Horizontal Wavenumber Spectra of Vertical Vorticity and Horizontal Divergence in the Upper Troposphere and Lower Stratosphere

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ABSTRACT

The author shows that the horizontal two-point correlations of vertical vorticity and the associated vorticity wavenumber spectrum can be constructed from previously measured velocity structure functions in the upper troposphere and lower stratosphere. The spectrum has a minimum around $k = 10^{-4}$ cycles per kilometer (cpkm) corresponding to wavelengths of 100 km. For smaller wavenumbers it displays a $k^{-1}$ range and for higher wavenumbers, corresponding to mesoscale motions, it grows as $k^{1/3}$. The two-point correlation of horizontal divergence of horizontal velocity and the associated horizontal spectrum is also constructed. The horizontal divergence spectrum is of the same order of magnitude as the vorticity spectrum in the mesoscale range and show similar inertial range scaling. It is argued that these results show that the mesoscale motions are not dominated by internal gravity waves. Instead, the author suggests that the dynamic origin of the $k^{1/3}$ range is stratified turbulence. However, in contrast to Lilly, the author finds that stratified turbulence is not a phenomenon associated with an upscale energy cascade, but with a downscale energy cascade.

1. Introduction

A problem of great theoretical and practical importance is the dynamic origin of the wavenumber spectrum of kinetic energy in the upper troposphere and lower stratosphere. The spectrum (Vinnichenko 1970; Nastrom and Gage 1985) displays two regions. In the mesoscale range, which is wavenumbers corresponding to wavelengths of the order of 1–100 km, the spectrum falls off as $k^{-5/3}$ and for smaller wavenumbers corresponding to longer wavelengths, the spectrum approaches a $k^{-3}$ dependence. Until recently, the discussion on the dynamic origin of the spectrum has been confined to a rather limited group of theoretically oriented dynamicists. However, as the resolving capacity of numerical models has dramatically increased during the last 10 years and the horizontal scale of resolution is entering into the mesoscale range, the problem has also become a matter of practical concern. What causes the spectrum to bend off from the $k^{-3}$ dependence to the less step $k^{-5/3}$ dependence? What is the nature of the mesoscale motions producing this transition and how does the dynamics of these, which in most numerical models still are very poorly resolved, influence the dynamics of the resolved motions? To develop efficient subgrid models and accurately optimize resolution, it has become more relevant to ask these questions. As a matter of fact, it has also been demonstrated (Koshyk and Hamilton 2001; Skamarock 2004; Kitamura and Matsuda 2006) that highly resolved numerical models are actually capable of reproducing the transition from a $k^{-3}$ to a $k^{-5/3}$ spectrum range.

The rather limited $k^{-3}$ range at larger scales is consistent with the theory of quasigeostrophic turbulence of Charney (1971), which explains this spectrum range as arising from a downscale cascade of potential enstrophy, defined as half of the squared potential vorticity. Although this theory is quite simplified, there is no reason to believe that it should not catch the essentials of the dynamics behind the $k^{-3}$ spectrum range. It has also been demonstrated (e.g., Tung and Orlando 2003) that quasigeostrophic numerical models reproduce this spectrum range reasonably well. In this paper, we will not question this interpretation, but focus on the dynamic origin of the $k^{-5/3}$ range.

Two completely different hypotheses have been proposed to explain this spectrum range: the quasi-two-
dimensional stratified turbulence hypothesis (Gage 1979; Lilly 1983) and the gravity wave hypothesis (Dewan 1979, 1997; VanZandt 1982; Bacmeister et al. 1996). The stratified turbulence hypothesis was elegantly outlined by Lilly (1983), who showed that under certain conditions fluid motions influenced by strong stratification will decouple into layers. He suggested that the dynamics within each layer should be governed by the two-dimensional vorticity equation. In addition to energy, this equation conserves enstrophy, defined as half the square of vorticity, and for this reason there are solutions exhibiting an upscale cascade of energy with an associated $k^{-5/3}$ spectrum range (Kraichnan 1970). The essential features of this hypothesis are the dynamic decoupling into layers, the vortical nature of the motion, and the strong nonlinearity leading to an upscale energy cascade. The gravity wave hypothesis, on the other hand, says quite the opposite. The mesoscale spectrum originates from a downscale energy cascade of weakly nonlinear gravity waves.

