A Toy Model of the Instability in the Equatorially Trapped Convectively Coupled Waves on the Equatorial Beta Plane

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ABSTRACT

The equatorial atmospheric variability shows a spectrum of significant peaks in the wavenumber–frequency domain. These peaks have been identified with the equatorially trapped wave modes of rotating shallow water wave theory. This paper addresses the observation that the various wave types (e.g., Kelvin, Rossby, etc.) and wavenumbers show differing signal strength relative to a red background. It is hypothesized that this may be due to variations in the linear stability of the atmosphere in response to the various wave types depending on both the specific wave type and the wavenumber. A simple model of the convectively coupled waves on the equatorial beta plane is constructed to identify processes that contribute to this dependence. The linear instability spectrum of the resulting coupled system is evaluated by eigenvalue analysis. This analysis shows unstable waves with phase speeds, growth rates, and structures (vertical and horizontal) that are broadly consistent with the results from observations. The linear system, with an idealized single intertropical convergence zone (ITCZ) as a mean state, shows peak unstable Kelvin waves around zonal wavenumber 7 with peak growth rates of \(-0.08\) day\(^{-1}\) (e-folding time of \(-13\) days). The system also shows unstable mixed Rossby–gravity (MRG) and inertio-gravity waves with significant growth in the zonal wavenumber range from \(-15\) (negative indicates westward phase speed) to \(+10\) (positive indicates eastward phase speed). The peak MRG \(n = 0\) eastward inertio-gravity wave (EIG) growth rate is around one-third that of the Kelvin wave and occurs at zonal wavenumber 3. The Rossby waves in this system are stable, and the Madden–Julian oscillation is not observed. Within this model, it is shown that in addition to the effect of the ITCZ configuration, the differing instabilities of the different wave modes are also related to their different efficiency in converting input energy into divergent flow. This energy conversion efficiency difference is suggested as an additional factor that helps to shape the observed wave spectrum.

1. Introduction

If the satellite record of outgoing longwave radiation (OLR; Liebmann and Smith 1996)—a good proxy for deep tropical convection (see, e.g., Arkin and Andanuy 1989)—within \(15^\circ\) of the equator is analyzed in zonal wavenumber–frequency space (Wheeler and Kiladis 1999, hereafter WK99) it shows a number of statistically significant peaks (Fig. 1). These propagating disturbances form a large part of the tropical synoptic-scale variation, organizing individual convective elements (typically 100 km across, persisting for a few hours) on large spatial (thousands of kilometers) and temporal (days) scales (e.g., Chang 1970; Nakazawa 1988). The wave activity peaks have been identified with the equatorially trapped waves of shallow water theory (see, e.g., WK99; Yang et al. 2007). Many of the properties of these waves are well described by rotating shallow water wave theories (Matsumo 1966). Here we attempt to address the observation that the various wave types (e.g., Kelvin, Rossby, etc.) and wavenumbers show differing amplitudes. It is hypothesized that this is due in part to the (linear) stability of the atmosphere to perturbations of the various wave types depending on the specific wave type and the wavenumber in question.

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We explore this using the simple model of the convectively coupled waves using the first two baroclinic vertical modes, midtropospheric moisture, and the subcloud layer developed in Kuang (2008b, hereafter K08b) extended to the equatorial beta plane. This model is part of a continuing effort in this field to construct models of minimal complexity to identify the basic instability mechanisms of these waves. There is a long history of such simple models, going back at least as far as 1969 (e.g., Yamasaki 1969; Hayashi 1970; Lindzen 1974; Neelin et al. 1987; Wang 1988; Mapes 2000; Majda and Shefter 2001; Khouider and Majda 2006a; Raymond and Fuchs 2007). The early models emphasized a first baroclinic structure that is of one sign over the full depth of the troposphere (e.g., Wang 1988; Neelin et al. 1987; Emanuel 1987). The importance of the second baroclinic component was recognized and included in the model of Mapes (2000), and an instability mechanism involving the second baroclinic temperature anomaly and deep convective and stratiform heating was identified (Mapes 2000). More recently, Khouider and Majda (2006a) included the interaction between free tropospheric moisture and deep convective and congestus heating in their model and found that moisture plays a major role in destabilizing the system. This is an important finding. However, the actual instability mechanism was not clearly identified. Khouider and Majda (2006a) described the process of a dry episode causing more congestus heating, which moistens and preconditions the free troposphere, leading to deep convection, which then returns the free troposphere to a dry condition. This description, while correct, does not reveal how the initial perturbation gets amplified and thus falls short of an instability mechanism. This motivated the study of K08b, which continued the effort by Khouider and Majda (2006a) but sought to develop conceptually simple treatments of convection and to identify the basic instability mechanisms.

In K08b, the convective parameterization is based on the quasi-equilibrium (QE) concept—namely, that convection responds quickly to the changes in the large-scale flow and so can be considered to be in statistical equilibrium with the flow (Arakawa and Schubert 1974; Emanuel et al. 1994) by keeping the convective available potential energy (CAPE) constant for parcels rising from the boundary layer to the midtroposphere. The tropospheric moisture is included as an input to the convection calculation to represent the effect of entrainment of the drier tropospheric air into a rising convective cloud. This acts as a control on the depth of convection because with a moister midtroposphere, updrafts lose less buoyancy from the entrainment of en-

![Fig. 1. OLR power divided by background spectrum for signals that are (a) antisymmetric and (b) symmetric about the equator. The power spectrum is averaged over the region 15°S to 15°N and is constructed from National Oceanic and Atmospheric Administration (NOAA) daily OLR data running from June 1974 to September 2007. Contours begin at 1.1 and are drawn at intervals of 0.1, only for statistically significant values. Shallow water dispersion curves drawn are for equivalent depths of 8, 12, 25, 50, and 90 m, in order of increasing frequency. (After WK99).](image-url)
environmental air and can reach higher to form deep convection and subsequent stratiform rain; thus, for a given amount of lower-tropospheric convective heating, the upper-tropospheric heating will be greater.

Analysis in K08b identified a moisture–stratiform instability, in addition to the stratiform instability of Mapes (2000). To see it in its most basic form (as illustrated in Fig. 11 of K08b), consider the horizontal propagation of a sinusoidal wave with a second-mode temperature structure in the vertical. The propagation of the wave modulates deep (first baroclinic) convective heating by perturbing the statistical equilibrium between the lower troposphere and the subcloud layer. More specifically, deep convection is enhanced behind a warm lower-tropospheric temperature anomaly (as large-scale ascent cools the lower troposphere) and is reduced behind a cold lower-tropospheric temperature anomaly. The modulation on deep convection peaks a quarter-cycle behind temperature anomaly peaks. The combined effect of an enhanced deep convective heating and its induced vertical advection of moisture is to moisten the midtroposphere, leading to deeper (i.e., more stratiform) convection in a region where the initial temperature anomalies are positive in the upper troposphere and negative in the lower troposphere, amplifying the initial perturbation. It was also found that in addition to the moist convective damping effect (MCD; e.g., Emanuel 1993; Neelin and Yu 1994) whereby the finite response time of convection damps high-frequency waves, the tendency of the second baroclinic mode convective heating to reduce existing midtropospheric moisture perturbations tends to reduce the instability at low wavenumbers. These two effects were suggested as contributors to the wavenumber selection in convectively coupled waves.

Extension of the model of K08b to the equatorial beta plane allows investigation of the equatorial Rossby (ER), mixed Rossby–gravity (MRG) and inertio-gravity (IG) wave types. These waves have significantly different convection/temperature/wind structures and observed amplitudes from the Kelvin-like waves studied in most previous work that are confined to a line along the equator. Our emphasis will be on identifying the basic factors that contribute to the differing instability of the different wave types and wavenumbers.

The linear instability spectrum of the resulting coupled system is found by eigenvalue analysis. We use realistic model parameters estimated from Cloud System Resolving Model (CSRM) studies of the convectively coupled waves. The instability analysis produces unstable waves with phase speeds, growth rates, and structures (vertical and horizontal) that compare reasonably well with observations.

