simulations with a value of

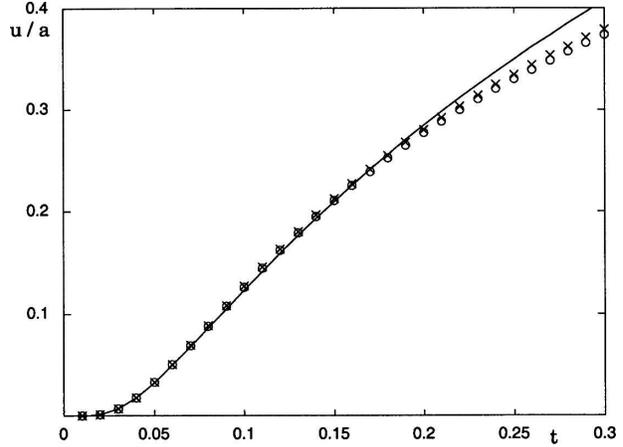


FIG. 3. Same as in Fig. 1 but in the presence of a linear source term in the diffusion equation [Eq. (32)] with  $k = 0.1$  (circles) and  $k = 0$  (crosses). The full line stands for the analytical prediction [Eq. (34)]. The other parameters are the same as in Fig. 1.

$k = 0.1$  (circles) equal to that of the diffusion coefficient. For comparison the pure diffusion case  $k = 0$  (crosses) and the analytical result, Eq. (34), (full line) are also depicted. One observes that the presence of the linear source term plays practically no role during a substantial time interval, in accordance with the theoretical predictions. This provides a further justification of the general conclusion drawn in section 2 based on the observation that source terms as opposed to terms coming from boundary terms build up more slowly, as they are initially zero. As expected, the deviations from the purely diffusive behavior turn out to be more marked as the value of  $k$  is increased with respect to the diffusion coefficient (not shown). For instance, for  $k = 0.2$  the behaviors remain indistinguishable during a time interval up to about 0.12 time units (to be compared with the time of about 0.17 units in Fig. 3). They subsequently deviate to reach a difference of about 0.01 at time 0.3 (cf. the difference of about 0.005 at time 0.3 in Fig. 3). As for the theoretical curve, it agrees remarkably well with the numerical result for a time interval comparable to Fig. 3, and subsequently deviates slightly to values below the numerical ones.

*d. Time-dependent boundary conditions*

Again we consider Eq. (20) in a semi-infinite space, but now allow the value of  $a$  at the left boundary  $r = 0$  to be time dependent. Equation (22) is then replaced by the following relation [cf. also Eq. (14)]:

$$u(r, t) = \sqrt{\frac{\alpha}{\pi}} \int_0^t dt' \frac{1}{(t-t')^{3/2}} \exp\left[-\frac{\alpha}{(t-t')}\right] a(t'). \quad (36)$$

It is instructive to derive the explicit form of (36) in the following two representative cases.

- The boundary error increases slowly in time (a case usually referred as “ramp”)

$$a(t) = a_0 + a_1 t, \quad a_1 \ll 1. \quad (37)$$

In this case one derives the exact result for short times

$$u(r, t) = \frac{a_0 + a_1 t}{\sqrt{\alpha}} t^{1/2} \exp\left(-\frac{\alpha}{t}\right). \quad (38)$$

- The boundary error is a square pulse of duration equal to  $\Delta$ ,

$$a(t) = a\theta(\Delta - t), \quad (39)$$

where  $\theta(\Delta - t)$  is the Heaviside function. In the short time regime one then obtains the following:

$$u(r, t) \approx \frac{a}{\sqrt{\alpha}} t^{1/2} \exp\left(-\frac{\alpha}{t}\right) \quad \text{for } t < \Delta, \quad (40a)$$

$$u(r, t) \approx \frac{a}{\sqrt{\alpha}} \left[ t^{1/2} \exp\left(-\frac{\alpha}{t}\right) - (t - \Delta)^{1/2} \exp\left(-\frac{\alpha}{t - \Delta}\right) \right] \quad \text{for } t > \Delta. \quad (40b)$$