Lindborg (1999, hereafter L99) suggested that it should be possible to decide whether the energy cascade is upscale or downscale by calculating third-order velocity structure functions from airplane data. Structure functions is a central concept in turbulence theory, and in particular the third-order structure function law of Kolmogorov (1941) is, maybe, its most central theorem (see Frisch 1995). The third-order longitudinal structure function $D_{LLL}$ is defined as the third-order statistical moment of the difference, $\delta u_L = u_L' - u_L$, between the longitudinal velocity components at two points separated by a distance $r$ (see Fig. 1). Kolmogorov’s four-fifths law says that for three-dimensional isotropic turbulence

$$D_{LLL} = -\frac{4}{5} \epsilon r,$$

in the inertial range of separation distances, where $\epsilon$ is the mean dissipation. L99 argued that, if the mesoscale dynamics were controlled by a downscale energy cascade, the third-order structure function should exhibit a similar negative linear dependence in this range of separations, while if it were controlled by an upscale energy cascade it should exhibit a positive linear dependence on $r$. L99 also calculated second-, third-, and fourth-order structure functions from aircraft data reported in the measurement of ozone and water vapor by an Airbus in-service aircraft (MOZAIC) database (see Marenco et al. 1998). However, as for the third-order structure functions, the results were seriously contaminated by a “bug” in the data program that was used. This was discovered when J. Cho reproduced all of Lindborg’s result, except the results for the third-order structure functions, using his own data program and the same database. Cho and Lindborg (2001) could correct the error and show that the third-order structure function is, in fact, linear in $r$ and negative in the range of mesoscale separation. From this observation the downscale flux of energy through the mesoscale range could also be estimated and it matched well with previous measurements of the mean dissipation in the upper troposphere and lower stratosphere. This was strong observational evidence supporting the forward energy cascade hypothesis and thereby, indirectly, the gravity wave hypothesis.

However, only because there is strong reason to believe that there is a downscale energy cascade, it is not certain that the motions undergoing the cascade consist of gravity waves. There are other possibilities. One possibility, which has been proposed by Tung and Orlando (2003), is that not only the $k^{-1}$ range of the atmospheric kinetic energy spectrum is produced by quasigeostrophic turbulence, but also the $k^{-5/3}$ range, or at least its lower wavenumber part. They argued that a fraction of the energy injected by baroclinic instability at large planetary scales will undergo a downscale cascade that is hidden behind the stronger cascade of potential enstrophy all the way down to a certain wavelength where it becomes dominant. Tung and Orlando also presented numerical results from a quasigeostrophic model calculation, exhibiting both of these spectral regions. It has been debated (Smith 2004; Tung and Orlando 2004) whether the resolution in their calculation was sufficient for the $k^{-5/3}$ range to be reliable as a robust result. In any case, the whole $k^{-5/3}$ range in the atmosphere can hardly be explained within a quasigeostrophic
model since, as argued in the conclusions, Coriolis effects are relatively weak in the high wavenumber end of this range. We will focus more on another type of explanation, which is that the mesoscale motions in some important respects actually have the properties suggested by Lilly (1983), except that the energy cascade is in the forward direction. This would mean that the motion could be described as vortices undergoing layer formation under the influence of strong stratification. The reason that the cascade would be in the forward direction would be that the shear instabilities, which are developing in between the layers, as also predicted by Lilly (1983), would break up the layers in the horizontal direction and prevent the quasi-horizontal motions to extend indefinitely in horizontal planes. The layers would develop into “pancake vortices,” and energy would start to cascade from large to small horizontal scales. In contrast to gravity waves, such a motion would carry a fair amount of vertical vorticity.

In this paper, we will investigate whether the mesoscale motions carry vertical vorticity in accordance with this hypothesis. We will do this by constructing the two-point correlation of vertical vorticity and the associated vorticity spectrum from the second-order structure functions calculated by L99. At the same time, we construct the two-point correlation of the horizontal divergence of the horizontal velocity and the associated spectrum, in order to make a comparison with recent results from numerical simulations (Koshykh and Hamilton 2001; Kitamura and Matsuda 2006).