We will show that the instability’s dependence upon wave type is related to the projection of mean-state intertropical convergence zone (ITCZ) heating onto the mode (as has been suggested in the past; e.g., Takayabu 1994; WK99) and to an energy feedback efficiency effect. This latter effect stems from the differing roles of divergent and rotational flows in the convective parameterization and the differing efficiencies with which the waves convert wave energy input from convective heating into divergent wind. We investigate this by analytical investigation of a simplified case to explore the details of the mechanisms responsible for wavenumber and wave type selection in the instability spectrum. The simplified case is chosen to remove the already understood mechanisms that exist within the model. Considering the instability in terms of the energy flow between the geopotential and wind fields shows that the instability strength is strongly influenced by the differing efficiencies of conversion from the energy input due to convective heating to divergent winds.

The paper is organized as follows: Section 2 provides a brief description of the model used in this paper. Section 3 describes the results of an eigenvalue analysis conducted to determine the spectrum of unstable waves present in our model, as well as tests of the sensitivity of the system to ITCZ configuration and to the parameters used. Section 4 describes a simplified limiting case of our model and an energy flow–based feedback mechanism that explains the differing instabilities of the different wave modes. This is followed by a summary and discussion (section 5).

2. Model description

a. Model equations

Our basic equations are a simple extension of the model of K08b to the equatorial beta plane. The relation to previous models was discussed in K08b and in the introduction. The model equations can be derived from the anelastic primitive equations for an atmosphere linearized about a resting mean state:

\[ \partial_x u' - \beta y v' = -\frac{1}{\rho} \partial_x p', \]  
\[ \partial_t v' + \beta y u' = -\frac{1}{\rho} \partial_z p', \]  
\[ \partial_z (\bar{\rho} u') + \partial_y (\bar{\rho} v') + \partial_x \bar{\rho} \frac{\partial y}{\partial y} = 0, \]  
\[ \partial_t T' + w' \left( \partial_z T + \frac{\bar{g}}{c_p} \right) = J' - \epsilon T', \]  
\[ \partial_z p' = \frac{\bar{g}}{\partial_x T', \]  

where $\bar{\rho}$ is the mean density, $\bar{g}$ is the mean gravity, and $c_p$ is the specific heat at constant pressure.
where $\rho$ and $T$ represent the density and temperature; $u$, $v$, and $w$ represent the velocities in the zonal, meridional, and vertical directions, respectively; and $J$ represents the convective heating. Overbars represent mean quantities and primes denote deviation from this mean state (assumed to be small). A linear damping on temperature, denoted by $\varepsilon$, represents radiative cooling/heating toward the mean temperature profile. The buoyancy frequency $N$ is given by

$$N^2 = \frac{g}{T} \left( \partial_z T + \frac{g}{c_p} \right),$$

and $\beta y$ is the linear expansion of the Coriolis parameter $f$ for small displacements away from the equator.

Then, with the additional assumptions of a rigid lid and constant buoyancy frequency, we can expand Eqs. (1)–(5) in terms of the baroclinic modes numbered $j$, thus:

$$\frac{g\bar{\rho}}{T} \left[ T^\prime, J^\prime, w^\prime \left( \partial_z T + \frac{g}{c_p} \right) \right] = \sum_j \left( \hat{T}, J, w \right)(x, y, t) \left( \frac{\pi}{2} \right) \sin \left( \frac{\pi \bar{z}}{H_T} \right) \quad \text{and}$$

$$\left( \hat{u}^\prime, \hat{v}^\prime, \hat{p}^\prime \right) = \sum_j \left( \bar{u}, \bar{v}, \bar{p} \right)(x, y, t) \frac{H_T}{2j} \cos \left( \frac{\pi \bar{z}}{H_T} \right),$$

where $H_T$ is the height of the lid at the top of the troposphere. We note that despite the above derivation, the use of these modes is based on empirical evidence rather than on their being normal modes of the dry atmosphere because the atmosphere does not possess a rigid lid at the tropopause—a fact emphasized in the context of convectively coupled waves by, for example, Lindzen (1974, 2003) and Raymond and Fuchs (2007).

Substituting the expansion forms [Eqs. (7) and (8)] into our perturbation velocity equations [Eqs. (1)–(3)] yields

$$\partial_t \mathbf{u}_j = -\beta y \hat{k} \times \mathbf{u}_j + \nabla_H T_j \quad \text{and}$$

$$w_j = -c_j^2 \nabla_H \cdot \mathbf{u}_j,$$

where $T$ is the temperature perturbation scaled to act similarly to a geopotential perturbation and $\nabla_H$ denotes the horizontal gradient vector operator $\nabla_H = (\partial_x, \partial_y)$. The wave speeds $c_j$ are given by

$$c_j = \frac{NH_T}{\pi}.$$

We can also make a modal expansion of the thermodynamic Eq. (4) by combining it with Eq. (10), generating

$$\partial_t T_j = -c_j^2 \nabla_H \cdot \mathbf{u}_j = J_j - \varepsilon T_j$$

For the rest of this paper, we restrict ourselves to $j = 1, 2$—the first two baroclinic modes, which are sufficient to explain most of the convective activity (Mapes and Houze 1995). The heating anomalies associated with these two modes, $J_1$ and $J_2$, represent the convective heating broken into deep convective and congestus/stratiform components, respectively. The modal decomposition of convection heating is based on cloud physics of convection. We note that the work by Raymond and Fuchs (2007) showed that one can produce a boomerang temperature structure with a first baroclinically convective heating profile and a radiating upper boundary condition, so the question of whether the two-mode structure observed in convectively coupled waves arises from cloud physics of convection or large-scale dynamics remains a matter of debate.

For the purposes of the convective parameterization, the heating is rewritten in terms of lower- ($L$) and upper- ($U$) tropospheric heating anomalies:

$$L = \frac{1}{2} (J_1 + J_2) \quad \text{and}$$

$$U = \frac{1}{2} (J_1 - J_2).$$

The total upper-tropospheric heating is considered to be a fraction of the lower-tropospheric heating:

$$\frac{U_0 + U}{L_0 + L} = r_0 + r_q q,$$

where $U_0$ and $L_0$ are the mean-state heating and $r_0$ is the mean-state ratio between $L_0$ and $U_0$; $q$ is the midtropospheric moisture anomaly and is evolved according to

$$\partial_t q = -a_1 c_1^2 \nabla_H \cdot \mathbf{u}_1 - a_2 c_2^2 \nabla_H \cdot \mathbf{u}_2 - d_1 J_1 - d_2 J_2,$$

where $a_j$ and $d_j$ give the midtropospheric moisture tendencies due to vertical advection and convection, respectively. We have neglected the effect of the horizontal advection of moisture in our formulation for simplicity and recognize that its effect should be evaluated in the future.
In a region of the midtroposphere that is anomalously moist with positive $q$, the entrained air is moister than average and the associated evaporative cooling is smaller, leading to relatively increased buoyancy of parcels in this region. This in turn leads to parcels rising to greater altitude and heating the upper troposphere more. As parcels rise to form deep convection, the subsequent stratiform heating further enhances heating in the upper troposphere relative to the lower troposphere. The opposite holds for a region with an anomalously dry midtroposphere.

The lower-tropospheric heating is considered to be relaxing quickly (over a time scale $\tau_L$, typically a few hours) toward the quasi-equilibrium situation given by the constancy of lower-troposphere nonentraining CAPE. In our context, quasi-equilibrium will be enforced by keeping the difference between the boundary layer moist static energy (MSE) $h_b$ and the vertically averaged lower-tropospheric saturation moist static energy ($h^*$) approximately constant. Note that $h_b$ and ($h^*$) both vary under the action of convection—a positive convection anomaly dries and cools the boundary layer through the action of downdrafts and turbulent transport and modifies the lower-tropospheric saturation MSE primarily (in our model) by heating the atmosphere—to reach a consistent equilibrium state. Thus, for equilibrium to be reached instantaneously,

$$\partial_t h_b = \langle \partial_t h^* \rangle_{LT}, \tag{17}$$

where LT is the region of the lower troposphere in which this QE formulation is assumed to hold, extending vertically up to the level where the effect of entrainment begins to become important. We shall neglect the surface flux anomalies in this paper and write

$$\partial_t h_b = -b_1 J_1 - b_2 J_2. \tag{18}$$

The term $b_j$ describes the reduction of boundary layer moist static energy by the convection $J_j$.