Both results, Eqs. (38) and (40) clearly exhibit the singular dependence on  $t$ . Figures 4a,b depict the time evolution of the response to the square pulse for the full problem [Eq. (36)] with a pulse given by (39) and for the asymptotic result (40) for two different  $\Delta$  values, setting  $r = 0.1$  (to mimic the semi-infinite system of the analytic calculations) and the other parameters as in Fig. 1. As can be seen the response goes through a maximum at a time slightly larger than  $\Delta$  and subsequently decays slowly to zero, revealing a significant “aftereffect” extending well beyond the range of the pulse. Figure 4c provides the spatial distribution of the error for different times found numerically from the full diffusion equation submitted to an error assimilated to a pulse of an amplitude  $a = 0.01$  and duration  $\Delta = 0.1$  time units, and for a system of size  $\ell = 1$ . As time grows the error invades progressively the interior of the system, to reach a value of about  $10^{-5}$  in the middle at a time equal to the duration of the pulse (not shown) and a level of about  $10^{-3}$  at time  $t = 0.5$ .

We close by deriving an expression for the mean-square error when  $a(t)$  is a random process. We first consider the case of a white noise:

$$\langle \mathbf{a}(t) \rangle = 0, \quad \langle \mathbf{a}(t) \mathbf{a}(t') \rangle = Q^2 \delta(t - t'), \quad (41)$$

where the angle brackets denote the average over different realizations of the noise.

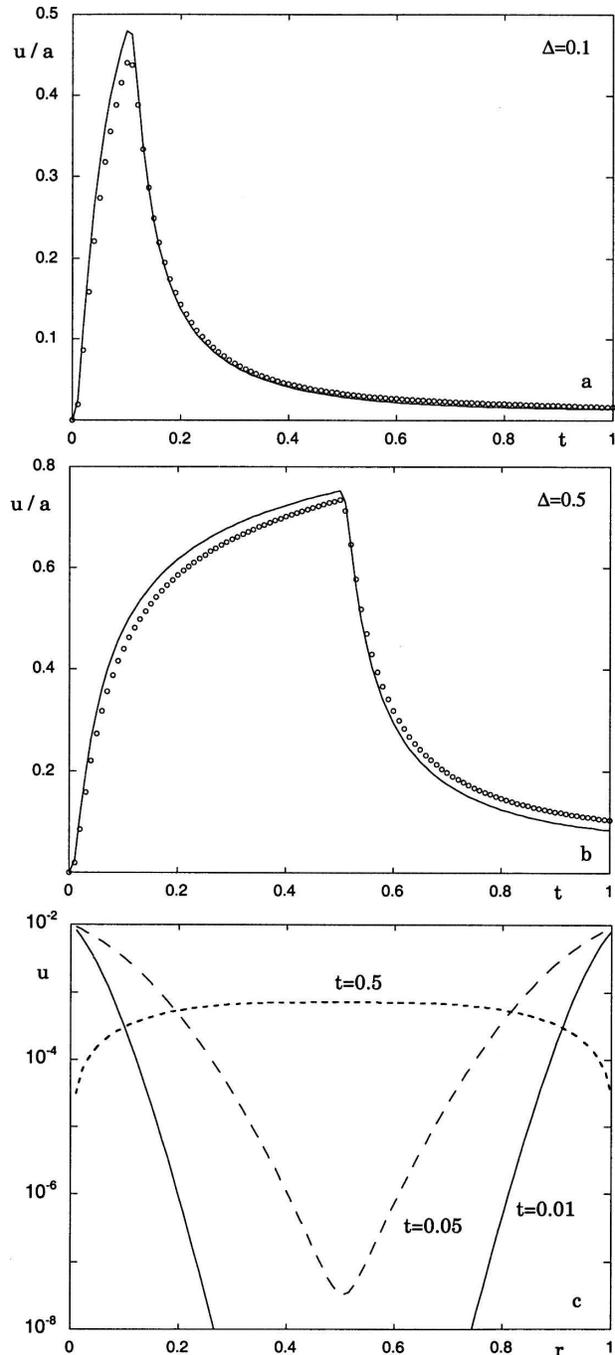


FIG. 4. (a), (b) Time evolution of the response of the diffusion equation to an error of the form of a square pulse [Eq. (39)] at the boundaries. Circles stand for the numerical solution and the full line for the analytical prediction [Eqs. (40)] with  $r = 0.1$ . (c) Spatial distribution of the error for different times as obtained numerically with  $\Delta = 0.1$ . The other parameter values are the same as in Fig. 1.