2. Construction of the two-point correlations and the associated spectra

The two-point correlation tensors of velocity and vorticity can be defined as (Batchelor 1953)

\[ R_{ij}(x, r) = \langle u_i(x) u_j(x + r) \rangle, \]

\[ Q_{ij}(x, r) = \langle \xi_i(x) \xi_j(x + r) \rangle, \]

where \( u \) is the fluctuating velocity field, \( \xi \) is the fluctuating vorticity field, and \( \langle \ldots \rangle \) can either be interpreted as an ensemble average or a time average. We shall assume that the velocity field is statistically homogenous in horizontal planes in which case \( R_{ij} = R_{ij}(x, r) \), where \( x_{z} \) is the altitude. In this case \( \langle \ldots \rangle \) can also be interpreted as a space average over a horizontal plane, and \( Q_{zz} = Q_{zz} \) is related to \( R_{ij} \) according to (Batchelor 1953)

\[ Q_{zz} = -\epsilon_{ijk} \epsilon_{jkl} \frac{\partial^2 R_{kl}}{\partial r_i \partial r_j} \]

with contraction over repeated indices. We denote by \( \mathbf{e}_i \), the unit vector in the vertical direction, \( \mathbf{e}_x \) the unit vector in the direction of the projection of \( r \) onto the horizontal plane and \( \mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y \). The three unit vectors form the traditional orthogonal basis of the cylinder coordinate formulation. Using dyadic notation the two-point velocity correlation tensor can be written as

\[ \mathbf{R} = \mathbf{e}_z \mathbf{e}_z R_{zz} + \mathbf{e}_x \mathbf{e}_x R_{xx} + \mathbf{e}_y \mathbf{e}_y R_{yy} + \mathbf{e}_z \mathbf{e}_x R_{xz} + \mathbf{e}_z \mathbf{e}_y R_{yz} + \mathbf{e}_x \mathbf{e}_y R_{xy} + \mathbf{e}_z \mathbf{e}_y R_{x}, \]

where the tensor components are functions of the altitude \( z \); the horizontal distance \( r \), between the two points; the vertical distance \( z \), between the two points; and \( \phi \), which is the angle that \( \mathbf{e}_z \) takes to a given horizontal direction, the east to west zonal direction, say.

A convenient way of keeping track of the contractions in (4) when \( Q_{zz} \) is to be calculated is to express \( \epsilon_{ij} \) and \( \epsilon_{ik} \) in Cartesian tensor notation (see Lindborg 1995) and use the relations

\[ \frac{\partial}{\partial r_i} = \epsilon_i^p \frac{\partial}{\partial p} + \epsilon_i^\phi \frac{\partial}{\partial \phi}, \]

\[ \frac{\partial}{\partial p} e_i = \frac{\partial}{\partial r_i} e_p = 0, \quad \frac{\partial}{\partial \phi} e_i = e_{i\phi} = -e_{\phi i}. \]

\[ \epsilon_{3k} = \epsilon_{ik}^p e_{3} - e_{i\phi}^p e_{3}. \]

A straightforward calculation gives

\[ Q_{zz} = \left( 1 - \frac{1}{\rho^2} \frac{\partial^2}{\partial r^2} \right) \left( \frac{1}{\rho^2} \frac{\partial R_{\phi \phi}}{\partial p} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \left( R_{\phi \phi} + R_{\phi \phi} \right). \]

In the case of axisymmetry, that is when all statistical quantities are invariant under rotations of \( r \) with respect to the vertical axis, the tensor components cannot depend on the angle \( \phi \) and in this case the two last terms including derivatives with respect to \( \phi \) are both zero. These terms also cancel if the mean value of (9) is formed with respect to \( \phi \). In both cases we can write

\[ Q_{zz}(x, \rho, z) = \left( 1 - \frac{1}{\rho^2} \frac{\partial^2}{\partial r^2} \right) \left( \frac{1}{\rho^2} \frac{\partial R_{\phi \phi}}{\partial p} \right) \]

Here \( Q_{zz} \) and \( R_{\phi \phi} \) may either be interpreted as two-point correlations measured in any horizontal direction in a statistically axisymmetric field or as mean values with respect to \( \phi \). It is worth pointing out that, although we have used the classical machinery of the
theory of homogeneous turbulence, we have neither assumed that the velocity field is divergence free due to incompressibility nor that the vertical direction is a homogeneous direction.