Meanwhile, we also assume that the average saturation moist saturation energy in the lower troposphere can be well approximated by a linear combination of the two baroclinic mode temperature anomalies:

$$\langle \partial_t h^* \rangle_{LT} = \partial_t [F(\gamma T_1 + (1 - \gamma) T_2)]. \tag{19}$$

Here, $F$ is a proportionality constant relating the change in lower-troposphere temperature to the change in moist static energy in the same region, and $\gamma$ describes the relative influence of the two modes on the temperature of the lower troposphere. It is, loosely speaking, equivalent to the fractional depth of the troposphere through which this QE formulation can be assumed to hold before entrainment becomes an important effect.

Substituting these two relations [(18) and (19)] into Eq. (17) allows us to quantify the quasi-equilibrium in terms of our model:

$$-b_1 J_1 - b_2 J_2 = F[\gamma \partial_t T_1 + (1 - \gamma) \partial_t T_2]. \tag{20}$$

By expanding this equation using Eqs. (12)–(15), we can find a lower-tropospheric heating $L_{eq}$ that satisfies the quasi-equilibrium assumption for the large-scale variables ($q$, $u$, and $T$):

$$L_{eq} = \frac{A L_0 q}{B} - \frac{F \gamma c_2^2}{B} (\nabla_h \cdot \mathbf{u}_1 - \frac{e}{c_1^2} T_1)$$

$$- \frac{F (1 - \gamma) c_2^2}{B} (\nabla_h \cdot \mathbf{u}_2 - \frac{e}{c_2^2} T_2), \tag{21}$$

where

$$B = F + (b_1 + b_2) - A (r_q q + r_0) \quad \text{and} \quad \tag{22}$$

$$A = (b_2 - b_1) + F (1 - 2\gamma). \tag{23}$$

We can improve the realism of the model by assuming that the convection does not respond to changes in the large-scale environment instantly. Rather, we allow $L$ to relax toward $L_{eq}$ over a short time scale $\tau_L$:

$$\partial_t L = \frac{1}{\tau_L} (L_{eq} - L). \tag{24}$$

By introducing the meridional velocity fields $v_j$ and the Coriolis force terms $\beta y k \times \mathbf{u}_j$, this extension prevents the simple reduction of the dry part of the prognostic equations to temperature fields that occurs in K08b, leading to the presence of velocity fields in the equations for $L_{eq}$ (21) and $q$ (16) [cf. his Eqs. (23) and (11)].

b. Parameter choices

As described in K08b, our parameter values, such as $a_j$, $b_j$, $d_j$, $F$, $\gamma$, $r_0$, $r_q$, $e$, and $\tau_L$, are estimated from the CSRM simulations of Kuang (2008a) which showed spontaneously developing convectively coupled waves. The estimation is conducted by projecting the CSRM results onto the vertical structure of our modes and interpreting regression results in the framework of our convective parameterization. We consider this approach to be somewhat more rigorous than tuning our model to match observations, as well as more efficient given the pre-existence of the CSRM results.

Parameter values are given in Table 1 and are essentially identical to those used by K08b. The units of the

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1 We would like to note that, unknown to Kuang (2008a), a similar modeling strategy was used by Brown and Bretherton (1995), although their method is more applicable to a convective parameterization than to CSRM.
parameters are based on measuring energies in the equivalent temperature change (in K) for 1 kg of dry air gaining that amount of energy. Moisture content is likewise measured by the temperature change that would result if all the stored latent heat in a volume of air were converted to thermal energy in an identical, but dry, air mass.

### 3. Beta-plane linear instability analysis

In WK99, the observations are of waves of finite amplitude. If the waves are indeed unstable, in the observations the linear growth of the unstable waves is balanced by nonlinear terms and an equilibrium is reached. In this case, the equilibrium is of a statistical nature, with waves growing and dying as they move around and enter regions of differing forcing (due to the changing background state and wave–wave interactions), and is also time varying on various scales. In this paper we investigate the linear regime of infinitesimal waves with a zonally and temporally fixed background state. We see this as a first step toward an understanding of the spectrum. Further investigation of the nonlinear effects is planned. Also planned is an investigation of other interpretations of the observations, such as stable waves continually excited by stochastic or extratropical forcing (e.g., Zhang and Webster 1992; Hoskins and Yang 2000) in the context of this model.

#### a. Linearization

Calculation of the instability spectrum assumes that our system is growing from infinitesimal anomalies, so we can remove any nonlinear terms from the equation set. In our case, linearization is simply achieved by ignoring the first term in the second parentheses from the definition of \( B \) [Eq. (22)],

\[
B = F + (b_1 + b_2) - Ar_D, \tag{25}
\]

and linearizing Eq. (15) into

\[
U = L_0\frac{\partial q}{\partial y} + r_0L, \tag{26}
\]

Our linearized model consists of Eqs. (9), (12), (13), (14), (16), (21), (23), (24), (25), and (26).

Our basic state, represented by the parameter \( L_0(y) \), is an idealization of the zonal mean state. As shown in Fig. 2, the control profile consists of a Gaussian peak (of amplitude 0.7 K day\(^{-1}\)) in mean-state convection centered on the equator, with a width of 5°, above a small, nonzero uniform offset that extends to the edges of the domain.

The linear system is then Fourier transformed to zonal wavenumber–frequency space and the resulting eigenvalue problem is solved on a meridionally discrete domain extending approximately 6000 km to the north and south, with 181 regularly spaced points. This large domain is utilized to reduce the influence of any edge-related errors and can be implemented without significant computational cost. The resulting complex frequency eigenvalues can be separated into the real part \( \omega \), which gives the modal frequency, and the imaginary part \( \Gamma \), which gives the modal growth rate. The corresponding eigenvectors give the modal structure in the prognostic fields (\( u_1, u_1, \) etc).

#### b. Results

The results from the linear analysis are shown in Figs. 3 and 4. Unstable waves are indicated in the frequency–wavenumber diagram, with the marker radius proportional to the growth rate. Unstable modes are assigned to an equivalent dry class (dispersion curve ranges shown dashed) based on their position in Fig. 3, relative to the dry dispersion curves indicated. Investigation of the modal structure (discussed below) shows that these identifications are warranted. The system contains unstable modes corresponding to Kelvin waves, \( n = 1 \) and

### Table 1. Parameter values as used in the control cases discussed in section 3.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Normative values</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_1, b_2 )</td>
<td>1.0, 2.0</td>
<td>Tendency for reduction in boundary layer moist static energy per unit heating ( J_1 ) and ( J_2 )</td>
</tr>
<tr>
<td>( a_1, a_2 )</td>
<td>1.4 K, 0.0 K</td>
<td>Increase in ( q ) tendency per unit vertical velocity ((\partial \bar{u} + \hat{a}_y)) by advection</td>
</tr>
<tr>
<td>( d_1, d_2 )</td>
<td>1.1, −1.0</td>
<td>Decrease in ( q ) tendency per unit heating ( J_1 ) and ( J_2 )</td>
</tr>
<tr>
<td>( r_0 )</td>
<td>1.0</td>
<td>Background mean ( U/L ) ratio</td>
</tr>
<tr>
<td>( r_q )</td>
<td>1.0 K(^{-1})</td>
<td>Linear dependence of ( U/L ) ratio on moisture deficit</td>
</tr>
<tr>
<td>( F )</td>
<td>4</td>
<td>Ratio between moist static energy and temperature in the lower-tropospheric “non-entraining” convection region</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0.5</td>
<td>Relative contribution of the first mode temperatures to the lower-tropospheric temperature anomaly</td>
</tr>
<tr>
<td>( \tau_L )</td>
<td>2.0 h</td>
<td>Adjustment time to approach QE over the lower troposphere</td>
</tr>
<tr>
<td>( c_1, c_2 )</td>
<td>(50.0, 25.0) m s(^{-1})</td>
<td>Dry gravity wave speeds for the first and second modes</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>0.15 day(^{-1})</td>
<td>Temperature anomaly damping coefficient</td>
</tr>
<tr>
<td>( \sigma_L )</td>
<td>5.0°</td>
<td>Meridional width of ( L_0 ) profile, representing the ITCZ structure</td>
</tr>
</tbody>
</table>
FIG. 2. Background state lower-tropospheric convective heating used in the control instability experiment. The background state is an idealization of the zonal mean state, with an ITCZ-like Gaussian peak over the equator and a small mean convection over the rest of the domain.