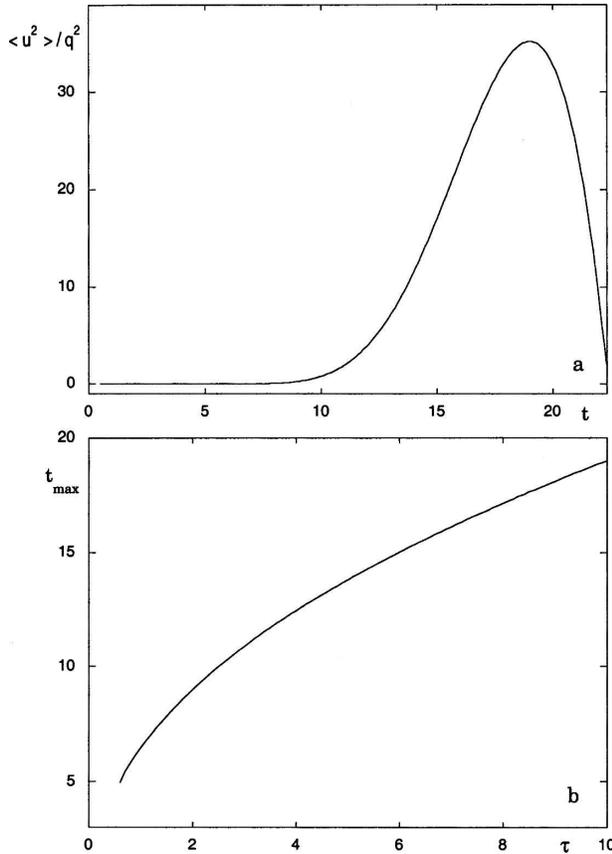


FIG. 5. Time evolution of the normalized mean quadratic error (a) when the system is subjected to an Ornstein–Uhlenbeck noise perturbation at the boundary with  $\tau = 10$  as predicted from the analytic result [Eq. (44)]; (b) times when the error attains its maximal value vs the correlation parameter  $\tau$  of the noise. Parameter value  $\alpha = 50$ .

We do so just to have a reference since in the context of limited-area forecasting models, this does not reflect adequately the idea that the outside domain is treated coarsely and hence varies on a slower scale than the system itself. Equation (36) leads then to the following exact result:

$$\langle u^2 \rangle = \frac{Q^2}{4\alpha} \left( \frac{2}{t} + \frac{1}{\alpha} \right) \exp\left(-\frac{2\alpha}{t}\right). \quad (42)$$

We next consider the more realistic case of red noise (Ornstein–Uhlenbeck process) of variance  $q^2$  and correlation time  $\tau$ :

$$\langle a(t) \rangle = 0, \quad \langle a(t)a(t') \rangle = \frac{q^2\tau}{2} \exp\left(-\frac{|t-t'|}{\tau}\right). \quad (43)$$

Contrary to the previous case expression (36) cannot be evaluated analytically in the most general case. We

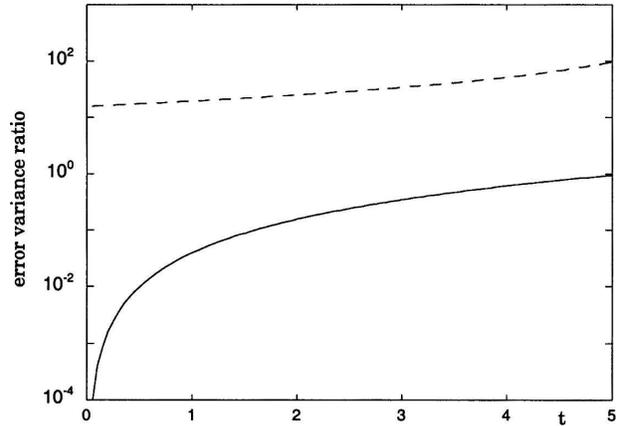


FIG. 6. Ratios of the variances of the errors of the deterministic and stochastic systems vs time as obtained analytically in the case of a white noise error at the boundary (full line) and an Ornstein–Uhlenbeck noise (dashed line). The parameter values are the same as in Fig. 5a.