To express $Q_{zz}$ in quantities that have been measured we introduce the velocity structure function tensor defined as

$$D_{ij}(x, r) = \langle \delta u_i \delta u_j \rangle,$$

where $\delta u_i = u_i(x + r) - u_i(x)$. In the homogeneous case, $D_{ij}$ is related to $R_q$ through the relation

$$D_{ij}(x, r) = -R_{ij}(x, r) + \langle u_i(x)u_j(x) \rangle$$

$$+ \langle u_i(x + r)u_j(x + r) \rangle.$$  

(12)

Applying the assumption of horizontal homogeneity we find that (10) also can be written as

$$Q_{zz}(x, r, \rho, z) = -\frac{1}{2} \left[ \frac{\partial D_{ij}}{\partial \rho} - \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial D_{ij}}{\partial \rho} \right) \right].$$

(13)

We will also study the two-point correlation function of horizontal divergence, $\nabla_h \cdot u_h$, where $u_h$ is the horizontal velocity component. We define this correlation function as

$$P = \langle \nabla_h \cdot u_h(x + r) \nabla_h \cdot u_h(x) \rangle = -\frac{\partial^2 R_{ij}}{\partial r_i \partial r_j}.$$

(14)

where repeated indices are summed over the two horizontal coordinate direction. The last equality follows from horizontal homogeneity. The divergence correlation function $P$ can be calculated as

$$P(x, r, \rho, z) = -\nabla_h^2 (R_{pp} + R_{\phi\phi}) - Q_{zz}$$

$$= -\frac{1}{2} \left[ \frac{\partial^2 D_{ij}}{\partial \rho} - \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial D_{ij}}{\partial \rho} \right) \right].$$

(15)

where we again have made the assumption of axisymmetry. It is interesting to note that the expression (15) has the same form as (13) with $D_{pp}$ and $D_{\phi\phi}$ interchanged.

L99 calculated the functions $D_{pp}$ and $D_{\phi\phi}$ from wind data records in the upper troposphere and lower stratosphere from over 6000 commercial flights reported in the MOZAIC database. L99 used the subscripts $L$ and $T$ instead of the subscript $\rho$ and $\phi$, with $L$ designating the longitudinal direction and $T$ designating the transverse direction. This is the same notation as traditionally used in the theory of homogeneous, isotropic turbulence (see Monin and Yaglom 1975) with the difference that, in this case, $r$ can point in any direction, while in our case we assume that $r$ lies in the horizontal plane; that is, we put $z = 0$. Using this notation we can write

$$D_{pp} = D_{LL} = \langle \delta u_L \delta u_L \rangle, \quad D_{\phi\phi} = D_{TT} = \langle \delta u_T \delta u_T \rangle,$$

(16)

where $\delta u_L = u_L - u_L$ and $\delta u_T = u_T - u_T$ (see Fig. 1). Here $D_{LL}$ and $D_{TT}$ were calculated by calculating $\delta u_L$ and $\delta u_T$ for each pair of data points reported along each flight and then squaring and averaging. To investigate the latitudinal and altitudinal dependence of these functions Cho and Lindborg (2001) carried out the same type of calculations on subsets of the whole dataset. It was found that the latitudinal dependence is sufficiently weak to justify the assumption of spatial homogeneity for separations up to 2000 km. The averaging was performed over a large set of flights in which $r$ can be regarded as almost randomly distributed in all horizontal directions, so we can apply the formalism developed here, interpreting the correlation functions as averages over all directions. It is also very likely that axisymmetry is a justified assumption. However, we do not need this to compute $Q_{zz}$ and $P$. The flights were performed on constant pressure surfaces, which means that a horizontal plane in this context should be interpreted as a plane of constant pressure.