FIG. 3. Eigenfrequencies for the linear system with an idealized zonal mean state. Data point diameter corresponds linearly to growth rate. Stable modes are omitted. Lines show theoretical dispersion curves for dry waves with equivalent depths of 40 and 60 m (in order of increasing frequency).
Kelvin waves are the most unstable wave type. Wavelengths between about 13,000 and 2,000 km (planetary wavenumbers 3 and 20) are unstable, with a maximum growth rate of \( \sim 0.1 \text{ day}^{-1} \) occurring at wavelengths of around 5,700 km (wavenumber 7). These waves have phase speeds between 20 and 25 m s\(^{-1}\), slightly slower than the second baroclinic wave, with the speed deficit becoming stronger as the wavenumber decreases. The growth rate curve rises sharply between wavenumbers 3 and 7 and then decays more slowly as the wavenumber is further increased to 17.

Figure 5 shows the structure of this mode, along with vertical structure of temperature. The first baroclinic mode is approximately in quadrature with the second, leading to a tilted vertical temperature structure, similar to observations. Zonal winds blow in the standard dry wave directions. The meridional wind has a small nonzero component. This is not unexpected because the moist physics that couples our system mixes the modes together so that the convectively coupled “Kelvin” wave observed, although dominated by a Kelvin wave profile, will still contain a small amount of other waves in superposition with it. The wind fields are plotted for the lower troposphere at \( z = 0 \).

Figure 3 shows unstable MRG waves, approximately between wavenumbers \(-15\) and 10, with a peak growth rate of \( \sim 0.03 \text{ day}^{-1} \) occurring at wavenumber 3 (\( \sim 13,000 \) km wavelength). The phase speed of the MRG waves also appears to drop below the dry speed as the wavenumber decreases, although this become less visible at large negative wavenumbers as the dispersion curves converge. The modal structure of the (eastward) planetary wavenumber-7 mixed Rossby–gravity wave is shown in Fig. 6.

The system also shows instability in the \( n = 1 \) (eastward and westward) IG waves, with a bias toward westward IG (WIG) waves.

The \( n = 2 \) and IG modes are only marginally unstable in this model. We consider growth rates below a few percent of the peak values to be only marginally unstable and far too dependent on assumptions about parameters for us to draw solid conclusions about them. The sensitivity of our model to parameter variation is explored in section 3d below.

The reconstruction of the planetary wavenumber-7 eastward IG (EIG) wave is shown in Fig. 7. In Fig. 8, the \( k = -7 \) WIG wave structure is also shown for comparison. These modes also show horizontal and vertical structures that compare well with observations.

All the reconstructed modes show first- and second-
Fig. 5. Structure of the coupled Kelvin wave at planetary wavenumber 7. (top), (middle) Temperature (contours) and wind anomalies (vectors) for the (top) first and (middle) second baroclinic modes; (bottom) reconstructed vertical temperature anomaly above the equator.

Fig. 6. Structure of the coupled MRG $n = 0$ EIG wave at planetary wavenumber 7. (top), (middle) Temperature (contours) and wind anomalies (vectors) for the (top) first and (middle) second baroclinic modes; (bottom) reconstructed vertical structure at $y = 1000$ km (north of the equator), approximately the peak of the MRG profile.
mode winds with similar amplitudes, broadly consistent with observations (see, e.g., Haertel and Kiladis 2004), as well as a vertically tilted structure, again generally consistent with observations (see, e.g., Straub and Kiladis 2002). Most noticeably absent from the instability spectrum are significant equatorial Rossby waves and the Madden–Julian oscillation (MJO). The complete absence of the MJO signal indicates, not unexpectedly,
that the current simple model is missing the physics responsible for the MJO. The Rossby waves exist in our system but are barely unstable within a range of parameter choices around our nominal values (discussed below in section 3d). Because both these missing wave types are at similar, low frequencies, the model may have either a too strong damping at low frequencies or may be missing some other physics that destabilizes waves at these frequencies.

Comparison of Figs. 3 and 4 with Fig. 1 shows that

(i) peak Kelvin wave growth occurs at a wavelength comparable to the most active waves in the observations;
(ii) the ranges of wavenumber and wave type showing instability in our model match the ranges of waves that are observed in the OLR signals;
(iii) the Kelvin wave dispersion curves shift to higher equivalent depth at higher wavenumbers, opposite to the trend observed (the reason for this discrepancy is unclear);
(iv) the model possesses a preference for unstable westward $n = 1, 2$ IG modes (as opposed to eastward $n = 1, 2$ IG), similar to observations; and
(v) The waves possess a tilted vertical structure that is generally consistent with the observations, with low-level temperature anomalies preceding a brief period of deep anomaly followed by a period of “stratiform” temperature anomaly.

c. Varying ITCZ configurations

To address the question of whether the dominance of Kelvin waves is due to the amount of overlap between the mean-state heating and the Kelvin wave profile being greater than that for other waves, we have conducted several additional experiments with different ITCZ configurations in our mean state.

Figure 9 shows a double-ITCZ mean state, with the two peaks approximately a second baroclinic deformation radius north and south of the equator, collocated approximately with the MRG temperature anomaly peaks determined from Fig. 6. Unstable modes for this case are shown in Fig. 10. Although moving the peak heating away from the equator does reduce the overall growth rate of the unstable Kelvin waves and increases that of the unstable MRG waves, the growth rates of the Kelvin waves are still strongest. The MRG growth rates surpass the $n = 1$ IG wave in the small wavenumber region of the spectrum. The ER waves are slightly more unstable, but still not significant. The reconstructed mode structures (not shown) are not significantly different from the single ITCZ case.
Two further experiments (not shown) placed (i) a double ITCZ at the off-equatorial peaks of the \( n = 1 \) IG temperature anomaly field and (ii) a single ITCZ at the northern MRG peak temperature anomaly position. Although the wave that had the greatest overlap with the convective envelope showed some enhancement of growth rate, neither of these experiments altered the general dominance of the Kelvin waves.

Another aspect of the observed activity profile that stands out as possibly being related to ITCZ configuration is the asymmetry between eastward and westward inertia-gravity waves—the observed activity of WIG waves is much stronger than that of EIG waves. It has been suggested that this is primarily because the latitudinal structure of the EIG waves is significantly more complex and a greater proportion of the EIG temperature anomaly is located away from the equator and the ITCZ (see, e.g., Takayabu 1994; WK99). To explore this possibility, a further experiment is shown in which the ITCZ is replaced with a meridionally constant mean-state heating of 0.75 K day\(^{-1}\).

The results from the linear analysis are shown in Fig. 11. Kelvin waves are still the most unstable wave type. Wavelengths between about 13 000 and 2300 km (planetary wavenumbers 3 and 17) are unstable, with a maximum growth rate of \( \sim 0.11 \) day\(^{-1}\) occurring at wavelength of around 5000 km (wavenumber 8). Unstable MRG waves (also referred to as \( n = 0 \) EIG waves for the \( k > 0 \) part of the spectrum) appear in the region approximately between wavenumbers \( -5 \) and \( 15 \), with a peak growth rate of \( \sim 0.05 \) day\(^{-1}\) occurring at wavenumber 7 (\( \sim 5700 \) km wavelength). The system also shows instability in the (eastward and westward) IG waves, with a significant instability bias toward WIG waves. Because of the extreme poleward extent of the mean-state convection in this case, there is significant instability in high meridional wavenumber (IG) modes.