therefore resort to an asymptotic evaluation in the limit of small diffusion coefficient  $D$  ( $\alpha \gg 1$ ), yielding

$$\langle u^2 \rangle = q^2 \left[ \frac{2e^{-t/\tau}}{(\alpha)^2 - (t^2/\tau)^2} - \frac{1}{\alpha} \frac{1}{\alpha - t^2/\tau} \right] \exp\left(-\frac{2\alpha}{t}\right). \quad (44)$$

Notice that Eq. (42) is recovered, as it should, in the limit of  $D$  small  $q^2 \rightarrow \infty$ ,  $\tau \rightarrow 0$ ,  $q^2\tau/2 = Q^2 = \text{finite}$ .

Both Eqs. (42) and (44) predict a singular behavior of the response for short times, owing to the presence of the  $e^{-r^2/2Dt}$  term and the  $t$  factor, which will again produce a  $t^{1/2}$  dependence for the standard deviation  $\langle u^2 \rangle^{1/2}$ . In other words the noise does not exert a smoothing action, contrary to what might have been expected. On the other hand, the response is no longer monotonic but goes through an extremum as seen in Fig. 5a. The location of this extremum as a function of the values  $\tau$  is depicted in Fig. 5b for  $\alpha = r^2/4D = 50$ . As can be seen the maximum is postponed for increasing correlation times.

It is interesting to compare the response to the noisy boundary error with the response to a fixed boundary error, whose magnitude is equal to the standard deviation of the noise. Figure 6 depicts the way the relative errors as measured by the ratios of the variances of the deterministic and the noisy systems, grow in time for  $\alpha = 50$ . The full and the dashed lines correspond, respectively, to the white noise and to the correlated noise cases. Clearly the presence of correlated noise acts as a decelerating mechanism for the error, the opposite being true for white noise. Figure 7 shows how the boundary error builds in time for a particular real-

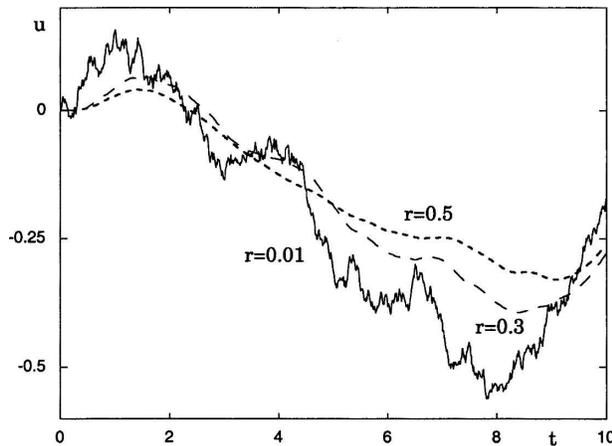


FIG. 7. Time evolution of the error at different distances from the boundary subject to an error given by Eq. (43) as obtained numerically by stochastic simulation of the full problem. The parameter values are  $\tau = 20$ ,  $D = 0.1$ ,  $q^2 = 0.01$ , and  $\ell = 1$ .

ization of a stochastic simulation of the full-diffusion equation subject to an Ornstein–Uhlenbeck noise at the boundaries for  $\tau = 20$ ,  $D = 0.1$ ,  $q^2 = 0.01$ , and  $\ell = 1$ . Interestingly, as one gets farther from the boundaries the effect of the noise is “rectified” in the sense of the response becoming increasingly smooth.