L99 assumed that the second-order structure functions can be written as superposition of two functions, of the form $\rho^{2/3}$, whose dynamical origin is an energy cascade, and one of the form $\rho^{2}(c - \ln \rho)$, whose dynamical origin is an enstrophy cascade. In accordance with this assumption the calculated structure functions could be fitted to the curves

$$D_{LL} = a_1 \rho^{2/3} + b_1 \rho^2 - c_1 \rho^5 \ln \rho,$$

(17)

$$D_{TT} = a_2 \rho^{2/3} + b_2 \rho^2 - c_2 \rho^5 \ln \rho,$$

(18)

with a high degree of accuracy in the separation interval between 2 and 2000 km. The six parameters $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, and $c_2$ varied slightly with latitude and altitude. Inserting these expressions into (13) and (15) and carrying out the differentiation we find

$$Q_{zz} = q_1 \rho^{-4/3} - q_2 \rho^{-5/3} - q_3 \rho^{2/3} \ln \rho,$$

(19)

$$P = p_1 \rho^{-4/3} - p_2 \rho^{2/3} \ln \rho + p_3,$$

(20)

where

$$q_1 = \frac{1}{3} \left( \frac{5}{3} a_2 - a_1 \right),$$

(21)

$$q_2 = 3c_2 - c_1,$$

(22)

$$q_3 = 3b_2 - \frac{5}{2} c_2 - b_1 + \frac{1}{2} c_1,$$

(23)
and $p_1$, $p_2$, and $p_3$ are calculated in the corresponding way with $a_1 \leftrightarrow a_2$, $b_1 \leftrightarrow b_2$, and $c_1 \leftrightarrow c_2$.

Using the empirical values of the parameters $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, and $c_2$ calculated by L99 on the basis of the whole MOZAIC dataset, we find that $q_1 \approx 1.0 \times 10^{-3}$ s$^{-2}$ m$^{4/3}$, $q_2 \approx 1.13 \times 10^{-9}$ s$^{-2}$, and $q_3 \approx 1.61 \times 10^{-8}$ s$^{-2}$. As for the parameters in the expression (20) we find that $p_1 = 0.66 \times 10^{-3}$ s$^{-2}$ m$^{4/3}$, while $p_2$ and $p_3$ both are very small compared to $q_2$ and $q_3$. $q_2/p_2q_2 \approx 0.04$ and $p_3/q_3 \approx 0.03$. Thus, there is an almost perfect cancellation of the leading order large-scale contribution to $P$. The two last terms in (17) and (18), supposedly associated with an enstrophy cascade, originate exclusively from the vortical part of the velocity field. The fact that there is not a 100% cancellation of the two last terms in (20) must be attributed to uncertainties in the measurements and to the curve fit procedure itself. In Fig. 2 we have plotted (19) and (20) calculated from the measured structure functions. The dotted lines are the zero line and the logarithmic part, $-q_2 \log \rho + q_3$, of (19). The logarithmic region for $Q_{zz}$ for separations between 100 and 2000 km is consistent with the theory of quasi-geostrophic turbulence of Charney (1971), which predicts that the vorticity spectrum, or more precisely the potential enstrophy spectrum, should exhibit a $k^{-4}$ range for wavenumbers corresponding to these scales. As we shall see, this is also the spectrum form at these wavenumbers. For separations greater than 200 km the two-point divergence correlation, $P$, is close to zero. As for the exact functional behavior of $P$ it is not possible to say what this should be for these separations.

It seems that $P$ and also $Q_{zz}$ take negative values for large-scale separations. Actually, it can be argued that $P$ and $Q_{zz}$ should behave in this way. Multiplying (10) by $\rho$ and integrating we find

$$\int_0^\infty Q_{zz}(x, \rho) \rho \, d\rho = \left[ R_{pp} - \frac{\partial R_{\phi \phi}}{\partial \rho} - \rho \frac{\partial R_{\phi \phi}}{\partial \rho} \right]_0^\infty = 0,$$

where we have assumed that $R_{pp}$ and $R_{\phi \phi}$ go to zero as $\rho \to \infty$. If this is true and if $R_{\phi \phi}$ is monotonic and twice differentiable as $\rho \to \infty$, then it follows that $\rho \partial R_{\phi \phi}/\partial \rho$ also goes to zero as $\rho \to \infty$. We have also used the fact that $R_{pp} \to R_{\phi \phi}$ as $\rho \to 0$, by axisymmetry or by the interpretation of $R_{pp}$ and $R_{\phi \phi}$ as mean values with respect to the angle $\phi$. A corresponding relation can be derived for $P$. Since $Q_{zz}$ and $P$ must be positive for small separations, the relation (24) and the corresponding relation for $P$ show that they must take negative values for large separations.