In this situation, the projection of the ITCZ heating profile onto the EIG and WIG structures should be essentially the same. As can be seen, some but not all of the east–west asymmetry has been removed. It will be demonstrated below that some of this asymmetry is due to the differing efficiencies with which the eastward and westward IG waves convert convective heating into more convective heating.

\(d\). Parameter sensitivity

As mentioned above, the model does display some sensitivity to parameter variations. Figure 12 shows the growth rate curves for sensitivity experiments conducted with the damping \( \varepsilon \) and the convective equilibrium relaxation time \( \tau_L \) increased and decreased by 10%. The instability for the Kelvin, equatorial Rossby, mixed Rossby–gravity and \( n = 1 \) inertia-gravity waves
are seen to be generally robust, with the location of the peak instabilities in these cases remaining approximately the same. The peak growth rates do change slightly, as expected, when the damping is varied. However, the $n = 2$ and higher IG modes can be observed to vary greatly in their instability character in this range of parameters. As would be expected from the moist convective damping theory, decreasing the convective relaxation time decreases the moist convective damping effect, which then decreases the damping applied to high-wavenumber instabilities, leading to an increase in instability for the $n = 2$ and $n = 3$ IG modes. For the same reason, the Kelvin wave peak instability can be seen to shift very slightly to higher frequencies when $\tau$ is decreased.

A similar sensitivity test was applied to all of the semiempirical parameters contained within our model. Generally, for a 10% change in parameter value, variations in the spectrum were similar in character and amount to those discussed for damping above (not shown). The parameters $r_0$ and $c_2$ showed interesting sensitivity:

(i) When $r_0$ is varied, the position of the peak instability shifts to higher wavenumber for higher $r_0$. The reverse holds for reduced $r_0$.

(ii) A variation in the $c_2$ parameter leads to a similar variation in the speed of the unstable “Kelvin” modes. Any unstable mode speed variation caused by modification of $c_1$ is negligible. This indicates that the speed of the unstable modes is largely determined by the second mode.

The equatorial Rossby waves are still not observed under any of these parameter variations.

These results are similar to the more detailed sensitivity study conducted for the 1D version of this model included in K08b.

e. Midtropospheric moisture versus free tropospheric moisture

As in K08b, we have used midtropospheric moisture instead of the vertically averaged free tropospheric moisture $\langle q \rangle$ in Eq. (16). This choice was based on the notion that $\langle q \rangle$ is not necessarily the most relevant quantity as a control on convection, which may be tied more strongly to certain weighted averages of free tropospheric moisture. We have used midtropospheric moisture although other and potentially better choices are certainly possible. The use of the column moist static energy conservation constraint was not necessary in our case because the effects of convection on the midtropospheric moisture and on the boundary layer...
moist static energy were estimated separately from the CSRM simulations. It is also not an essential element for identifying the basic instability mechanisms.

However, for other issues such as understanding the column MSE budget of the system, it is desirable to use so that the column MSE conservation constraint is automatically satisfied, as done in Khouider and Majda (2006). Let us (in this subsection only) replace \( q_{\text{mid}} \) with \( q \) in Eq. (15), so that the column MSE budget equation can be written as

\[
\begin{aligned}
\partial_t [ \langle q \rangle + & T_1 M_T + h_b M_b ] = M_T [(1 - \langle a_1 \rangle)c_1^2 \nabla_H \cdot \mathbf{u}_1 - \langle a_2 \rangle c_2^2 \nabla_H \cdot \mathbf{u}_2] + [(1 - \langle d_1 \rangle)M_T - b_1 M_b] J_1 \\
+ & [ - \langle d_2 \rangle M_T - b_2 M_b ] J_2,
\end{aligned}
\]

where \( M_T \) and \( M_b \) are the mass in the troposphere and the boundary layer, respectively, and the other parameters retain their meanings defined above, except with a \( \langle \rangle \) to denote that they are associated with \( \langle q \rangle \). Column MSE conservation requires that

\[
(1 - \langle d_1 \rangle)M_T = b_1 M_b \quad \text{and} \quad (28)
\]

\[
-\langle d_2 \rangle M_T = b_2 M_b \quad \text{and} \quad (29)
\]

so that one can therefore infer values of \( \langle d_1 \rangle \) and \( \langle d_2 \rangle \) from \( b_1 \) and \( b_2 \). We estimate that \( M_T/M_b \) is \(-10\), which gives \( \langle d_1 \rangle \sim 0.9 \) and \( \langle d_2 \rangle \sim -0.2 \). The CSRM results of Kuang (2008a) can also be viewed in this framework following the procedure described in the appendix of K08b. Regression of the column MSE tendencies against vertical velocities of the two vertical modes give a value for \( 1 - \langle a_1 \rangle \) that is effectively zero and a value of \( \langle a_2 \rangle \sim 0.4 \). This indicates, allowing for uncertainties

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**Fig. 12.** Damping sensitivity studies. As in Fig. 2, but for modification of the damping parameters as follows: (a) \( \epsilon \rightarrow 1.1 \times \epsilon \), (b) \( \tau_l \rightarrow 0.9 \times \tau_l \), (c) \( \tau_l \rightarrow 1.1 \times \tau_l \), and (d) \( \tau_l \rightarrow 0.9 \times \tau_l \). Peak growth rates for each case are (a) \( 10 \times 10^{-2} \text{ day}^{-1} \), (b) \( 11 \times 10^{-2} \text{ day}^{-1} \), (c) \( 9 \times 10^{-2} \text{ day}^{-1} \), and (d) \( 11 \times 10^{-2} \text{ day}^{-1} \).
of the estimates, a close to neutral gross moist stability to the first baroclinic mode vertical motion, and the column MSE variations are almost entirely associated with second baroclinic mode vertical motion. When \( q \) is used, we have also estimated that \( L_0 \times r_0 \) is about 0.55. The resulting instabilities (not shown) are very similar to those resulting from the instability analysis of model using midtroposphere moisture.

4. The role of energy flow efficiency

This model has previously been analyzed for Kelvin-like waves on the equator (K08b), revealing some of the mechanisms determining the instability spectrum. Specifically, the MCD effect (e.g., Emanuel 1993; Nee- lin and Yu 1994) damps high-frequency waves, and the tendency of the second baroclinic mode convective heating to reduce existing midtropospheric moisture perturbations tends to reduce the instability at low wavenumbers. Expansion of the model to the beta plane in this paper has revealed the existence of at least one more mechanism within the model, responsible for the differences in growth rates for waves with similar frequencies. We have already shown that the ITCZ configuration and projection of the mean-state heating on to the modes’ temperature anomaly profiles exert some control on the shape of the growth rate curves. However, this control does not seem to explain all the aspects. We hypothesize that the growth rate curves for the various wave types are further shaped by the relative efficiency with which the various waves convert input energy into the divergent winds, which in turn generate more energy input. In this section, we will show that the growth rates can be directly linked to the generation of divergent winds in the modes. To explore the mechanisms that control the shape of the instability spectrum, in this section of the paper we consider a simplified system that has been modified to remove the mechanisms already identified in K08b and any ITCZ convective projection amplitude effects (discussed in section 3, above) through the choice of specific parameters. Some more detailed exploration of limiting cases within the model (in 1D) is conducted in K08b.

a. Physical description of simplified system

Our simplified system is based on the second limiting case of K08b, in which the convective mass flux is dominated by entraining parcels and the boundary layer is in quasi-equilibrium with the second mode temperature \( (\gamma = 0) \). The first temperature mode is essentially uncoupled from our system and can be ignored. The system is further simplified by setting \( b_2 = 0 \) to remove the influence of the second mode heating on the boundary layer moist static energy and \( \epsilon = 0 \) to remove the radiative damping; \( d_2 \) is also set to zero to remove the effect of second mode heating upon the midtropospheric moisture \( q \). These changes remove the scale selection of the moisture–stratiform instability.