#### 4. Advective dynamics of the boundary error

In this section we consider systems in which the operators acting on the spatial coordinates in Eqs. (1), (3), or (6) are associated to advection. The principal qualitative difference with respect to the situations studied in section 3 is that the evolution equations have now a hyperbolic structure. A typical consequence will be the existence of wavelike behavior such as, for instance, gravity waves or shocks (Landau and Lifshitz 1959). It should be realized that wavelike behavior may also arise in parabolic equations of the type considered in section 3, provided that the source terms are nonlinear functions of  $X$ . Examples are provided by the Kolmogorov–Petrovsky–Piskounov equation familiar from chemical kinetics and population dynamics (Fife 1979) and the Burgers equation modeling one-dimensional turbulence (Burgers 1974; Calogero and De Lillo 1989). Here we focus on systems having a purely hyperbolic structure, since our purpose in this work is to identify the different universality classes of boundary error dynamics. Combinations of these qualitatively different situations would blur the main issue although, as it will be mentioned briefly later on, they may induce some interesting effects.

A reference model that contains already much of the

physics of this class is the advection of a passive scalar by an incompressible flow:

$$\frac{\partial X}{\partial t} = -\mathbf{v} \cdot \nabla X. \quad (45)$$

We shall first consider the simplest implementation of (45), namely, a one-dimensional medium confined in the positive semiaxis ( $0 < r < \infty$ ), and take the velocity  $\mathbf{v}$  to be a positive constant. Translated in terms of regional modeling this means that the system of interest is located downstream with respect to an external environment (the region of  $r < 0$ ), which affects the system through the values of the field  $X$  that it generates on the boundary  $r = 0$ .

The evolution for the error corresponding to Eq. (45) is described by [cf. Eqs. (9)–(10)]

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= -v \frac{\partial u}{\partial r} \\ u(r, 0) &= 0 \\ u(0, t) &= a(t) \end{aligned} \right\}. \quad (46)$$

Green’s function associated to this problem can be calculated straightforwardly from Eq. (11) using Laplace transforms, and turns out to be expressed in terms of the Heaviside step function. In the present case it is actually simpler to solve (46) directly. Switching to Laplace transforms  $\tilde{u}(r, s)$  and using the second Eq. (46) one obtains

$$s\tilde{u}(r, s) = -v \frac{d\tilde{u}(r, s)}{dr}$$

or

$$\tilde{u}(r, s) = A(s) \exp\left(-\frac{s}{v} r\right), \quad (47a)$$

where the integration constant  $A(s)$  is determined from the boundary conditions [third relation (46)]:

$$A(s) = \tilde{a}(s). \quad (47b)$$

The inverse Laplace transform can now be written out explicitly for any given form of  $a(t)$  (Abramowitz and Stegun 1972):

$$u(r, t) = a\left(t - \frac{r}{v}\right) \theta\left(t - \frac{r}{v}\right). \quad (48)$$

We see that the boundary condition gradually invades the interior of the system, such that the values of the error at successive points  $r$  suddenly switch at  $t = r/v$  from the initial value of zero to values given by the boundary condition  $a(t)$  evaluated at the lagged times  $t' = t - r/v$ . The error front propagates at constant velocity  $v$ .

Owing to the vanishing of the  $u$  for  $t < r/v$  and the jump occurring at the switching times  $t = r/v$  the behavior of the error is singular, and thus not expressible as a power of  $t$  for short times. This singularity is of a different kind to the one identified in section 3, where the behavior, although nonanalytic, remained continuous. The presence of a slight amount of dissipation, for instance in the form of a second space derivative added to the right-hand side of (45) would tend to roundoff the wave front, but its effect would only show up beyond a certain time interval related to the magnitude of this term.

We next consider the more interesting situation where two fields advected by a uniform basic current  $v$  are coupled via the conservation laws of fluid dynamics. Such a coupling provides the basic mechanism of generation of gravity waves (Phillips and Shukla 1973):

$$\left. \begin{aligned} \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial r} &= -\frac{\partial \phi}{\partial r} \\ \frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial r} &= -c^2 \frac{\partial u}{\partial r} \end{aligned} \right\} \quad (49)$$

Here  $u$  denotes a perturbation of advection velocity around the basic streaming current  $v$  along the (unique) space direction  $r$ ,  $\phi$  is the geopotential, and  $c$  is the wave velocity. Since Eqs. (49) are linear the error field that is now a two-component vector will satisfy the same set of equations, but will be subjected to zero initial conditions and to boundary values at  $r = 0$  given by functions of  $t$  of small amplitude:

$$\left. \begin{aligned} u(r, 0) = \phi(r, 0) &= 0 \\ u(0, t) &= a_1(t) \\ \phi(0, t) &= a_2(t) \end{aligned} \right\}, \quad (50)$$

where we used for the two error components the same notation as for the original fields.