The two-dimensional horizontal spectrum of vertical vorticity is defined as

$$\Phi(k) = 2\pi k \mathcal{F}[Q_{zz}],$$

where

$$\mathcal{F}[Q_{zz}] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_{zz} \exp(-i \mathbf{k} \cdot \mathbf{r}) \, dr_x \, dr_y.$$
separations we can only say that \( P \) goes to zero quite fast. To obtain reasonable approximations of \( \Phi \) and \( \Psi \) in the corresponding wavenumber ranges, \( \mathcal{K} := [2\pi/2000, 2\pi/2] \) km\(^{-1}\) and \( \mathcal{K}' := [2\pi/200, 2\pi/2] \) km\(^{-1}\), we apply the method of generalized Fourier transforms. This method is justified if it can be assumed that the contributions to the spectra from outside \( \mathcal{K} \) and \( \mathcal{K}' \) are small when \( k \in \mathcal{K} \) and \( k \in \mathcal{K}' \), respectively. In turbulence theory the method is standard (see Monin and Yaglom 1975). It has also proved to work very well to construct the atmospheric kinetic energy spectrum using this method, which was shown by L99 who obtained a kinetic energy spectrum from the measured structure functions (17)

\[
2\pi k J_q(k \rho^{-1/3}) = k \left[ \frac{1}{2\pi} \int_0^{2\pi} q_1 \rho^{-1/3} \exp(-ik\rho \cos \theta) d\theta dp \right] = \int_0^{2\pi} q_1 \rho^{-1/3} J_q(k \rho) dp = \int_0^\infty \tau^{-1/3} J_q(k \rho) \tau \ d\tau k^{1/3}.
\]

where \( J_q \) is the zero-order cylindrical Bessel function. The last term in (32) can be found in standard mathematical handbooks and takes the value given in (31).

The second term is calculated as

\[
2\pi k J_q[-q_2 \ln \rho] = 2\pi k^{-1} J_q[\nabla^2 \ln \rho] q_2 = 2\pi k^{-1} J_q[2\pi \delta(r)] q_2 = q_2 k^{-1}.
\]  

The last constant term in (19) does not give any contribution to \( \Phi \) when \( k \in \mathcal{K} \) since the generalized Fourier transform of a constant function is a delta function. Correspondingly, the two-dimensional divergence spectrum is calculated as

\[
\Psi(k) \approx 1.57 \rho_2 k^{1/3},
\]

for wavenumbers larger than 1/200 cpkm. For smaller wavenumbers \( \Psi \) should go to zero monotonically with decreasing wavenumber since the magnitude of \( P \) is very small at large separations.

In Fig. 3, we have plotted the vorticity and the divergence spectra. In this plot we have adopted the convention of measuring \( k \) in cpkm, which means that 1/k gives a corresponding wavelength measured in km. As can be seen, the vorticity spectrum reaches a minimum around \( k = 0.01 \) cpkm. To the left of the minimum it approached a \( k^{-1} \) curve, consistent with quasigeostrophic turbulence, and to the right of the minimum it grows as \( k^{1/3} \). The divergence spectrum has the same form and is of the same order of magnitude as the vorticity spectrum in the mesoscale region. However, its magnitude is a little bit lower.

If the mesoscale energy spectrum were produced by internal gravity waves the divergence spectrum would have dominated over the vorticity spectrum. In the ideal case of linear internal gravity waves without system rotation, the vorticity spectrum should be identical to zero. In the case of inertia–gravity waves we have

\[
\Phi'(k, \omega) = \frac{f^2}{\omega^2} \Psi'(k, \omega),
\]

where \( \Phi' \) and \( \Psi' \) are the vorticity and the divergence spectral densities in wavenumber–frequency space and
$f$ is the inertial frequency. Integrating over all frequencies we find

$$\Phi(k) = \frac{f^2}{\omega} \int^f \frac{f^2}{\omega} \Psi'(k, \omega) d\omega \approx \Psi(k),$$  \hspace{1cm} (36)$$