Let us further tighten quasi-equilibrium to “strict quasi-equilibrium” by enforcing instantaneous relaxation to the equilibrium values of \( L \) (by setting \( \tau_L = 0 \)). This change removes the MCD effect.

A uniform \( L_0 \) profile is used, removing the ITCZ projection effects.

Importantly, the parameter \( r_q \), which controls the coupling between moisture and heating, is set to a very small value so that the heating can be approximated as an infinitesimal perturbation on the dry system.

This modified case is evaluated numerically (Fig. 13) in the same manner as the control case. The simplified system shows many unstable modes, but for this section we will concentrate on the Kelvin, MRG, \( n = 1 \) ER, and \( n = 1 \) IG modes, so the modes with \( n > 1 \) are deliberately omitted in Fig. 13. The Kelvin wave has a constant growth rate for all wavenumbers greater than zero. The MRG wave growth rate increases monotonically from the very small for large negative wavenumbers to relatively large at high positive wavenumbers, with rapid increase in the region between \( k = -5 \) and \( k = +5 \). In the domain evaluated, the MRG growth rate never reaches that of the Kelvin wave. The system also shows unstable ER and IG waves, with ER growth rates increasing from a low value for large (negative) wave-numbers to close to the Kelvin value for wavenumber zero, and \( n = 1 \) IG waves have a large growth rate for all wavenumbers, with growth increasing as \( |k| \) increases.

Within the spectrum of unstable modes are several modes located at \( k = 0, \omega = 0 \). These modes indicate an instability in our mean state that would lead to a mean-state drift away from the values we use here on a time scale comparable to the growth of the waves we are interested in. This is not considered a serious shortcoming here because we are looking at a very specific limiting case. In our more realistic cases treated earlier, such “climate drift” modes are not observed—all unstable \( k = 0 \) modes possess nonzero frequency and thus constitute oscillations about the mean state that, within the linear wave regime, do not affect other waves.

b. Energy flow from heating into divergent winds and heating feedback

To demonstrate our hypothesis, let us consider how the energy that flows into the wave due to our convective heating goes into increasing the divergence of the
flow. As we will show, the amplitude of the convective heating is proportional to the amplitude of the divergence, so this component of the energy drives the positive feedback, whereas energy that flows into the rotational component or the potential energy of the wave is essentially trapped there and cannot contribute to the further growth of the wave.

For this section, let us consider a wave that is essentially dry in character and that is forced by a diagnostic heating that approximates the simplified case discussed above. We will further consider how this wave changes with time over a short interval while the heating acts. The heating can be approximately written as a diagnostic function of the dynamical variables as follows:

\[ J_2 = -r_0 L_0 q \]  

(32)

(letting \( r_0 = 1 \)).

Combining the above equations, we have

\[ \partial_t J_2 = -r_0 L_0 \partial_t q = -r_0 L_0(a_1 - d_1)J_1 = \frac{r_0 L_0(a_1 - d_1)F}{b_1} \partial_t T_2. \]  

(33)

We can see that \( J_2 \) keeps the same phase as \( T_2 \). Further, because \( J_2 \) is a very small perturbation on the dry wave, we can also write

\[ \partial_t T_2 \approx c_2^2 \delta_2, \]  

(34)

where the divergence \( \delta_2 \) is defined by

\[ \delta_2 = \nabla_H \cdot \mathbf{u}_2; \]  

(35)

thus,

FIG. 13. As in Fig. 2, but for the simplified system described in section 4 and with the dispersion curves for \( h_{eq} = 80 \) m also plotted. All modes with meridional index \( n > 1 \) are deliberately not plotted, despite being unstable within this system.
\[ \partial_t J_2 = \frac{r_g(a_1 - d_1)Fc^2}{b_1} \delta_2. \]  

(36)

So we now have

\[ |J_2| \approx \frac{|\delta_2|}{\omega}. \]

(37)

Combining our above relations for the heating field, we define

\[ J_2 = -\frac{B|\delta_2|T_2}{\omega T_2}, \]

(38)

where \( B \) is a constant of proportionality, considered to be very small, so that the heating is only a small perturbation to the waves:

\[ B = -\frac{r_g(a_1 - d_1)Fc^2}{b_1}. \]

(39)

In this limiting case, our equation set reduces to:

\[ \begin{align*}
\partial_t u_2 &= \beta y v_2 + \partial_x T_2, \\
\partial_t v_2 &= -\beta y u_2 + \partial_y T_2, \quad \text{and} \\
\partial_t T_2 &= c_e^2 \delta_2 + J_2.
\end{align*} \]

(40-42)

From this point, the first mode is mathematically superfluous, so we can drop the subscripts without ambiguity.

c. Kelvin wave heating–divergence feedback

We start with a neutral Kelvin wave, with the standard form (assuming the dry dispersion relationship to be sufficiently accurate given the arbitrarily small heating) for the complex wave:

\[ \tilde{u} = u_0 e^{\left(-\frac{y^2}{2c}\right)} e^{ikx - \omega t}, \]

(43)

\[ \tilde{v} = 0, \quad \text{and} \]

\[ \tilde{T} = -c_e^2 k u_0 e^{\left(-\frac{y^2}{2c}\right)} e^{ikx - \omega t} \]

\[ = cu_0 e^{\left(-\frac{y^2}{2c}\right)} e^{ikx - \omega t}, \]

(45)

where the amplitude of \( \tilde{T} \) is determined using the standard wave relations (in wavenumber–frequency space) and the dry dispersion equation (see, e.g., Gill 1982).

We can then calculate the total energy (per unit length in the \( x \) direction) stored in the wave (\( H \) is the equivalent depth of the second mode):

\[ E = \frac{1}{2} u''H + \frac{1}{2} \frac{T^2}{g} = \frac{H}{2} \left[ u_0^2 + \frac{(cu_0)^2}{c^2} \right] \left[ e^{\left(-\frac{y^2}{2c}\right)} \cos^2(kx - \omega t) \right] \]

\[ = Hu_0^2 \frac{\omega}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \cos^2(kx - \omega t) \, dx \, dt \int_{-\infty}^{\infty} e^{\left(-\frac{y^2}{2c}\right)} \, dy = \frac{H}{2} \sqrt{\frac{c\pi}{\beta}} u_0^2. \]

(46)

The overbar denotes the time and zonal mean of a quantity, integrated over its meridional extent. The energy input in time \( \Delta t \) is given by

\[ \Delta E = \int \int \Delta t = \frac{-B|\delta|T^2}{\omega T} \Delta t = \frac{B|\delta|}{\omega cu_0} c^2 u_0 \left(\frac{\eta^2}{\beta}\right) \cos(kx - \omega t) \Delta t = \frac{B|\delta|u_0 \Delta t}{2k} \sqrt{\frac{c\pi}{\beta}} \]

(47)

\((T) = -cu_0, \ |\delta| \) is defined similarly).