It is convenient to write Eq. (49) in the following form:

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ \phi \end{pmatrix} = - \begin{pmatrix} v & 1 \\ c^2 & v \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial \phi}{\partial r} \end{pmatrix}. \quad (51)$$

The matrix in the right-hand side possesses the eigenvalues

$$\sigma_1 = v + c, \quad \sigma_2 = v - c, \quad (52a)$$

the corresponding eigenvectors being

$$(1, -c)^+, \quad \left(\frac{1}{c}, 1\right)^+. \quad (52b)$$

Acting on both sides of (51) with the inverse of the matrix  $\mathbf{R}$  formed by these two eigenvectors one is led to a problem in which the two variables

$$\begin{pmatrix} u' \\ \phi' \end{pmatrix} = \mathbf{R}^{-1} \begin{pmatrix} u \\ \phi \end{pmatrix} \quad (53)$$

obey to separate closed equations,

$$\left. \begin{aligned} \frac{\partial u'}{\partial t} &= -\sigma_1 \frac{\partial u'}{\partial r} \\ \frac{\partial \phi'}{\partial t} &= -\sigma_2 \frac{\partial \phi'}{\partial r} \end{aligned} \right\}, \quad (54)$$

each of which has the form of Eq. (45), complemented by the initial and boundary conditions following from Eqs. (50) and (53).

The first of these equations can be handled exactly as earlier in this section leading to a solution of the form (48), that is, to an error wave invading the system at a speed  $v + c$  inducing a discontinuous switching from zero to a value dictated by the boundary conditions. We thus recover the result of the previous example and, in particular, the nonanalytic behavior of the error for short times.

The second, Eq. (54), is more subtle. Taken out of context, it still admits a solution of the type of Eq. (48). As long as the speed  $\sigma_2 = v - c$  is positive the behavior is exactly as before. The situation changes for  $\sigma_2 < 0$  because now the Heaviside function in Eq. (48) is non-zero for every positive  $r$  for  $t = 0$ , contrary to the initial condition prescribed. Stated differently, the Cauchy (initial value) and Dirichlet (boundary value) problem are mutually incompatible for the second Eq. (54), a property related to the hyperbolic character of the model equations. It is worth pointing out that a solution of the form of Eq. (48) makes sense in the region of  $r < 0$ . The role of “system” and “environment” are thus inverted, as the system now becomes a source of excitations invading the environment. We are witnessing a “two-way” interaction calling for a careful matching of the limited-area and external models at the boundary, as observed already in the pioneering work of Phillips and Shukla (1973). This problem is beyond the scope of the present work.

Part of the above analysis can be extended to include nonlinearities. Since the error equations in the short time regime are, by definition, linearized this will be manifested by the appearance of space–time-dependent coefficients—for instance a  $v$  being a function of  $r$  and  $t$  in Eq. (46). To see the type of effect that may arise in this case consider constant boundary conditions [ $u(0, t) = a$ ] and  $v$  depending only on time. One can

then check that the solution of (46) compatible with the initial and boundary conditions becomes

$$u(r, t) = a\theta\left(\int_0^t v(t') dt' - r\right) \quad (55)$$

as long as the integral over  $t$  remains nonnegative for all  $t$ . We see that the error still propagates in a pulselike fashion from the boundaries to the interior that is, again, not amenable to an expansion in powers of  $t$  for short times. The difference with Eqs. (46) and (49) is that the times at which the points  $r$  are attained by the pulse are now given by the following more involved relation:

$$\int_0^t v(t') dt' = r. \quad (56a)$$

As an example, for a positive periodic flow of the form  $v(t) = v_0 \sin^2 \omega t$  one obtains for the arrival times the transcendental equation:

$$t - \frac{\sin 2\omega t}{2\omega} = \frac{2r}{v_0}, \quad (56b)$$

entailing a variable speed of propagation. This is depicted in Fig. 8 where we plot the time of arrivals of the perturbation versus the distance from the boundary. The situation is qualitatively similar, though technically more involved, when  $v$  is taken to be a random function of time to emulate some short-scale features of atmospheric turbulence.