with equality only if the frequency is equal to $f$ for all waves contributing to the spectrum. Empirically, it is found that typical frequencies of inertia–gravity waves in the upper troposphere and the stratosphere with horizontal wavelengths of the order of 100 km are considerably larger than $f$, with typical periods not greater than a few hours [see Dewan (1997) and references therein]. From (36) we should thus have $\Phi(k) \ll \Psi(k)$ for a realistic frequency distribution. In Fig. 3 we see that the vorticity spectrum instead is a little bit larger in magnitude than the divergence spectrum. Therefore, it is not likely that the mesoscale spectrum is produced by inertia–gravity waves. If, on the other hand, the mesoscale spectrum were produced by a two-dimensional inverse energy cascade, as suggested by Lilly (1983), the divergence spectrum would be much smaller in magnitude than the vorticity spectrum. Instead, we find that these spectra are of the same order of magnitude. So, our analysis gives additional evidence suggesting that the mesoscale energy spectrum does not originate from an inverse energy cascade. The result that the divergence and the vorticity spectra are of the same order of magnitude in the mesoscale range [see Davis (1994)] who found that the divergence spectrum was totally dominating. However, it is consistent with the simulation results of Koshyk and Hamilton (2001) who found that energy is approximately equipartitioned between vortical and divergent modes in the mesoscale range.

Putting the left-hand side of (13) to zero and integrating, we find that for a gravity wave field

$$D_{LL} = \frac{\partial}{\partial p} (pD_{TT}).$$  \hspace{1cm} (37)$$

For future investigations it could also be of some interest to write down the corresponding relation

$$E_L(k_1) = -k_1 \frac{dE_T}{dk_1},$$  \hspace{1cm} (38)$$

where $E_L$ and $E_T$ are the one-dimensional power spectra of the longitudinal and transverse velocity components, $u_L$ and $u_T$, respectively, and $k_1$ is the one-dimensional wavenumber. It is straightforward to derive (38) from (37). In case of a power-law dependence

$$D_{LL} \sim D_{TT} \sim \rho^m, \quad E_L \sim E_T \sim k_1^{-m-1},$$  \hspace{1cm} (39)$$

we should have

$$\frac{D_{LL}}{D_{TT}} = \frac{E_L}{E_T} = m + 1.$$  \hspace{1cm} (40)$$

For a two-dimensional nondivergent field the corresponding relations hold with $D_{LL} \leftrightarrow D_{TT}$ and $E_L \leftrightarrow E_T$. Thus, the parameters $a_1$ and $a_2$ in (17) and (18) would be related as $a_1/a_2 = 5/3$ for a gravity wave field and as $a_1/a_2 = 3/5$ for a two-dimensional vorticity field. Instead, L99 found that they are related as $a_1/a_2 \approx 0.9$. Cho and Lindborg (2001) made calculations where the data were divided into four latitude bands and two altitude bands, one covering the upper troposphere and one covering the lower stratosphere. In all eight subsets the ratio $a_1/a_2$ took values close to or slightly smaller than unity, suggesting that the energy contents in vortical and divergent modes are of the same order of magnitude. As a matter of fact, Cho et al. (1999) did already study the ratio between the longitudinal and the transverse kinetic energy spectra, $E_L/E_T$, as a measure of the relative importance of divergent and vortical modes. They argued that gravity waves would give a value larger than unity for this ratio due to Doppler shifting. They calculated this ratio from aircraft wind data over the Pacific Ocean and obtained values close to unity. From this observation they concluded that the mesoscale dynamics cannot be dominated by gravity waves. Our analysis confirms this conclusion.

### 3. Conclusions

The fact that the vorticity and the divergence spectra are of the same order of magnitude in the mesoscale range rules out gravity waves as well as a 2D inverse energy cascade as the single cause behind the measured spectra. It is perhaps still possible to argue that the mesoscale motions can be a mixture between these two modes of motions or one of these two modes and some third type of motion. However, from a dynamical point of view we find such a hypothesis not very plausible. The functional form of the spectra and the results from the data analysis of Cho and Lindborg (2001) suggest that the spectra are generated by a dynamics that is strongly nonlinear. It does not seem very likely that two very different modes of motion can independently survive side by side in the presence of strong nonlinearities. It also seems unlikely that two independent modes of motions would produce spectra of the same functional form and the same order of magnitude. It is more likely that the vorticity and the divergence spectra are footprints of a single creature. What can that be?