With the increased energy in the wave after this interval, the amplitude of the wave (given by \( u_0 \)) will have increased:

\[ \Delta u_0 = (\partial_{u_0} E)^{-1} \Delta E \]

\[ = \frac{B\|\delta\|\Delta t}{2HK}. \]

(48)

Now, the divergence in the wave can also be determined:

\[ \delta = \partial_x u = -ku_0 e^{\left(-\frac{y^2}{2c}\right)} \sin(kx - \omega t). \]

(49)

We can calculate how the divergence increases in this time interval:

\[ \Delta \delta = -k\Delta u_0 e^{\left(-\frac{y^2}{2c}\right)} \sin(kx - \omega t). \]

(50)

Because \( \Delta \delta \) is exactly in phase with \( \delta \), we can also say that

\[ \Delta |\delta| = k\Delta u_0 \]

\[ = k \frac{B\|\delta\|\Delta t}{2HK}, \]

\[ \frac{\Delta |\delta|}{\Delta t} = \frac{Be}{2H |\delta|}. \]

(51)
As shown, the amplitude of the Kelvin wave divergence grows exponentially with time at a rate that is independent of wavenumber, as observed in the numerical analysis above.

d. Mixed Rossby–gravity wave heating–divergence feedback

The basic dry MRG wave is well described by its y-velocity field

$$\tilde{v} = u_0 e^{-\frac{(\beta y)}{2\omega}} e^{i(kx - \omega t)}$$

(52)

and a dispersion relationship

$$\omega = \pm \frac{k c}{2} \left( 1 + \sqrt{1 + \frac{4\beta}{k^2 c}} \right).$$

(53)

Then, using the standard forms,

$$\tilde{T} = -\frac{i\omega c^2}{\omega^2 - c^2 k^2} \left( -\partial_y + \frac{k}{\omega} \beta y \right) \tilde{v}$$

$$= i c u_0 \beta ye^{-\frac{(\beta y)}{2\omega}} e^{i(kx - \omega t)}, \text{ and}$$

(54)

$$\tilde{u} = \frac{\beta y \tilde{v} + i k \tilde{T}}{-i\omega}$$

$$= -\frac{i c u_0 \beta ye^{-\frac{(\beta y)}{2\omega}}}{ck - \omega} e^{i(kx - \omega t)}. \quad \text{(55)}$$

Similarly to the Kelvin wave example, we can find the energy per unit length stored in this wave:

$$E = \frac{H}{4} \frac{u_0^2 [c(ck^2 + \beta) - 2c k \omega + \omega^2]}{(\omega - ck)^2} \sqrt{\frac{\pi c}{\beta}}. \quad \text{(56)}$$

Then, the increase in the wave amplitude due to an increase in energy $\Delta E$ is given by

$$\Delta \tilde{u} = \frac{2 \Delta E (\omega - ck)^2}{H u_0 [c(ck^2 + \beta) - 2ck \omega + \omega^2]} \sqrt{\frac{\beta}{\pi c}}, \quad \text{(57)}$$

and the increase in energy over $\Delta t$ is

$$\Delta E = \frac{Bc^4 u_0 |\Delta t|}{4\beta \omega (\omega - ck)} \sqrt{\frac{\pi c}{\beta}} \quad \text{(58)}$$

$$|\Delta t| = cu_0 \beta (ck - \omega)$$

leading to an increase in the wave amplitude of

$$\Delta \tilde{u} = \frac{bc^3 |\Delta t| (\omega - ck)}{2H \omega [c(ck^2 + \beta) - 2ck \omega + \omega^2]}, \quad \text{(59)}$$

the divergence of the MRG wave is

$$\delta = \frac{u_0 \omega \beta}{c^2 k - c \omega} ye^{-\frac{(\beta y)}{2\omega}} \cos(kx - \omega t) \quad \text{(60)}$$

and the magnitude of the divergence is

$$|\delta| = \frac{v_0 \beta \omega}{c(\omega - ck)} \quad \text{(61)}$$

(note the change of sign so that the magnitude of the divergence is positive—this simplifies interpretation). Thus,

$$\frac{\Delta |\delta|}{\Delta t} = \frac{Bc}{2H} \frac{|\delta|}{|\delta|} \left[ \frac{c \beta}{(c k^2 - \beta - 2ck \omega + \omega^2)} \right]$$

$$= \frac{Bc}{2H} \left[ \frac{\beta}{2\beta + \frac{k^2 c}{2} \left( \sqrt{1 + \frac{4\beta}{k^2 c}} \right)} \right], \quad \text{(62)}$$

where the upper sign applies for positive wavenumbers and the lower sign applies for negative wavenumbers.

e. Equatorial Rossby and inertio-gravity wave heating–divergence feedback

A similar analysis for the $n = 1$ Rossby and inertia-gravity waves yields

$$\Delta |\delta| = \frac{Bc}{2H} \left[ \frac{c(3c^2 k^2 + 2ck \omega_n + \omega_n^2)}{c^4 k^4 + 3c^2 k^2 \beta + 3c \beta \omega_n^2 + \omega_n^2 + 2c^2 \omega_n (\beta - k \omega_n)} \right], \quad \text{(64)}$$

where

$$\omega_{n=1} \approx \left( -\frac{\beta k}{k^2 + \frac{3\beta c}{c}} \right) \sqrt{k^2 c^2 + 3\beta c} \quad \text{(65)}$$

for the Rossby and IG waves respectively (see, e.g., Gill 1982), obtained from the dispersion relationship

by ignoring the first and last terms on the left-hand side, respectively.

f. Energy feedback results

Figure 14 shows the growth rate of the MRG, $n = 1$ ER, and $n = 1$ IG waves in this construction, normal-
ized by the constant growth rate of the Kelvin waves. The growth rate of the MRG increases as the divergent component of the wave increases from a small value at negative wavenumbers (where the wave is very much like the low-divergence ER waves) to a large value at positive wavenumbers, where the wave most resembles the high-divergence Kelvin wave. Just as observed in the numerical study (Fig. 13), the growth rate increases rapidly between \( k/H = 5 \) and \( k/H = 15 \).

The growth rate of the \( n = 1 \) ER wave increases as \( k \) increases from \( -\infty \) to zero, as observed in the numerical results. The growth rate of the \( n = 1 \) IG wave increases as \( |k| \) increases from a small nonzero value at \( k = 0 \), as observed.

It is interesting to note that the Rossby waves achieve growth rates very close to those of the Kelvin waves because the Rossby waves are typically considered to be dominated by vorticity. However, this can be explained by visualizing a very long-wavelength Rossby wave \( (k \approx 0; \) refer to the left column of Fig. 15). This wave will consist of nearly uniform motion away from and toward the equator, with zonally uniform meridional velocity and near-zero zonal velocity. In this case, the wave clearly has a large divergence and very small vorticity. As the wavenumber increases, the contribution of the zonal divergence increases, offsetting the meridional divergence (because of the phase difference). At large wavenumbers, the two components cancel, leading to reduced divergence and a wave that has significantly larger rotational flow (right column of Fig. 15).

The inertio-gravity wave spectrum is explained by similar logic, although the details are different because of the different structures of the waves, particularly the nonzero vorticity at zero wavelength and the increasing divergence with wavelength.

The numerical values of the corresponding growth rates (also normalized to the numerical Kelvin wave growth rate) are also plotted on this figure for comparison. The MRG and ER growth rates are essentially indistinguishable from the analytic solutions. The discrepancy for the inertio-gravity waves stems from the approximation used for the wave frequency in Eq. (65), which is only accurate to within 13% in this case (see, e.g., Gill 1982), whereas the error in the ER frequency is less than 2%.

The physical content of this argument is contained more in the method than the final expressions. The construction of instability mechanism here emphasizes the differing efficiencies with which the modes can convert input energy into divergent flow, driving more energy input. As the growth rates derived from this con-
struction of the model resemble the numerical results closely, we feel that this vindicates our hypothesis, illuminating a further mechanism that influences the linear instability spectrum of the equatorial convectively coupled waves.

The specifics of our derivation assume the basic form of the moisture–stratiform instability where low-level divergence leads to second baroclinic convective heating a quarter-cycle later as discussed in the introduction. The detailed result is thus dependent on the existence of the moisture–stratiform (or similar) instability mechanism.

5. Summary and discussion

In this paper, we have demonstrated that our extension of the model of K08b to the equatorial beta plane allows us to produce a linear system with a spectrum of unstable modes that bear a good resemblance to the modes visible in the global OLR data, both in terms of excited wavenumber range and also in terms of the wave types present. The unstable Kelvin, MRG, and IG waves visible in the results of WK99 are reproduced well using this model.

Our model is closely related to that of Khouider and Majda (see, e.g., Khouider and Majda 2006a,b, 2008), especially in the inclusion of midtropospheric moisture as a control of convection depth. However, by making our model conceptually simpler than their model, the various instability and spectrum-shaping mechanisms are made clearer. The further extension of our model to the beta plane also allows the investigation of instability shaping due to the mean-state profile and the meridional winds of the various waves observed on the equatorial beta plane.