## 5. Conclusions

In this work we analyzed the different modes of response of a forecasting model to boundary error. We have shown that the behavior displays a nonanalytic structure that could in no way be reduced to a power of  $t$  for short times as it happens in classical model error. Two general classes of systems exhibiting such a behavior were considered, namely, systems in which the operators acting on the spatial coordinates of the fields of interest are diffusion-like (or more generally parabolic) and advection-like (or more generally hyperbolic). In the first class the dominant contribution to the error is expressed as a fractional power of  $t$  multiplied by a negative exponential of  $t^{-1}$ , while in the second class it is expressed as a wave front invading the system in a steplike fashion. In each case expressions for the dominant part of the space dependence of the error were also derived. A most important point has been that source terms, associated to a dependence of the evolution laws on the fields themselves rather than their space derivatives, give rise to higher-order terms mani-

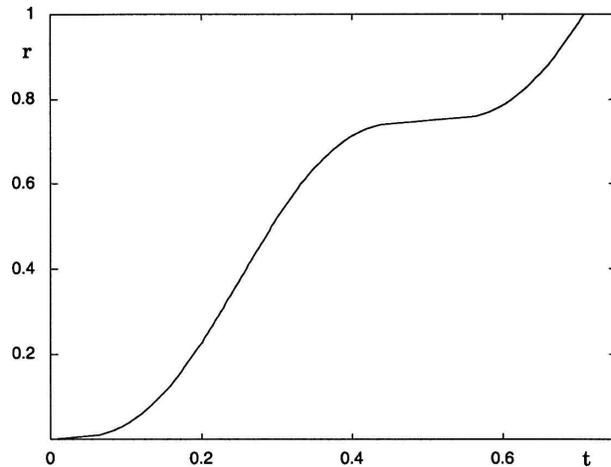


FIG. 8. Arrival times of the boundary perturbation at different distances  $r$  from the boundary, for the advection of a passive scalar by a time periodic flow in the form of a sine square pulse [Eq. (56b)]. The parameter values are  $v_0 = 3$  and  $\omega = 2\pi$ .

festated at a later stage of the evolution as compared to the above dominant contributions.

To reach these conclusions an approach accounting from the outset for finescale effects in space and time had to be performed. One may ask, what is the connection between such an approach and the one routinely practiced in numerical forecasting models, where the fields of interest are either discretized in space with a mesh given once far all or expressed in series of modes truncated to a finite order. In doing so boundary conditions are incorporated as additional parameters in the resulting equations and boundary error appears thus to be reduced to model error, suggesting a mean quadratic error dependence in  $t^2$ .

The contradiction is only apparent, and is removed by realizing that the coefficient multiplying such a  $t^2$  dependence becomes resolution dependent and is actually diverging in the continuous limit, as one can check straightforwardly by formally expanding the exponential in Eq. (25) in powers of  $t$  (see also comment following this equation). Put differently, even when the equations are expressed in discrete form (which is precisely what is done when solving a problem numerically, as in the numerical analysis leading to Fig. 1), the representation of the error in powers of  $t$  in the short time regime is inadequate and one must resort to different bases of functions. We have shown in this work that these are provided by the Green's functions and related quantities associated with the operators acting on the spatial coordinates.

Throughout our analysis the cases of even and odd spatial derivatives were treated separately, since our objective has been to delimit classes of qualitative dif-

ferent behaviors. Still, it would be interesting to extend the work in order to account for cross effects. For instance, adding a second-order spatial derivative term in a purely advective equation involving first-order derivatives constitutes a singular perturbation. Nontrivial effects can thus be expected such as the smoothing of an otherwise abrupt wave front. An interesting case where this type of effect is known to take place is provided by the Burgers equation and its two- and three-dimensional generalizations (Burgers 1974; Calogero and De Lillo 1989).