It can be argued that the total amount of vertical vorticity carried by the mesoscale motions is larger than
the vertical vorticity carried by the large planetary vortices. From (31) we can calculate a typical value of the vertical vorticity in the mesoscale range as

\[ |\xi_2| \sim \sqrt{\int_{k_1}^{k_2} 1.57 q_1 k^{5/3} \, dk} \approx 7f > f, \quad (41) \]

where \( k_1 = 0.01 \, \text{cpkm} \) and \( k_2 = 0.5 \, \text{cpkm} \). Here we have used the empirical value of \( q_1 \) and the 45° latitude value of \( f \). The factor 7 is only interesting because it allows us to say that the typical value is not smaller than \( f \). Using the somewhat lower value, \( k_2 = 0.1 \, \text{cpkm} \), of the larger cutoff wavenumber we still get \( |\xi_2| \sim 2f > f \). We can therefore conclude that the mesoscale motions cannot properly be described within the quasigeostrophic approximation since this is based on the assumption that \( |\xi_2| \ll f \). This can also be argued from the fact that the divergence spectrum is of the same order of magnitude as the vorticity spectrum, while in the quasigeostrophic approximation the divergence spectrum is zero.

It seems plausible that the minimum in the vorticity spectrum around \( k = 0.01 \, \text{cpkm} \) marks the transition between two dynamic regimes. The minimum is rather broad, which makes it difficult to define a single wavenumber as the transition wavenumber. The data analysis of L99 shows that the \( r^{2/3} \) terms in (17) and (18) can be identified as subleading terms up to scales \( O(1000 \, \text{km}) \), and the two other terms can be identified as subleading terms down to scales \( O(100 \, \text{km}) \). Thus, the transition is gradual. The wavenumber at which the two terms in (31) are equal corresponds to a wavelength of about 250 km. For wavenumbers a little bit to the left of the minimum it is likely that the quasigeostrophic approximation should describe the dynamics reasonably well. This regime is associated with a downscale flux of potential enstrophy in Fourier space, manifesting itself as vortex filamentation in physical space. However, for scales smaller than 100 km, it is not likely that vortex filamentation is a dominant process, as also has been pointed out by Bartello (2000). It is not a bold guess that there is some type of instability associated with the transition seen in the vorticity spectrum at these scales.

An interesting candidate is the zig-zag instability discovered by Billant and Chomaz (2000a,b). Under the action of strong stratification vortices are found to slide apart and split up into layers with typical vertical wavelength \( \lambda \sim u/N \), where \( N \) is the Brunt–Väisälä frequency. This was demonstrated, theoretically (Billant and Chomaz 2000a), as well as experimentally (Billant and Chomaz 2000b). The energy feeding the mesoscale cascade from its large-scale end may originate from quasigeostrophic vortices undergoing some similar type of instability in which they break up into layers. Billant and Comaz (2001) also presented an elegant scaling analysis of the Boussinesq equations in the limit of strong stratification and showed that nonlinear terms always play an important role, even though the stratification is very strong. Recently, Waite and Bartello (2004) and Lindborg (2005, 2006) carried out numerical simulations of the Boussinesq equations and confirmed that the Billant–Chomaz scaling is valid. Lindborg (2006) also showed that the combined action of layering and shear instabilities leads to a downscale energy cascade with an associated \( k^{-5/3} \) spectrum. A downscale energy cascade with an associated horizontal energy spectrum of the form \( k^{-5/3} \) was also obtained in direct numerical simulations of strongly stratified turbulence carried out by Riley and deBruynKops (2003). This is a kind of motion that in some respects is similar to what Lilly (1983) called stratified turbulence. Strong nonlinearities and layer formation are fundamental aspects of the motion, which also carries a substantial amount of vertical vorticity. However, the energy flux is in the direction from large to small scales, which is opposite to what Lilly suggested. Nevertheless, stratified turbulence is still a very appropriate name, as pointed out by J. Riley (2005, personal communication). The analysis presented in this paper provides additional support for the hypothesis of stratified turbulence associated with a forward energy cascade.

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