The model does fail to produce any significantly unstable ER or any MJO waves. As stated above, this may be due to our model possessing too great a low-frequency damping or lacking an effect that serves to destabilize the low-frequency waves. However, the observations of Yang et al. (2007) indicate that the Rossby wave signals detected in the OLR spectrum might have...
a different instability character. For example, they may be unstable only to finite disturbances, requiring a large extratropical Rossby or near-resonant MRG wave to excite them. Or possibly the waves are simply stable, with their decay in balance to near-constant stochastic forcing from the extratropics and the MRG activity. The reason for the lack of MJO activity is at present unknown.

Previous interpretations of the spectrum, like those of Takayabu (1994) and WK99, use the differing projections of the mean-state heating onto the wave structure to explain the east–west asymmetry. In the context of our model, it was also observed that the position and size of the ITCZ affects the growth rates of the different waves: modes that possess a large spatial overlap with the ITCZ profile are enhanced, whereas those with a small overlap are weaker. However, even when the ITCZ configuration provides large overlap for the MRG or IG waves and small overlaps for the Kelvin waves, the Kelvin wave growth rates remain large compared to those of the MRG and IG waves, indicating that additional factors affect the growth rates of different wave types. We suggest that the efficiency with which waves convert input energy into divergent winds is an important factor. We note that Lindzen (1974) also argued that the east–west asymmetry in the $n = 1$, 2 IG waves may be related to the differing amounts of convergence in the different propagation directions. Our mechanism shares some characteristics with his argument in that the growth of modes is related to the strength of the divergent winds of that mode, except that our mechanism is based on the moisture–stratiform instability instead of the classical wave–conditional instability of the second kind (CISK) as in Lindzen (1974). We return with more discussions of wave-CISK at the end of the paper.

With the addition of our energy flow divergence feedback mechanism, the basic shape of the wave spectrum is starting to become clearer. The linear part of the spectrum is shaped by the combined actions of moist convective damping, which damp high-frequency waves; the damping effect of the second mode convective heating on midtropospheric moisture anomalies, which damping more strongly the lowest-frequency waves; the ITCZ projection effect; and the energy flow divergence feedback, which selects the wave types with larger divergent components.

Analytical extension of the energy flow feedback to our full set of equations is expected to be nontrivial; furthermore, our discussion is based on a simple linear model of convectively coupled waves. However, it seems likely that the principle of “wasted energy” in the rotational component of the velocity fields of the equatorial waves will also apply in the real atmosphere, helping to shape the observed spectrum of convectively coupled equatorial waves.

The present model does not make use of the column MSE budget so that it does not make a statement of the gross moist stability. One could use the vertically averaged free tropospheric moisture in place of the midtropospheric moisture and take advantage of column MSE conservation, as done in Khouider and Majda (2006a) and discussed in section 3f. The gross moist stability to first baroclinic motions in the CSRM simulations of Kuang (2008a) is found to be close to neutral. Because we also neglect radiative and surface flux feedbacks, the moisture mode found in, for example, Fuchs and Raymond (2002), Raymond and Fuchs (2007), and Sugiyama (2008a,b, both manuscripts submitted to J. Atmos. Sci.) is absent here. We plan to include the radiative and surface feedbacks to examine the moisture modes in the future.

Last, given the long history of wave-CISK (Yamasaki 1969; Hayashi 1970; Lindzen 1974), we would like to briefly discuss the relation between wave-CISK, the direct stratiform instability in Mapes (2000), and the moisture–stratiform instability in K08b and the present study. The direct stratiform instability may be viewed as a two-mode discretization of the classical wave-CISK envisioned in Lindzen (1974), in which waves modulate convection, which then further generates waves, and for waves of certain structures and phase speeds, the two can reinforce each other. Indeed, as shown in K08b, when the lag time between stratiform heating and deep convective heating is neglected, the growth rate of the direct stratiform instability increases linearly with wavenumber, similar to that observed in classical wave-CISK. Our moisture–stratiform instability mechanism relies on the effect of midtropospheric moisture anomalies on convection, and hence differs from the classical wave-CISK ( unlike the classical wave-CISK, it has bounded growth rates at short wavelengths even without the MCD effect). The moisture–stratiform instability could be also viewed as wave-CISK in a very broad sense, if one defines wave-CISK as a self-exciting cooperative instability between wave and convection that does not require surface flux or radiative feedbacks (Bretherton 2003). However, it is unclear that this association is particularly meaningful because the definition of wave-CISK in this case would be so broad that it would not contain the actual physical processes that are at work.

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2 This was pointed out to us by Chris Bretherton.
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REFERENCES


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We have noticed that the following error was made in Andersen and Kuang (2008). The calculation of the Rossby wave divergence and vorticity fields as plotted in Fig. 15 was erroneous. A corrected version of Fig. 15 is shown below. While this change does not alter the major conclusions of the paper, there are some changes necessary to our interpretations in section 4f, including the addition of Fig. 16.

The second full paragraph on page 3754 (beginning “It is interesting to note …”) should be replaced with the following paragraph:

It is interesting to note that the growth rates of the Rossby waves are greater at longer wavelengths. This can be explained by considering how the wave energy generated per unit divergence and the divergence increase per unit energy increase vary with wavenumber. For Kelvin waves, the wave energy generated per unit divergence is proportional to \(1/k\), while the divergence increase per unit energy increase is proportional to \(k\), leading to constant growth rates. Although for very long wavelength Rossby waves the divergence per unit wave energy is small relative to the Kelvin waves, the growth rate is still high because the wave energy generated per unit divergence is higher relative to the Kelvin waves (Fig. 16). The wave energy generation per unit divergence decreases very rapidly with wavelength (relative to the Kelvin waves) because of the rapid decrease with wavelength in temperature anomaly per unit divergence (see Fig. 15; in this simplified case, temperature and heating are collocated). However, the fraction of energy that is sent to the divergent flow (again normalized by that of the Kelvin waves) actually increases with wavenumber until intermediate wavelengths (around \(k = 7\)), slowing the decrease in growth rate. At large wavenumbers, both these terms are small, leading to small growth rates.

Section 5 also requires a small amendment. The second paragraph on page 3756 (beginning “With the addition of our energy …”) should read as follows:

With the addition of our energy flow divergence feedback mechanism, the basic shape of the wave spectrum is starting to become clearer. The linear part of the spectrum is shaped by the combined actions of moist convective damping, which damps high-frequency waves; the damping effect of the second-mode convective heating on the midtropospheric moisture anomalies, which damps more strongly the lowest-frequency waves; the intertropical convergence zone (ITCZ) projection effect; and the energy flow feedback effect, which selects the wave types with larger divergent and/or geopotential components. In more realistic cases, the heating is not in general coincident with the temperature (or geopotential) field, so it will also be necessary to consider the spatial correlation between the two.

We have also noticed some typographic errors that do not affect the main results:
The final equality in Eq. (46) should read

\[ \ldots = \frac{H}{2} \sqrt{\frac{c \pi}{B}} u_0^2. \]

Equation (47) should read

\[ \ldots = \frac{H}{2} \sqrt{\frac{c \pi}{B}} u_0^2. \]
Then, the $c$ in the numerators of Eqs. (48) and (51) should be deleted.
Equation (58) should read
\[
\Delta E = \frac{B|\delta|T^2\Delta t}{\omega |T|} = \frac{B|\delta|u_0^4e^{(-\beta y^2/2c^2)}\cos^2(kx - \omega t)\Delta t}{2k \sqrt{\beta}}
\]

Then, the $c$ in the numerators of Eqs. (48) and (51) should be deleted.
Equation (58) should read
\[
\Delta E = \frac{Bc^4u_0|\delta|\Delta t}{4\beta\omega(\omega - ck)}.
\]

The final term in the numerator of Eq. (64) should be $3\omega_{n=1}^2$.
The authors apologize for any inconvenience these errors may have caused.

REFERENCE