The results reported are expected to carry through in large numerical forecasting models. Indeed, the building blocks of these models are provided, precisely, by diffusive and advective operators acting on the fields of interest and we have shown that any further contributions in the form of source terms will influence only the later stages of the error evolution. It would nevertheless be interesting to carry out such experiments in order to assess the range of validity of the short time regime. On the theoretical side, the crossover between this regime and the intermediate time one would also be worth analyzing.

*Acknowledgments.* This work is supported by the Belgian Federal Science Policy Office under Contract MO/34/004 and the European Commission under Contract 12975 (NEST).

## REFERENCES

- Abramowitz, M., and I. Stegun, Ed., 1972: *Handbook of Mathematical Functions*. Dover, 1046 pp.
- Buizza, R. M., J. Miller, and T. N. Palmer, 1999: Stochastic simulation of model uncertainties in the ECMWF ensemble prediction system. *Quart. J. Roy. Meteor. Soc.*, **125**, 2887–2908.
- Burgers, J. M., 1974: *The Nonlinear Diffusion Equation*. Reidel, 173 pp.
- Calogero, F., and S. De Lillo, 1989: The Burgers equation on the semi-infinite and finite intervals. *Nonlinearity*, **2**, 37–43.
- Chu, P., L. M. Ivanov, T. M. Margolina, and O. V. Melnichenko, 2002: On probabilistic stability of an atmospheric model to various amplitude perturbations. *J. Atmos. Sci.*, **59**, 2860–2873.
- Crank, J., 1956: *The Mathematics of Diffusion*. Clarendon Press, 347 pp.
- Fife, P., 1979: *Mathematical Aspects of Reacting and Diffusing Systems*. Springer, 185 pp.
- Kalnay, E., 2003: *Atmospheric Modeling, Data Assimilation and Predictability*. Cambridge University Press, 341 pp.
- Landau, L. D., and E. M. Lifshitz, 1959: *Fluid Mechanics*. Pergamon Press, 536 pp.
- Lorenz, E. N., 1984: Irregularity: A fundamental property of the atmosphere. *Tellus*, **36A**, 98–110.
- Morse, P. M., and H. Feshbach, 1953: *Methods of Theoretical Physics: Part I*. McGraw-Hill, 997 pp.
- Nicolis, C., 2003: Dynamics of model error: Some generic features. *J. Atmos. Sci.*, **60**, 2208–2218.
- , 2004: Dynamics of model error: The role of unresolved scales revisited. *J. Atmos. Sci.*, **61**, 1740–1753.
- , 2005: Can error source terms in forecasting models be represented by Gaussian Markov noises? *Quart. J. Roy. Meteor. Soc.*, **131**, 2151–2170.
- , and G. Nicolis, Ed., 1987: *Irreversible Phenomena and Dynamical Systems Analysis in Geosciences*. Reidel, 578 pp.
- Nutter, P., D. Stensrud, and M. Xue, 2004: Effects of coarsely resolved and temporally interpolated lateral boundary conditions on the dispersion of limited-area ensemble forecasts. *Mon. Wea. Rev.*, **132**, 2358–2377.
- Palmer, T. N., 2001: A nonlinear dynamical perspective on model error: A proposal for non-local stochastic-dynamic parametrization in weather and climate prediction models. *Quart. J. Roy. Meteor. Soc.*, **127**, 279–304.
- Phillips, N. A., and J. Shukla, 1973: On the strategy of combining coarse and fine grid meshes in numerical weather prediction. *J. Appl. Meteor.*, **12**, 763–770.
- Reynolds, C. A., P. J. Webster, and E. Kalnay, 1994: Random error growth in NMC's global forecasts. *Mon. Wea. Rev.*, **122**, 1281–1305.
- Tribbia, J. J., and D. P. Baumhefner, 1988: The reliability of improvements in deterministic short-range forecasts in the presence of initial state and modeling deficiencies. *Mon. Wea. Rev.*, **116**, 2276–2288.
- Warner, T. T., R. A. Peterson, and R. E. Treadon, 1997: A tutorial on lateral boundary conditions as a basic and potentially serious limitation to regional numerical weather prediction. *Bull. Amer. Meteor. Soc.*, **78**, 2599–2617.
- Widder, D. V., 1941: *The Laplace Transform*. Princeton University Press, 406 pp.